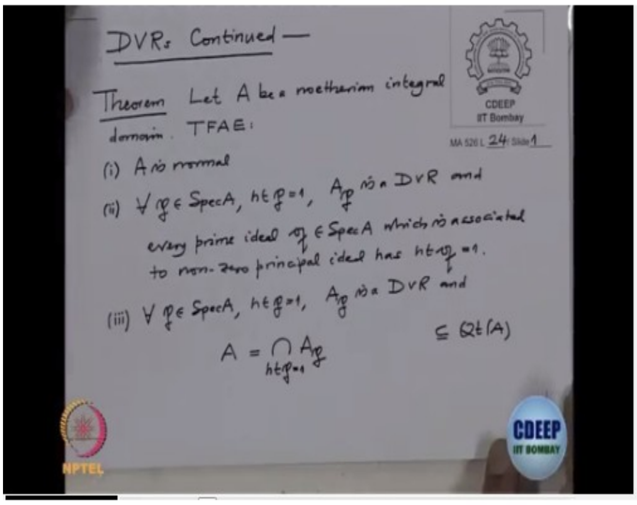
Lecture – 57

Dedekind Domains

Gyanam Paramam Dhyeyam: Knowledge is Supreme.

So we will continue sometime our study of DVRs, Discrete Values and Rings. I want to give a characterization of a normal domains in terms of Discrete Values and Rings. So the theorem we would prove fist is, Let A be neotherian integral domain then the following are equivalent. One, A is normal domain. Two, for every height one prime, for every p primary ideal of height one, the localization is, localization of A at p is a DVR and every prime ideal q which is associated to principle ideal to a non-zero principal ideal as height not $\mathbb{Q}$ is 1. That is second condition. Third one, same for every p prime ideal of height one, A localized at p is DVR and if I take the intersection or all height one primes in a localization at p this intersection is happening the coefficient field of A because A is the domain all localization the subring of the quotient field. So this intersection is here, so this is A. So then this three conditions are equivalent so normality you can characterize by Discrete values and rings at height one primes. Okay, let's prove this.

(Refer Slide Time: 3:44)

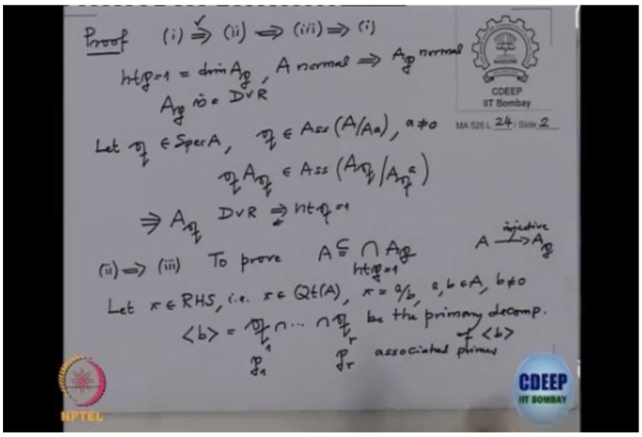


So proof, I going to prove one implies, two implies, three implies one. So first one implies two. One we are assuming normal. The first part is clear because if p is a height one, then this is also the dimension of localization and its one dimensional local ring and because A is normal localization is also normal. So it is one dimensional neotherian in those rings so therefore it is DVR. A normal so that implies Ap normal, and hence by earlier theorem that Ap is a DVR. Second condition we have to prove that every prime ideal q which is associated to a non-zero principle ideal should be height one. Okay, so let us take prime ideal q, and let q be a prime ideal. And suppose q is associated to $\frac{A}{AW}$ where a is a non-zero element that is the meaning that $A q$ is associated to a non-zero principle ideal. And therefore if I localize at q. $Q(Aq)$ this will be associated to A localized q, A localized q a for the localization associate, remains associated.

Look at this ring A localized q, this is a local ring where maximal ideal is associated to a non-zero principle ideal and that was one of the criterion in when we characterize DVS. If we have a local ring where the maximal ideal is associated to a principle ideal it's a DVR. It's a height one. So this is DVR and therefore height q is one. So that proves one implies two. Now two implies three, we have given that at height one primes, the localization is DVR. So the same condition is given in three also. So we have only have to prove that the intersection only to show A is the intersection of Aps where is p is running over height p equal to one. This is what you get. And in this also, this inclusion is clear because A to A_p is always injective. This is injective. And everything that happening in quotient field. So therefore we have only to prove the other way inclusion. So for that lets take X in the RHS, and I want prove x is in A. Okay, So X is in the coefficient field. Because this intersection is happening in the coefficient field. So therefore this x belongs to that is X belong to a coefficient of field of A, so we can x as a fraction a or b, where a, b are elements in A, and b is non-zero. All right, now I will give you the primary ideal composition for the principle ideal b.

So look at the ideal b, ideal generated by b and look at it primary ideal composition. So this is equal to Q1 intersection, intersection, intersection Qr, b the primary decomposition of the principle ideal b. And now we take the reduce primary decomposition. And the corresponding associated primes, let us denote them by $p_1, \ldots, p_r$, associated primes. Okay, but in two we are assuming that if we have a prime ideal q which is associated to a non-zero principle ideal it has to have height one. So all t these ps will have height one.

(Refer Slide Time: 9:30)



That is given by assumption into all height pi they are one for I equal to one to r. Because p either associated to the principle ideal. So when I localize b localized pi only qi will survive. Because there can't be any inclusion relation between them so therefore only pis will survive. This is true for every i. But this means, A , $\frac{a}{b}$, look at our $\frac{a}{b}$, this was our x and this belong to the all intersections running over height one primes. So in particular this one belongs to a localized pi for all i. So this simply means, that the ideal, the element a belongs to the ideal b generated by b in A localized p_i . But this is q_i A localized p_i and this is true for all i. But then on the other end this Al belongs to the ring a so therefore this A belongs to A intersection q_i A localized p_i , this is true for all i. But this is same as a primary component we are localized and intersects back. So this is get backs with q_i . This is true for every i. Bu then a belong to the intersection. So that is A belong to the intersection of, q one intersection, intersection, intersection q_r which was the primary decomposition of the principle ideal b. So a belongs to the principle ideal b, so that will imply $\frac{a}{b}$ which was x. this will belong to a because this a is the multiple of b, when I write a multiple of b, b gets canceled and integrated in a. So that proves two implies three. Now let us prove three implies one. So three is, we have given three that is all height one prime A_p s are DVRs and intersection of localization at p where intersection is running over height one primes should be the ring a and from here we want to prove the ring is normal. So that means, we want to show that a is an integral close domain. We have already given a is neotherian integral domain. So enough to prove that a is integral close. A is integrally close. That means we have to prove that if we have an element in the quotient field alfa belongs to the quotient field of a and in alfa is integral over a then it should be in a. So alfa is integral over a suppose, that alfa is integral over a, then remember we want to prove. This is what we want to prove alfa belongs to a.

Well, if alfa is integral over a then also it will be integral over A localized p for every p actually. Because a localized p is bigger than a. In particular it will belong to now what is given part of the three, if I take height one primes then it's a DVR, so therefore alfa belongs to a localized p for all p with height of p equal to one because they are DVRs and DVRs are really close. But then alfa will belong to the intersection of all A_p s where is this intersection is raining over height p equal to one. But by the second assumption in three, this intersection is given to be A. This is by assumption in three. So that proves that alfa belongs to a therefore a is ideally close, that a is normal. Okay, so I will mention one more theorem here, which I will not prove I think, you might have seen earlier that this theorem.

Let A be an integral domain then A is normal, if and only if polynomial ring in finitely mini variables $x_1, \ldots, x_n$ is also is normal. This is proof. I think, maybe I if I remember correctly I have written in the supplements in exercise one.

So now I want to prepare for Dedekind domains. So Dedekind domain definition I give but I want give a characterization of Dedekind domain. So first, we want to start with Dedekind domains. So give the motivation, Dedekind one, these are names after Dedekind. And Dedekind wanted to abstract the properties which are in the ring of integers. So ring of integers we know, elementary theorem of, elementary fundamental theorem on elementary number theory says that every integer, non-zero integer is a product of prime numbers. And this is product is essentially in a unique that means, up to an order of the primes and up to the associates. But associated in this ring are just plus minus one. The units are not too big. So now if I want to generalize this in a integral domain first. Let's say, integral domain but now, they'll several difficulties will occur namely, the unit group may be too big so we can't say-- Okay, that is one difficulty. Another one is, definition of prime, so prime numbers here will corresponds to the prime ideals. The definition, right? These are now denoted by the gothic letters p. And now the usual integers are replaced by the ideals you see the divisibility is replaced by the contentment. So then one would like to know, or one would like to characterize all integral domains which have the property that every ideal in that is a product of prime ideals and in essentially in a unique way. So this is the theorem we want to prove. So those rings are called Dedekind domains. So this is one of the characterization and then definitions then now people make always abstract definitions. So the definition was, we will take this as a definition. And what I said that will be the theorem.

So the definition says, a neotherian integral domain is called a Dedekind domain if it is integrally closed dimension of dimension it's a less equal to a. So obviously dimension will be one only because it's a domain so the only way that it will dimension zero it's in the field. So this allows to say that, field the Dedekind domains. Okay, so we have example that fields are Dedekind domain also PIDs are Dedekind domain. So let us see few examples. One, any PID is a Dedekind domain. I will write Dedekind. Or any DVR in a particular DVR implies it's a Dedekind domain. So we have many examples. So polynomial ring over a field. So polynomial ring in one variable over a field, powers in one variable over a field, this is typically a DVR, this typically a PID. The ring of integers is also Dedekind, okay. More than that now, another important aspect of the Dedekind domains was the following examples which do come from number theory. So take a number field, k is number field, number field simply means, its finite extension of q, finite field extension of q is called a number field. And we have here the ring of integers $\mathbb{Q}$ is quotient field of $\mathbb{Z}$ and now we take the integral closure of $z \in K$ so that is some people denote it by O_K or A_K . So this is integrally closure of $z \in K$. Now question is, whether is it a Dedekind domain? So let's see, what are the obvious facts? Obvious facts as it's a domain, dimension is one because this extension in integral and z is dimension one so dimension of A_k is one. Also A_k is integrally close. So the main difficulty is to prove it in neotherian and that is because of this is one of the motivation for the neotherian. Neotherian was the first who realize that it's important to study these examples. So it is neotherian in track. This more general or some remarks concerning this I have written in the exercise three, latest I have added this. So in fact, it would be a free modulo z of finite rank. So in fact, A_k is a free $\mathbb{Z}$ module of finite rank. This was proved by Stimke. This is not so difficult prove, I will add it in a, what is equal to a rank? Yes, rank will be equal to the degree of the field extension. Finite field of finite rank equal to K over $\mathbb{Q}$. Okay, so all these are Dedekind domains. But now, another very important question from number theory and many important questions, when is it a UFD? So this is a Dedekind domain, so it's a locally a PID but when it is a. So when is A_k UFD? This question still doesn't have a good satisfactory answer till today even when the number field of dimension two, even quadratic extensions. For imaginary quadratic field extensions there is a good answer for that this is by Gauss and Gauss have guessed it but it was proved later in 1960s by Stark and others. The proof is not completely algebra, it involves lot of algebraic number theory. And for the positive, I mean the real quadratic fields it is still wide open even for degree two extensions. So there is a lot of scope to do research.

(Refer Slide Time: 26:34)