Lecture – 56

Discrete Valuation Ring (Contd)

Gyanam Paramam Dhyeyam: Knowledge is supreme.

So, we have proved 4 in place 5, now 5 in place 1, this is the longest implication because you remember what is 5? 5 is a northeim local and its maximal ideal is non-zero and principle. And from this we want to cook up a valuation, discrete valuation. So you need to define a discrete valuation. So we have given the maximal ideal is non-zero and principle. So m is generated by t let us say and t is non-zero. This is what we have given and from here we want to define a discrete valuation. So from here first note that every element of A non-zero, every non-zero element of A can be uniquely written as ut^n, where u is a unit in A, and any some natural number. Unique that means this u and n are uniquely determined by that element. Okay, why that, so start with a non-zero element A. So let 0 not equal to a be an element in A. Now what do we know? We know that by Krull's intersection theorem At^n, $n \in \mathbb{N}$ this intersection is 0. This is by Krull's intersection theorem, because it is northeim local and M is the maximal ideal generated by t. Okay. So therefore, this non-zero element it belongs to, if there will exist, so there exist an $n \in \mathbb{N}$ such that a will belong to At^n and not the next one. And it doesn't belong to At^{n+1}, it cannot belong to all the powers, so you stop where it belong, where it doesn't belong and just one step before, that so now, because of this a you can write it as some ut^n. Now I want to check that this is unit and this n is unique and okay. This is what we want to check, where u is in A. Now, note that this u cannot be in, u cannot be belong to A t, because if it is a multiple of t, then this A will be a next multiple of this one. So because of this you cannot be this. So that means this u is not in the maximal ideal. So that means because we are in a local case. So that means u belong to the units, is a unit in A. Okay. Now if I have two different expression uniqueness now if I can write this as some wt^n, where w is a unit and m is natural number, then because of this you shift, so that will imply, because of this equality that will imply, let's assume n and m, I want to prove n equal to m. So let us assume that n is bigger equal to m, assume this. So then when you do this, then what do we get, you cancel power of t, so ut^{n-m}, will be equal to w, which is a unit. So this cannot happen unless $n=m$. Because if n is not m then it is in the maximal ideal. So n equal to m and once n equal to m this is and u equal to w. So we have proved that every non-zero element of the ring A is uniquely written in the form ut^n, for some unique natural number n.

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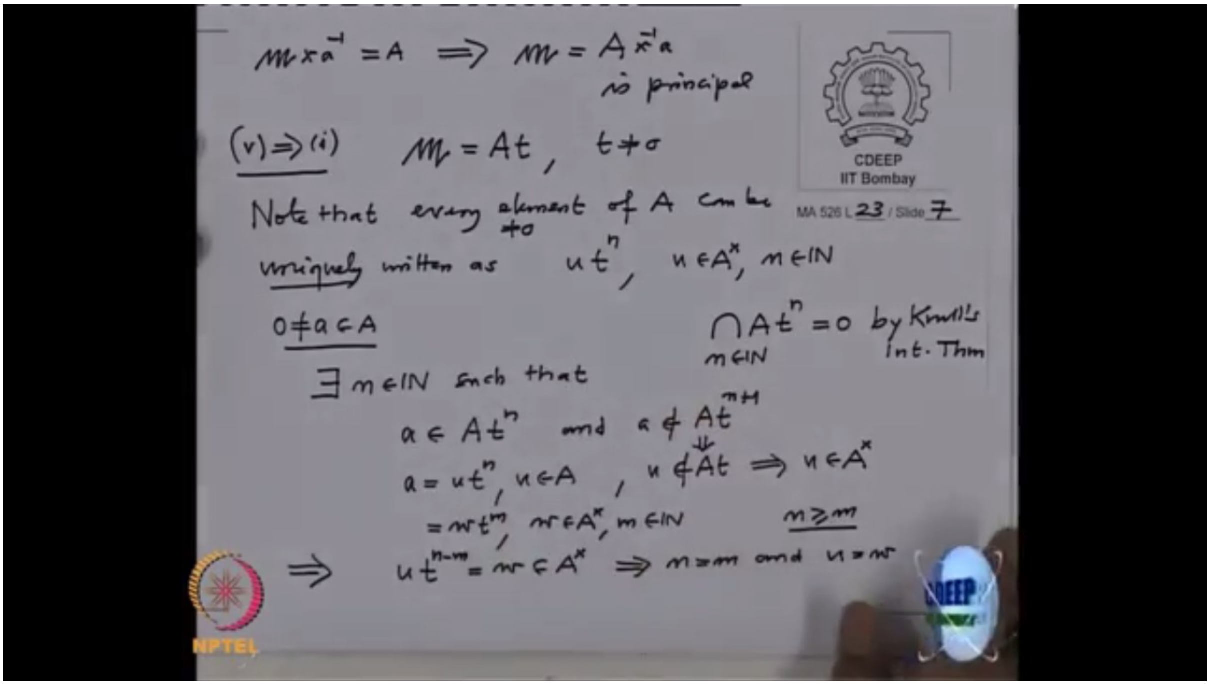

So, we can define v. Okay. So, before I define v, note that now the Coefficient field K, any element of the coefficient, any non-zero element of the Coefficient field I can write in the form some u unit in A and now I can allow negative powers also so that is every element $x \in K$ non-zero, I can write uniquely in the form x equal to ut^n, where u is in unit in A and n is a integer. Because any element of K is some element of A divided by some element of A and they knew club the units together and the powers also together and you will have to allow negative powers. So every element you can write like that now you can define V on $K \setminus \{0\}$, to $\mathbb{Z}$. To define any non-zero element x going to this unique power n, so that is $v(x)$, this is n. It makes sense. And let 0 go to infinity. So I get a valuation, this is a discrete valuation and we have to check that this satisfy those three properties of the valuation, namely it's this is a group homomorphism and $v(x+y)$ bigger equal to, but those are easy to verify that. Verify that v is a discrete valuation on K and also it is clear, the valuation ring this R_v is precisely A, because R v is precisely those elements of the Coefficient field where value is bigger good as here, but that is precisely means the elements are in A. So that means, that is A is a DVR. So this proves the theorem completely. Okay, now I want to deduce some corollaries. Okay. So I just want to make one remark in about the that proof. So in the just last implication, 5 implies 1 implication, the choice of t. See in the 5 it was given that the maximal ideal is principle. So that choice of t. So note that the choice. A generator of M is called a uniformizing parameter for A or v. It is a choice for a generator, because one may choose some other generator to be unit multiple of the given t. And then, but the valuation will be the same. Okay. So that v, that valuation v is called canonical discrete valuation of K.

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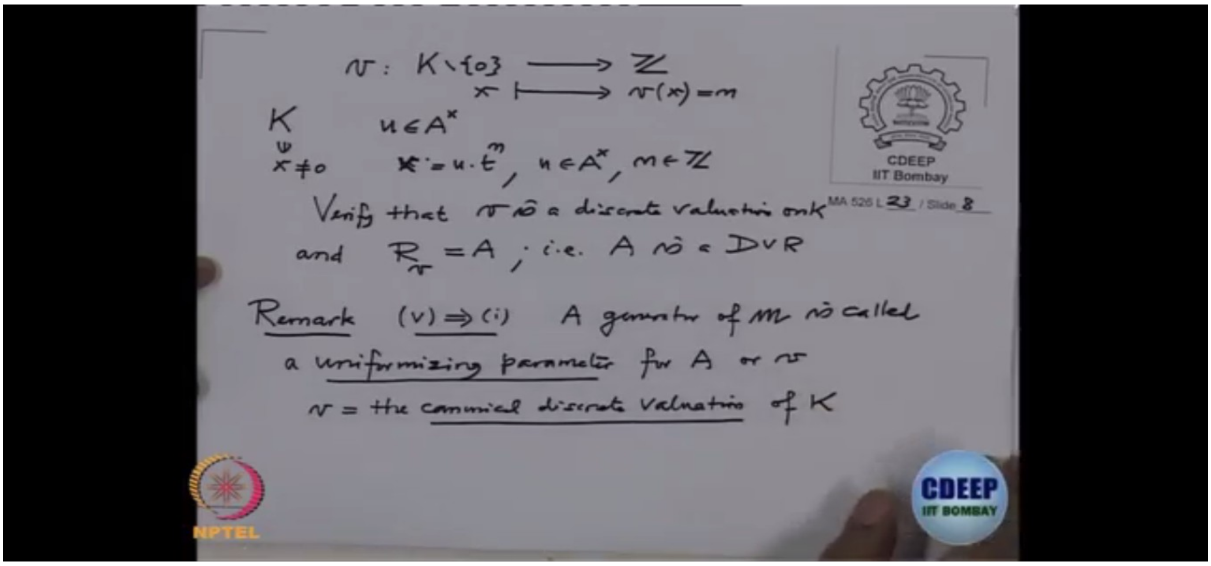

Okay. So let us deduce some corollaries. So corollary 1, okay, so let A equal to R_v be a DVR and m is the maximal ideal. So values are positive. And K is the Coefficient field of A. t is a uniformizing parameter. So it's really this is the data one gives, you know, R_v that is v maximal ideal uniformizing parameter. Okay, then 1 every element of K, so this is like a summary of many implication. Every non-element of K is uniquely written in the form u t power n, where n is $v(x)$, so every element x, $v(x)$ and u is the unit in A. Sorry, for non-zero element. 2, every ideal, every non-zero ideal a in A is a unique power of m, so that is a is m^n. Actually it is, in fact a is A x and if a is Ax then a is $m^{v(x)}$. We know A is principle, A is a principle ideal so this is the power of value of that x, generator of A. 3 ideals of, ideals in A is a totally ordered set under inclusion by inclusion. Totally ordered means, it's ordered that means reflexive, transitive and dissymmetric relation and its comparable 2 elements are comparable. So 4 an element $x \in K$ is a uniformizing parameter for A. if and only if value of that element is x. If value is x is 1, then obviously it is in maximal ideal and obviously it as to be generator if it is not a generator, it will be, you know, in square and so on Okay.

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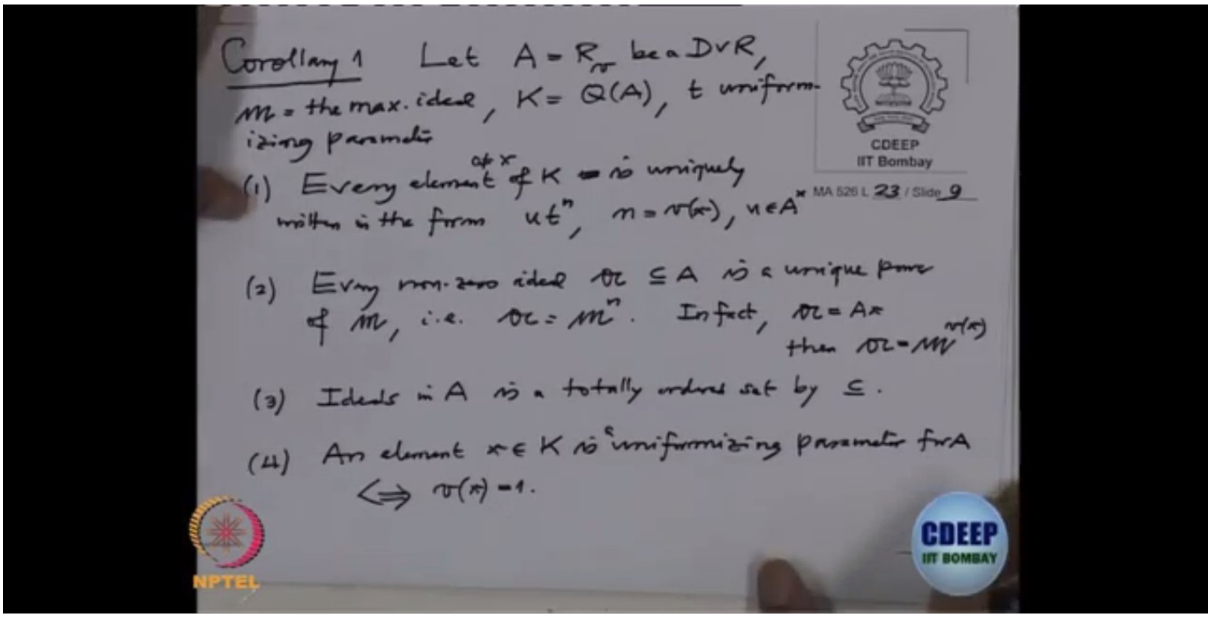

So this corollaries easy to, it is a summary of one of the implication what we have done but this is nice to memorize things, most important is ideals is a totally ordered set. Okay, next corollary. Let A be a normal domain and K is the Coefficient field of A. Then this corollary will give examples of discrete valuation ring. So 1 if I localize A, A localize at p and obviously we know for the discrete valuation ring dimension is 1, so we localize at height-one prime, so that this ring as dimension 1, so is a DVR for every height-one prime ideal, p in A. So for example when you take a polynomial ring over a field, then we know it is a normal domain and if you take a prime ideal of height-one there and localize there it will be a DVR or if you take a regular local ring of arbitrary dimension and height-one prime there the localization will be DVR, because this dimension 1. And we have proved regular imply is normal. I incidentally regular imply is normal, I have rewritten the proof in the corrected another proof today, which is much simpler, don't have to go through the graded ring. Okay second part, if I have a non-zero element in K, if 0 not equal to f is in K the Coefficient field. Then not here, I am giving you to the notation which will come from 1. Once this is a DVR, then it will have a discrete valuation and that discrete valuation is usually denoted by v suffix p, this is the discrete valuation, corresponding to this Ap. So if f is non-zero element in K and if then v p of this f is 0 for almost all height-one prime ideals, p contain in A and where this v p is where v p is the discrete valuation corresponding to, p. This is the generalization of that fundamental theorem of arithmetic. Okay. Proof: Okay. So, 1 we want to prove it's a DVR, so by our characterization we can prove either of the one of the five. So the easiest is to note that it's a local domain and also northeim, because A was northeim, A is normal so northeim and dimension is 1. So the 5 condition A northeim local and the maximal ideal is non-zero and principle. No. Yes. Height-one, so dimension is an, not the 5 but 3 is better, 3 says a local, normal and dimension is 1. So local A p is local and because A is normal A localization p is normal and the dimension is 1, because height is 1 we are considering. So 1 is clearly A p is local, normal and dimension is 1, Ap which is height p which is 1, so by 3 implies 1 over the theorem, Ap is DVR. Okay second part, what do you want to show, we want to show that for each f non-zero only finitely many $v_p(f)$, finite many values are non-zero. Okay, start with a non-zero element f, actually start first with an element in A and p prime ideal of height-one. Then, because f is in A f is also an A localized p. And therefore v_p of, f is non-negative. But I want to show that only for it is 0 what happens then it is non-zero.

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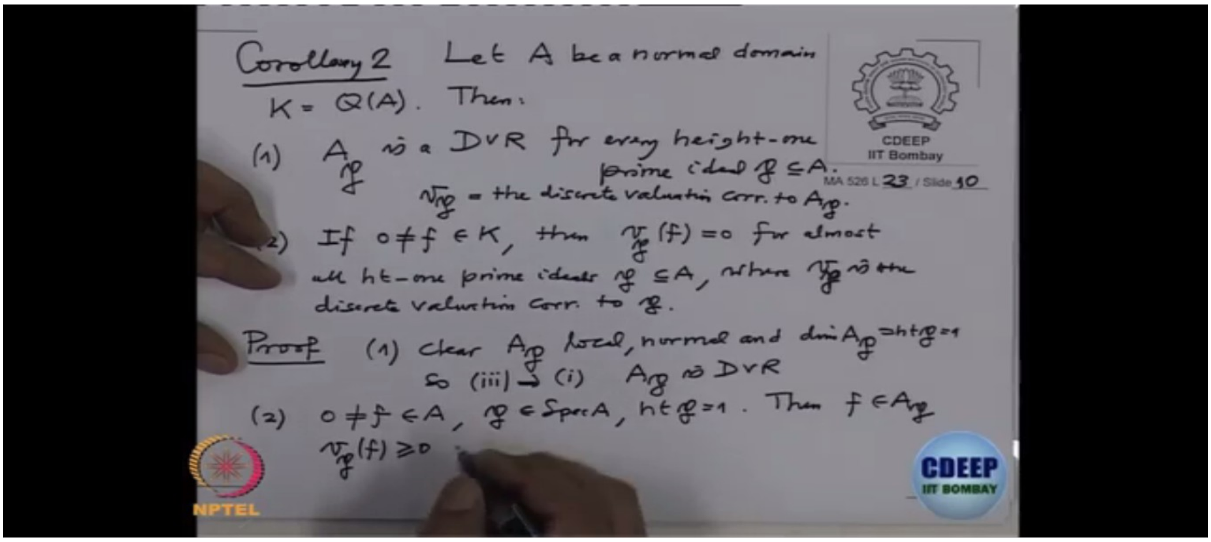

So therefore v_p of, f is non-zero if and only if v_pp of, f is positive, that is if and only if, f belongs to the maximal ideal of A p which is p Ap. But that is if and only if, f belongs to p. So how many p's it can belong to if that is what we have to analyse. So if, f belongs to p and height of p is 1, that only means that, that will mean that then p will be associated to A by A f. But we know that the number of associated prime ideals are finitely many, so therefore only finitely many p's will have this property, so that proofs that, so this finite set, so therefore the section follows so therefore v_p of, f is non-zero only for finitely many p's. p with height p 1. Now we are done only in A, but now take arbitrary element in the field K, so that look like $\frac{f}{g}$ in K and on 0, but then what is v_p of, f or g take any p with height-one, then this will be equal to $v_p(A)$, $v_p(f)-v_p(g)$, this is because it’s a group homomorphism. And the same this is non-zero for finitely many, this is non-zero for finitely many and at most f or g will be non-zero for finitely many p's.

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