Lecture – 59

Characterization of Dedekind Domain

So, today the last lecture and we are left to prove three important Characterization of Dedekind Domains. So the first theorem, this is theorem illustrated last time but we did not prove. So let A be an integral domain then the fallowing are equivalent.

One, A is normal. And Krull dimension of A is smaller equal to one. Let me remind you, when we see a normal that includes noetherian [1:24 inaudible]. SO, one A is normal domain that means A is an noetherian, it is closed domain.

Two, A is noetherian and for every prime ideal p A localize at p is a DVR for every non-zero p, p for prime ideal. [2:14 inaudible]

And three, for every fractional ideal in A is invertible. Okay, so, we just call a fractional ideal means it's a A sub-module of A quotient field of A, who has a common denominator. And invertible means, this fractional ideals [2:54 inaudible] and then a, the element is invertible means [2:58 inaudible]. So let us first finish off the proof of this equalance.

So, proof: I am going to prove one if and only two and two if and only three. These are [3:18 inaudible] So let us prove one implies two, A is normal so noetherian dimension is less equal to one, so there can only be prime ideal of height zero or height one. So, if it is non-zero prime ideal then it has have height one, because it's a domain and therefore localization is usually normal of dimension one, that [3:56 inaudible] do here. So this one is really clear but I will write few heights since A is normal, A is noetherian,
And let p non-zero prime ideal, then height of has to be one because height is at most one, so it can't be, if it is zero it is a zero prime ideal but its non-zero prime ideal, its height is one. So Ap, A localization p is normal because A is normal, localization of normal is normal. And hence DVR. Because normal it includes, dimension one, their DVS. So, one implies two, two implies one

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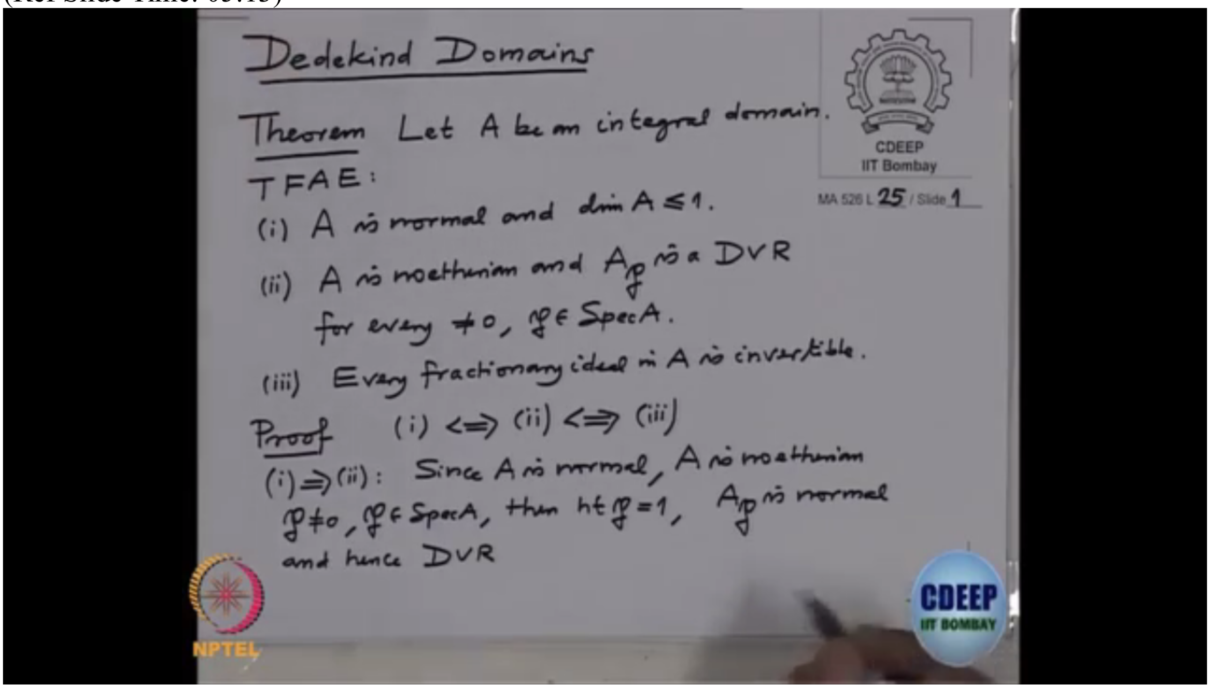

We want to prove that [5:16 inaudible] from this given conditions, we want to prove a is literally close and the dimension is less equal to one. So let see about dimension, so, dimension of A is by definition supremum of height p, as p varies in the spectrum, this is the definition of the Krull's dimension which is Sup instead of height I will write the Krull dimension of A localize at p, p in Spec A. So all this, for non-zero prime we have given that by two a localize at p is [6:09 inaudible] so for non-zero primes this dimension is one, and for zero prime it's a quotient field, so therefore it is actually equal to one, so it's less equal to one by two.

Now we need to prove that [6:30 inaudible] close and also, okay. So therefore every prime ideal associated to a non-zero ideal in A has height equal to one. Now there are two possibilities only, zero or non-zero. Zero or height one. So if it is associated to non-zero ideal it has to have height one, therefore we have proved earlier that A is normal. This has proved earlier. So that proves one if and only if two.

Now two implies three, [7:58 inaudible] invertible. And two we've given its noetherian every localization at non-zero prime ideal is a DVR.

Okay. SO, start with the fractional ideal, we want to prove it invertible. Let A containing K, this is quotient field of A be a fractional ideal. Now that means, A sub-module of K which has a common denominator, then by multiplying by common denominator d times a, this is contained in a , now this an ideal, usual ideal, this is a ideal in A, [8:57 inaudible] d, d is in A, d is non-zero. That is because it's a fractional ideal. Now, if I want to prove this fractional ideal is invertible it's enough to prove that this da is invertible because we have proved d a is invertible and d is already notable. Therefore multiplying by a inverse of d we'll get. So, enough to prove that d times a is invertible. In another words we can assume that the given fractional ideal is actually integral ideal. Okay. So, now since a is noetherian this, let's call this da to b, d is finitely generated ideal, but once it is finitely generated this colleen operation will come out with the localization. For this equality we need finitely generated. So once the colleen operation commutes for every prime ideal p, if p zero it's a field, and if p is non-zero this Ap is a DVR, that tis what we have given in two.

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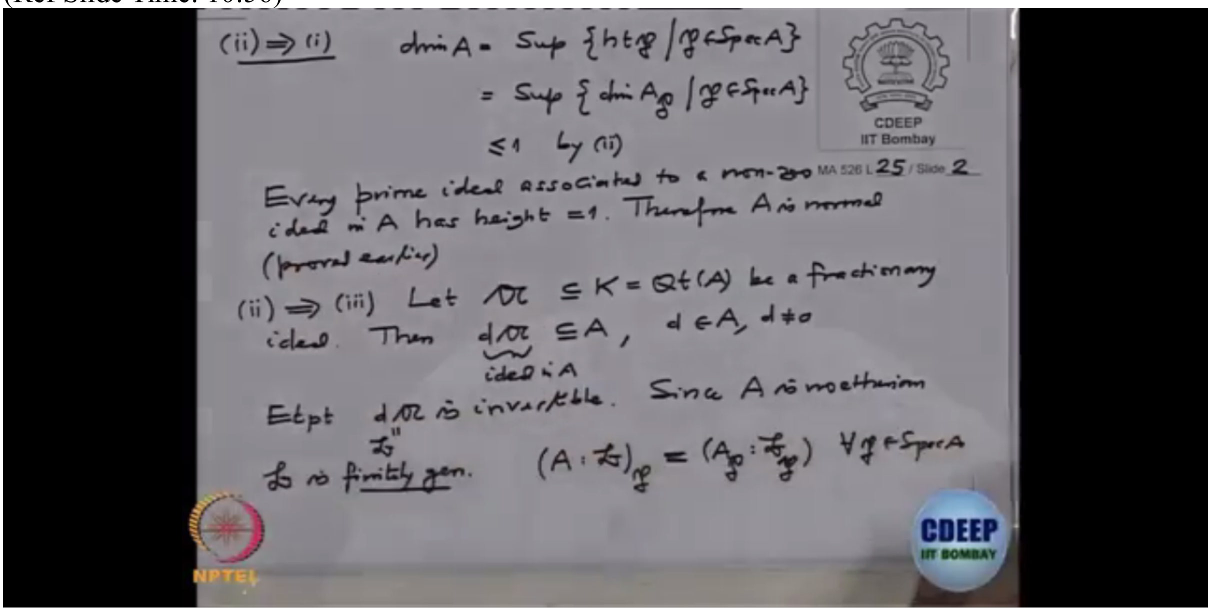

So in particular, when p is non-zero for p non-zero since Ap is a DVR, this ideal bp is principal, in particular ideal it is invertible, hence invertible. Not that quotient field of A and quotient field of bp is a same. So everything is happening in K. So non-zero pit's invertible. And we already know [11:47 inaudible] the expected inverse. So, A localize at p is same as bp times A colleen, A localize at p colleen b localize at p. This is because it's invertible but this now, localization commutes with the colleen operation because we are in a noetherian case, so this is same as b times A colleen b then whole thing localize at this. So at every localization there are equal and we've have already seen one is containing the other, namely this product was contained in A, therefore we can prove from [12:42 inaudible] that A is equal to b times A colleen b. So that show that b is invertible. So that proves two implies three.

Now three implies two we need to prove that A is noetherian and at every localization on 0 at prime ideal every localization is a DVR. All right. So 3 in place 2, so since agree we have proved every invertible ideal is been proved last time every invertible ideal is finitely generated. Just yesterday you proved this in particular every ideal is finitely generated, so A is noetherian, because every integral, if the assumption 3 is every factionary ideal is invertible. So in particular every ideal is invertible. And every invertible ideal is finitely generated, therefore every ideal is finitely generated, therefore A is noetherian. And now so we have proved A noetherian, now we want to prove that every localization at a non-zero prime ideal is DVR. So start with non-zero prime ideal. So p is factionary ideal, p is integral, therefore factionary and therefore by assumption 3, this is by 3, p is invertible, therefore the localization is invertible, but that shows that the maximal ideal is invertible so that show that A p is a DVR. Because yesterday we have seen that if you have a local noetherian local domain, then A is a DVR, if and only which maximal ideal is invertible that was the only one, we have to check only one ideal is invertible. So that proves 3 in place 2 and that finishes the proof of this theorem.

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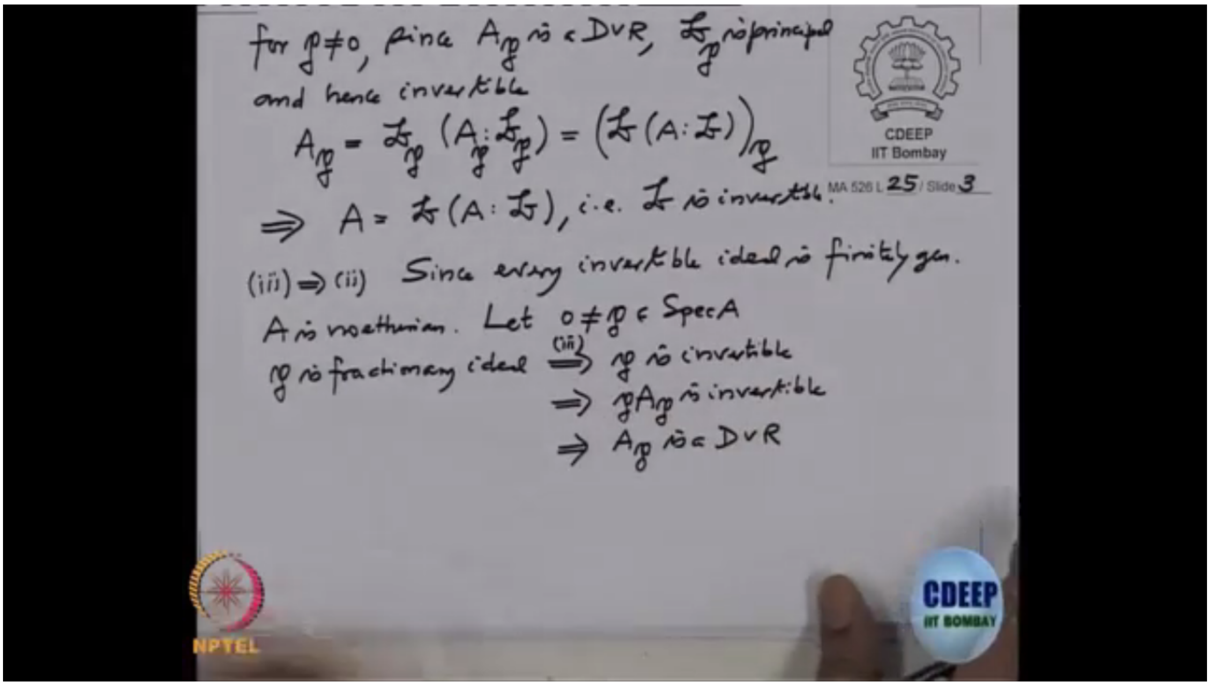

And now we want to prove that so this three conditions. So this is the 1 2 3. So in addition to this three I want to add one more condition here namely every ideal is a product of prime ideals and this decomposition is the unique, so which will generalize the standard theorem on prime characterization for the ring of integers. So this is the theorem we want to prove. So theorem A is integral domain, an integral domain, A is a Dedekind Domain, A is a DD if and only if, every ideal of A is a product of prime ideals. And a uniqueness, I will state in a more precise form in the corollaries. So proof: Okay. So we will prove this way first. So we are assuming A is a Dedekind Domain. So suppose that A is a DD and let A be a non-zero ideal in A. Now what do you want to do we want to prove that A is a product of prime ideals. For every non-zero prime ideal, because A is a DD and the earlier theorems said that for every non-zero prime ideal A localize at p is a DVR by earlier theorem. And this is a DVR where p A p is the maximal ideal there and if I choose this A, if A is containing p, then this A, extension of A in the localization, this is a proper ideal and this will be now, it's a power of that maximal ideal because A p is a DVR and if you were not containing A p, then it's a unit ideal. Then it's, we have nothing to worry about it. So in this case this is p A p and it's on power, that power I want to denote by v p a, and in this case, this v p is non-negative. And clearly that this v p is so clearly v p a is positive if and only if a is containing p if and only if a A localized p A is containing p A localized p all this are clear. So therefore locally it's a power of a, actually a prime ideal. Okay. So now we note that the Dedekind Domain as dimension less equal to 1. Since dimension of A is less equal to 1, minimal prime, a minimal prime containing a, will belongs to the associated primes of A by a. And therefore now you see it can have only the finite, this only the finitely many, associated primes are finitely many, so it as at most finite number of prime ideals so there are at most finitely many prime ideals containing a, remember a here assuming non-zero.

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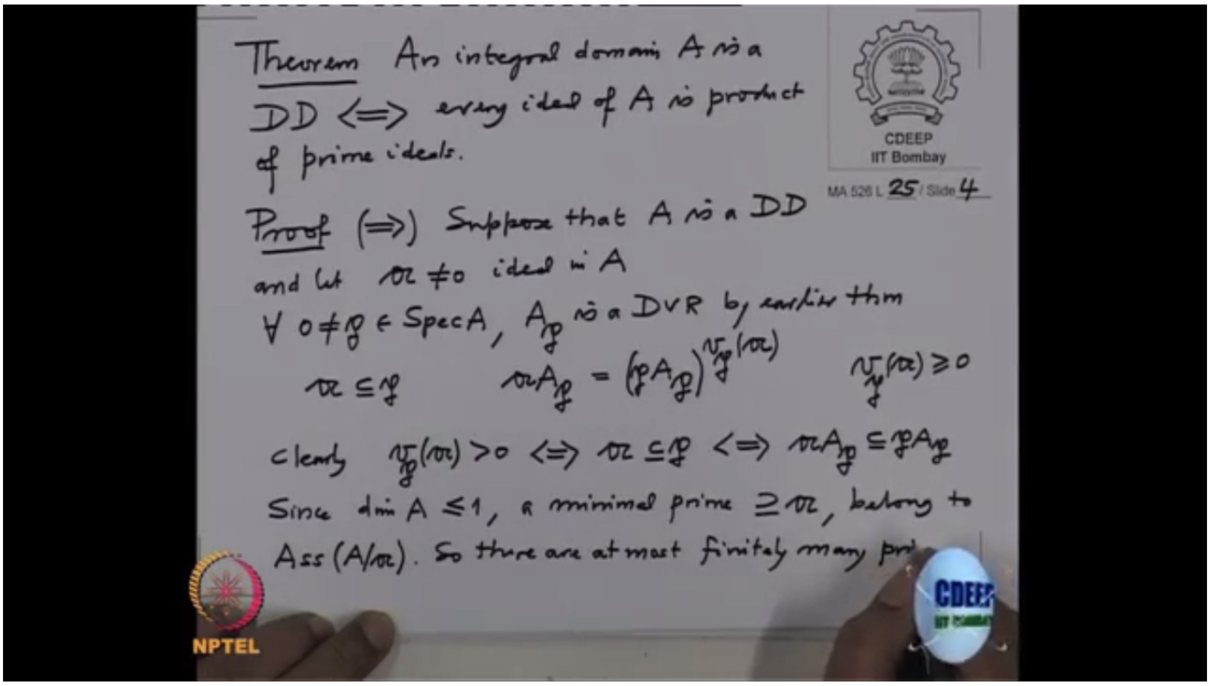

Look if a is zero there is nothing prove. So therefore, so this means, that is, v p of a is non-zero only for finitely many prime ideals or in another word is z p a is 0 for almost all. Therefore, thus, if I take the product, product is [23:00 inaudible] over p, p power v p a, this makes sense, because this product is only finitely many primes this, these are, so these are because this symbol is nonzero only for finitely many p. So this product is indeed finite product. And if I call this, this is clearly an ideal, if I call this ideal b, b is the product of prime ideals and know I will prove that v equal to a, once I proved v equal to a, we are done. So we will prove b equal to a. First of all note that this b is contained in a, or if I want to prove b equal to a, I will prove that the localizations are equal so you should localize, bap that means I have to localize this part, this is the finite product. This a finite products and if I localize this [24:27 inaudible] localization only one of them will survive because this p is different, so this is same as p power v p a but this is same as a localize p that is what we have noted above. That this is a DVR and therefore this ideal is a power of that maximal ideal and that power is denoted by this. So, this one. Yes. Yes, here. I have to write A p. So, therefore that proves b equal to a, that localization there equal. And therefore it proves this implication if it is dedicated domain then every ideal is a product of prime ideals. [25:30 inaudible] little bit more invert. So conversely we will use some Lemmas. So we will start doing by one by one, where two or three lemmas. There are simple but you need to prove them. And you observe very carefully, this proof is an imitation for the same proof [26:07 inaudible] fundamental theorem of athematic.

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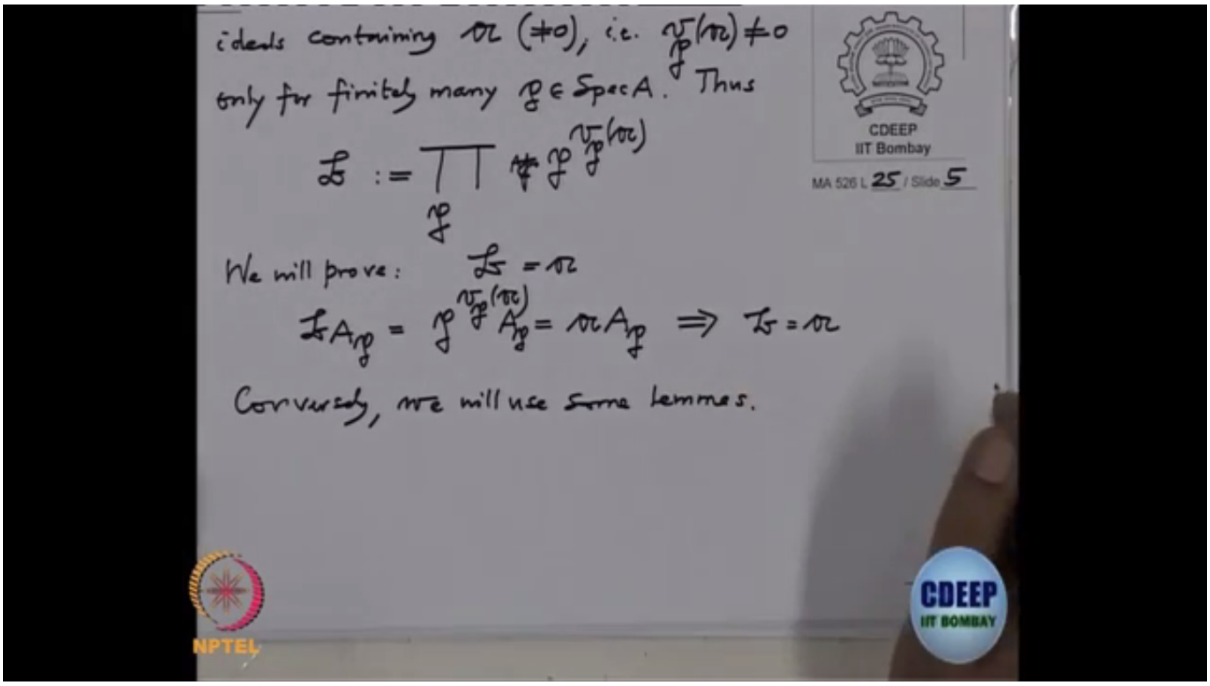

Same ideas only the numbers are replaced by ideals.