Lecture – 55

Discrete Valuation Ring

GyanamParamamDhyeyam: Knowledge is Supreme.

We will do today Discrete Valuation Ring. Some of this might be reparation for you but still I want to do it for the sake of completeness.

Let K be a field, so I will give connection between the discrete valuation ring and normal domains and then we will do next lecture probably we'll start Dedekind domains. K be a field and A discrete valuation v of K is a map. v from K to $\mathbb{Z} \cup \{\infty\}$ which satisfy the fallowing three properties.

One,v of x equal to ∞ if and only x is zero.

Two, if rest it v2 case, K^x, so it is a map from K^x to $\mathbb{Z}$. this is a non-trivial group of homo. Where K^x is a group under multiplication and $\mathbb{Z}$ is a group under addition. So this there is at least non-trivial so everybody can't be zero, so at least one element of K^x should go to other than zero that means this Image of v of K^x is non zero and also to spell out $v(xy)=v(x)+v(y)$ at the minimum group. Automatically it will go one will go to zero.

And the third one, $v(x+y)$ is bigger equal to the minimum of $v(x), v(y)$. Such a thing is called a discrete valuation ring. I just, couple of words about discrete, normally when studies general valuations but then this will, this side $\mathbb{Z} \cup \{\}$ will be different group. For example, real numbers or even bigger groups. So those are not discrete groups because of that this is called a discrete valuation.
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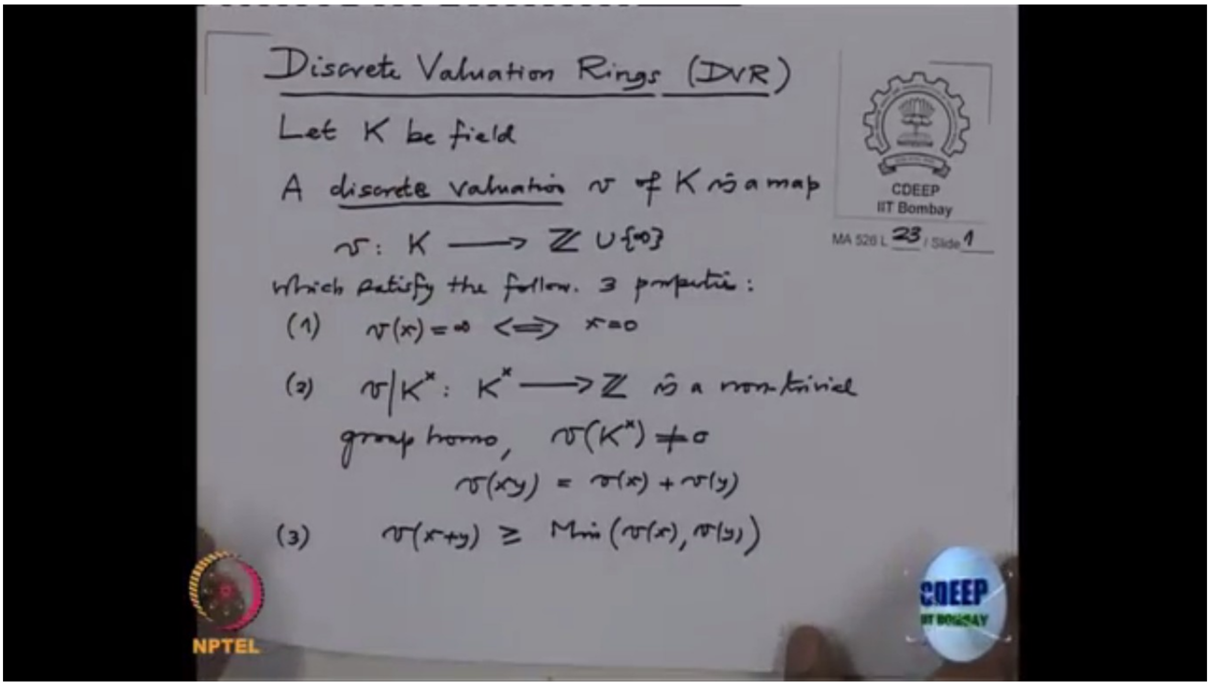

Okay. So examples, few examples. Exact definition came from the, this classical examples. So, suppose, I have prime number p, p is a prime number, then and our field is ℚ now. ℚis our field. You see it came from the fallowing. If, you know by fundamental theorem of arithmetic every integer n, we can write it as sign, that is plus minus one. And the product of prime numbers. So $p_1^{n_1} \ldots p_r^{n_r}$ and for the uniqueness this epsilon is plus minus one and $p_1, \ldots, p_r$ you are arranging this $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_r$ and this $n_1, \ldots, n_r$ are the positive integers. Then you might have seen in the school that the standard notation for this power n_1 which is the highest power of p_1 occurs in a n is a standard notation was $v_p\left(p^{n_1}\right)$ this is n_1 and so on. $v_{p_r}(n)$ is n_r. So in general we can write in a compact way any integer n we can write it as sign of n times product, product is p is in over p prime numbers, p power $v_p(n)$, where $v_p(n)$ is zero for almost all p in p only finitely any vp of nr are non zero. This gives you also nice identification ℤ with plus minus group, product cross product, product, product is running over p, $\mathbb{N}^{(P)}$. This is neotherian theorem. So this $v_p(n)$'s are the standard discrete valuations. This vp map, think of vp is a map from ℤ or also you can define it for the ℚ, so you can think it is generally it is a map from ℚ to actually $\mathbb{N} \cup \{\}$. Now for a ℚtherefore you have to take ℤ here. So these are the discrete valuations, each p will give you a discrete valuation on ℚ. And this is one extra that we will see. And this we can imitate, there is nothing special about ℤ, you can imitate for the polynomial ring in one variable over a field which is a PID.

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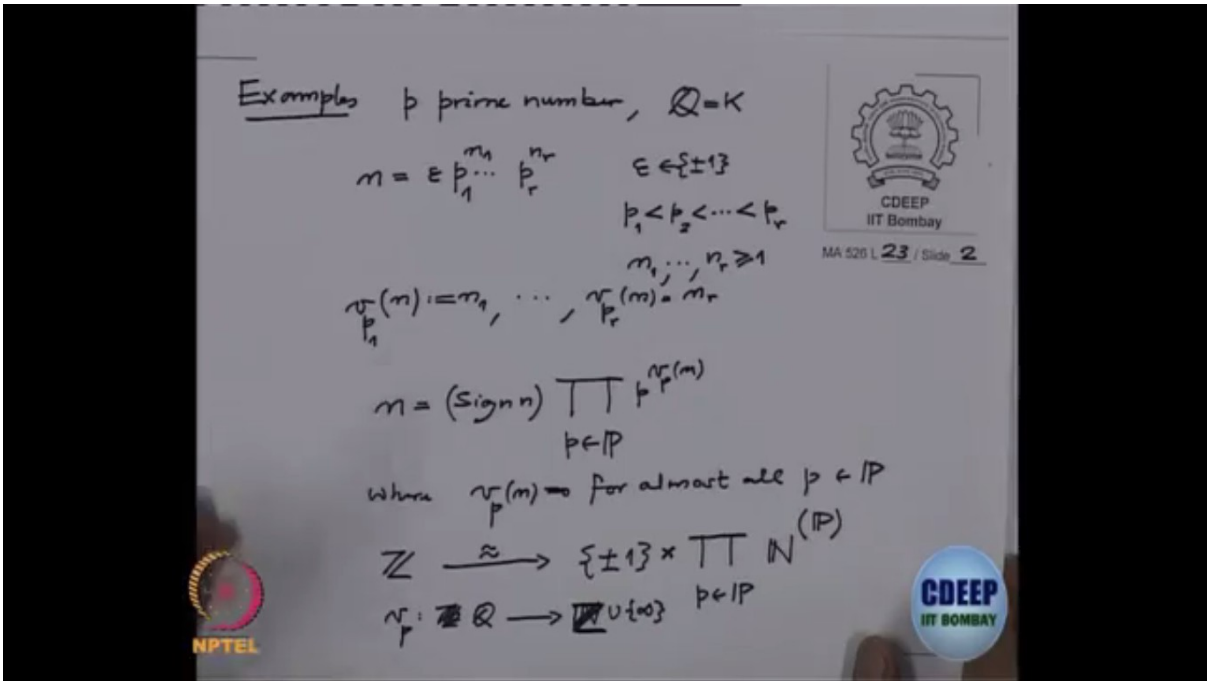

Okay. So little bit of notation. So for each discrete valuation on K, we define Rv, this is by definition, all those element in K, such that, so v of X is called a value of v at x. This is non-negative. And in this mv is all those x in K, such that vx is strictly bigger than zero. Then I want to note that this R_v is called valuation ring of v and it is now very easy to check that, first of all we know v of 1 is zero and if x is in R_v but not in mv then v of x must be zero. And once this is zero, v of x inverse is also zero. So x inverse will belong to R_v, so that means all those x for which v of x is zero, their unites in R_v. So therefore actually non unites form an ideal, therefore this shows that R_v is a local ring with the maximal ideal mv. We are going to study this rings more precisely. So, one final definition here, when I say a normal domain, no not normal, thediscrete valuation, I mean. An integral domain A is called a DVR, discrete valuation ring if A is equal to R_v for some discrete valuation v of the Quotient field of A. This is the standard notation we are using for the Quotient maps, such things are called. And as usual normal, when you say normal domain that includes the definition, includes it is noetherian. So I will just record again normal domain means,this is noetherianintegrally closed, integrally closed means in each Quotient field domain. That is called normal domain. So, now we want see what is relation between DVR, normal domains and dimension and so on.

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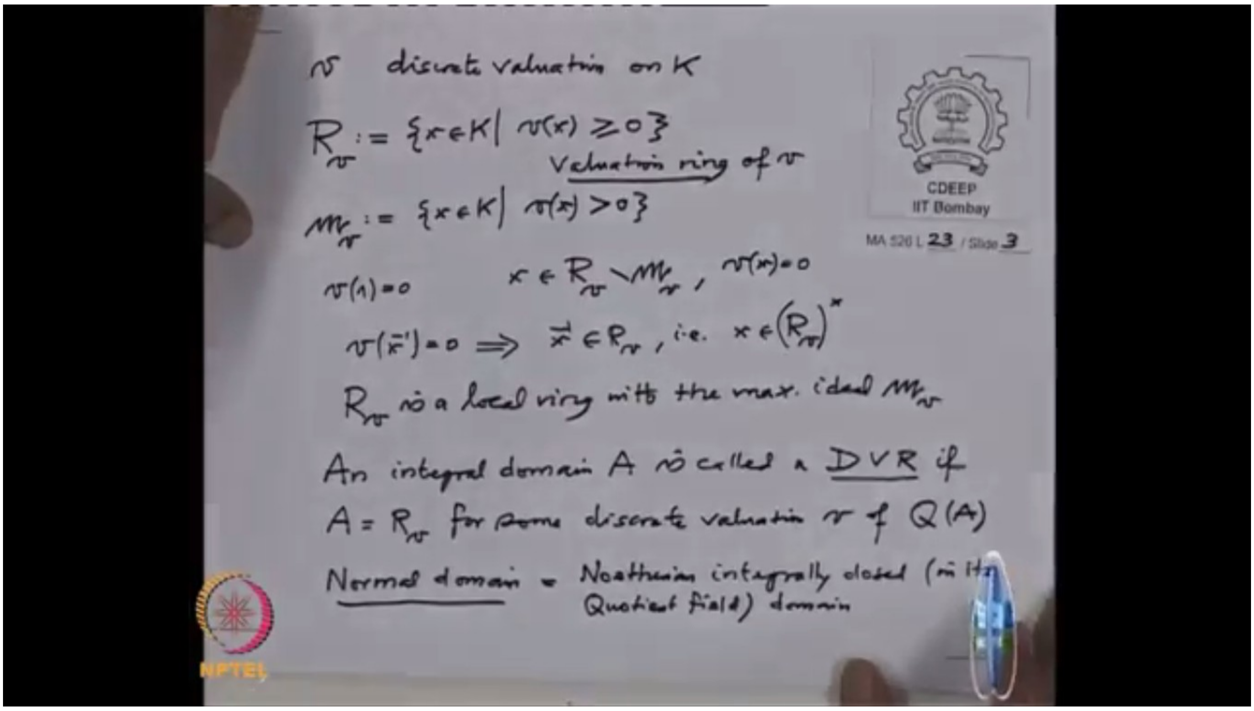

So that is the theorem. So this theorem we will prove. This gives the characterization of discrete valuation rings.So A is integral domain and $Q(A)$ is quotient field then the fallowing are the equivalent.

One, A is a DVR. That means A is a valuation ring of some discrete values.

Two, A is a local PID. A PID means every ideal is principal and local means there is only one maximal ideal and A is not a field.

Okay. Three, A is local, normal, and dimension one, krull dimension one. That means every non zero prime ideal is maximal.

Okay. Fourth one, A is local, normal, and the maximal ideal m is associated to some non zero element, this belong to associated primes of $\frac{A}{Aa}$ for some non-zero a in m.

Five, A is noetherian local and its maximal ideal m is non-zero and principal.

So, proof. One implies two. So many of these proofs are simple. So, we want to prove now A is aPID and its not a field. So, and we have given it's a discrete valuation. So write A equal to R_v. We have seen above that is local. R_v is local, we've seen above also we have seen that the Quotient field of R_v is same as Quotient field of A which is the field K, discrete valuation onthe Quotient field of K. So, by condition two of the valuation which say that it's a group of morphismsthat show that their exist at least one non-zero element, one element x in K^x, such that vx is positive. If x is not then x inverse will be positive. So, in any case there is one x whether it is positive, so therefore this x will belong to them, and therefore mv is non-zero. Therefore A cannot be field because the maximal ideal we saw that R_v has a maximal idea then v, so therefore A which is R_v is not a field. So what did we say, it's a local not a field, now I want to show it to the PID.

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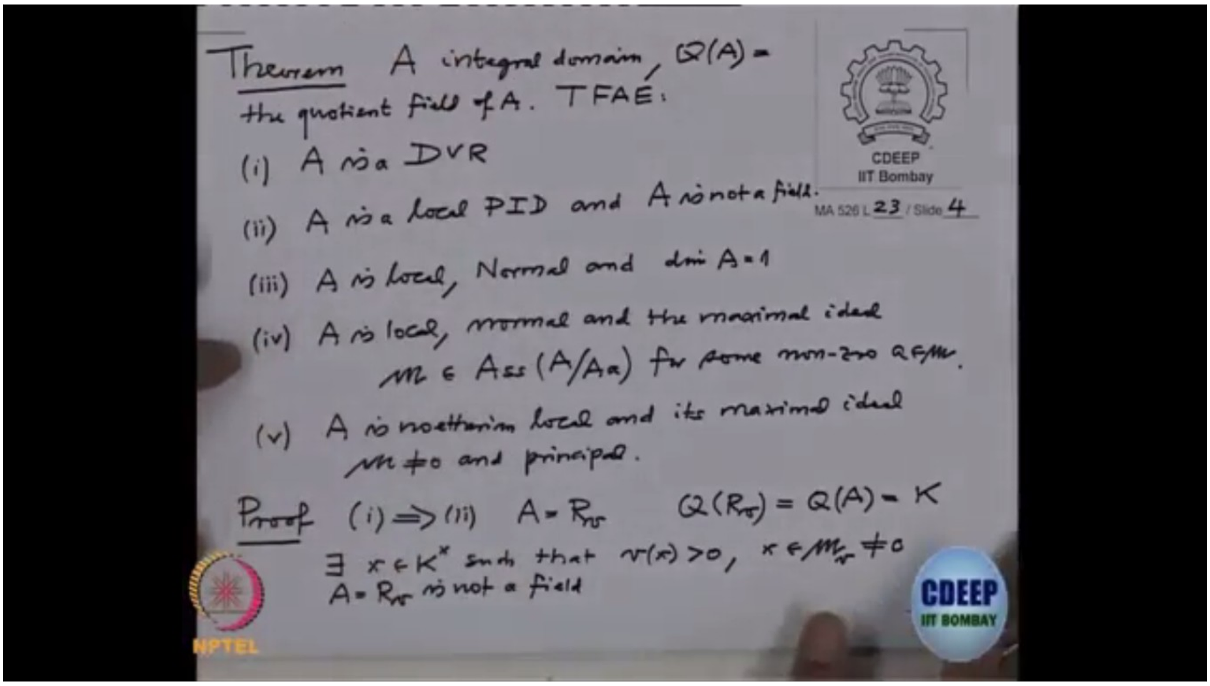

Okay. So take any non-zero ideal A, A is the ideal in A non-zero. I want to show it is principal. Now, let us take n is equal to Min of all the values of a where a varies in a. So because we've already showed that a is local and this is an ideal and we can also assume it is proper ideal because otherwise nothing to be proved. So A will be containing the maximal ideal, therefore all actually this guys is a sub set of ℕ, so this values are sub set of ℕ. So it will have a minimum. So minimum exist, so that means there is an a in a with v of a is equal to n. And then a cannot be zero and now you choose b, a arbitrary b, I want to show that this a is the generator of the ideal a. So take any b in a any b, and then look at vb, vb is bigger equal to n, n is v of a, so therefore by the property that group homomorphism property that v of ba^{-1}, which is $v(b) - v(a)$, which is bigger equal to zero, therefore this one is said, so this means, ba inverse, this belong to the ring discrete, valuation ring given by v. So, ba^{-1} is in A, but that we mean that we'll multiply by A^{-1}, whether I write it in this way or that is same as a inverse b, now multiplying A, so you will get b belong to ideal generated by Aa. So all together we have shown that a is generated by Aa, so a is principal.

Okay. Now, two implies three.So what is two, what is extra, so local PID is not a field. Local it is there, now I want show PID, therefore dimension is one. Now, I only have to show it in normal with this assumptions. So that means I want to show it is, we will show that it is a UFD. Okay. So it seems, A is a PID, that is given in two, it is noetherian and A is VFD because PIDs are UFD's and UFD's are normal we've already proved or already you know UFD's are normal. So that proves three.

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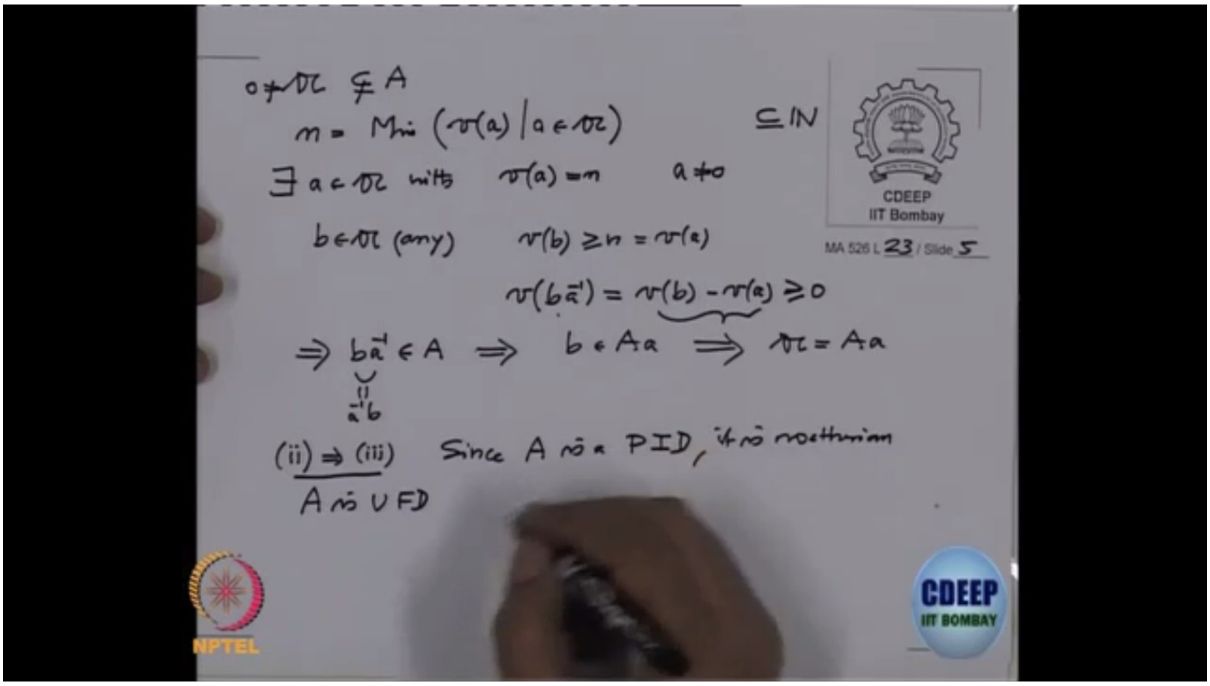

To implies three's proof, now three implies four. So, three is a local normal dimension one. So four is normal local and the maximal ideal is associated to some non-zero element. So take a maximal ideal m, the unique maximal ideal m and this is the only non-zero prime ideal in A. Because dimension is one and it is local. If there was prime ideal different from non-zero and different from m and it will containing m and that dimension we increase. So this only maximal ideal. This is only non-zero prime ideal. That means when I take any non-zero element a is an element in m and if a is non-zero element and if you look at the associated primes of $\frac{A}{Aa}$, the only element here will be m. But there can't be any other prime ideal because it will contain A and or will be containing m, so therefore that is only prime ideal. But this means, prissily this means m is associated to this, so that proves four.

Now, four implies five. Okay. For five, local, four is given local, normal and maximal ideal is associated to some non-zero element and five is noetherian local and its maximal ideal is non-zero and principal. So, noetherian is already there because we are assuming normal noetherian and locally also there, so I only have to prove that the maximal ideal is non-zero and it is principal. So, take the maximal ideal, we know this associated to some non-zero element of A, a non-zero. So particular m is non-zero because A contains this A which is non-zero and therefore there exist, this m is associated means, so there exist an element $x \in A$ so that this m is a colon ideal an elector of x mod A, but that means, m equal to ideal generated by a, colleen ideal generated by x. But this will mean that if I take any element of m and multiply by xa^{-1}, I will go in A, So, to check this, I have to check that m times x is continued in A but that is definition of this colleen. Because of this m times x is continued in the ideal generated by a, that means this multiplying a inverse from this set, so it is of this. So that shows that this one, so mxa^{-1}, this is an ideal in A. And I want to check that now, if suppose these are the proper ideal mxa^{-1} is not A proper ideal, then what happens, let us see. Then it actually contained in the maximal ideal, then mxa^{-1} will be contained in m but then that /means that this element xa^{-1} is integral over A. That is one of the characterization for integral elements if you have a faintly generated module, so that m times contained in this, then it is integral. So this the same. So to prove this we use the same trick as this Caley Hamilton trick Okay. But know, because A was in four we had assumed that it is normal and this is an element in the quotient field, therefore this will belong to A, since A is normal. But this will mean one belongs to this Az Ax, this means precisely this but this is m, so it's a contradiction. Maximal ideal is a proper ideal, one belong to here, so this cannot be true.

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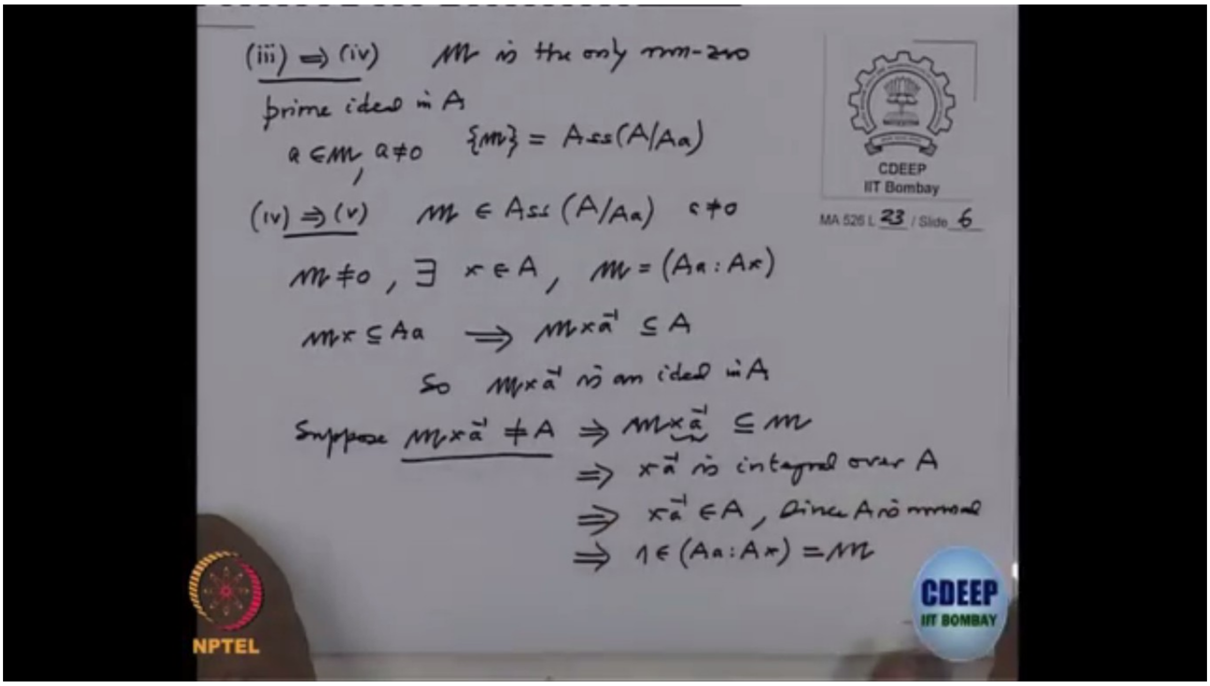

So that means mxa^{-1} has to be A, but this will mean m is generated by $Ax^{-1}a$, so we actually cooked up a generator so its principal.

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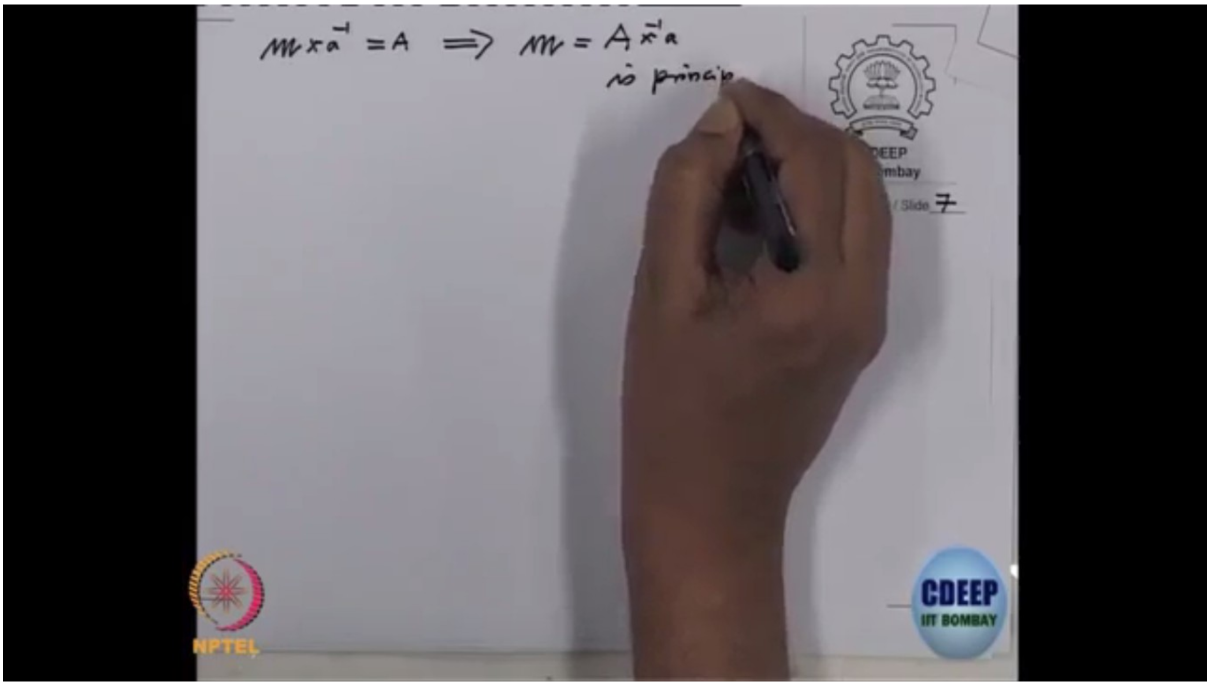
$m \times a^{-1} = A \implies m = A x^{-1} a$
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