Lecture – 54

RLR- Prime Ideals of Height 1

Gyanam Paramam Dhyeyam: Knowledge is Supreme.

Okay, step two. We have proved it is projective model. I will now prove it is principle, Extended Idealist Principle. Remember we wanted to prove that the p is principle. But now, the step two says, "When I localized that the B, the element we started with that extended ring it is principle and the third step will be the pre itself is principle. Okay. So now let us look at, the homological it all, we know that the global dimension of A is finite. Therefore, homological dimension of this P at the A module, this is a supremum, so this is less equal to this, so this is also finite. But what does that mean? That means this p has a finite projective resolution. So, that means p, we can write it as, resolution like this. p0, pn minus 1, pn, 0, these are finitely generated projective A-modules. But they are over a local ring A. So, finitely generated projective are free. So, therefore, actually I have finite free resolution for p. And therefore when I tensor these, tensoring with B. Remember B is $S^{-1}A$. And S^{-1} is exact. So, therefore after tensoring with B it will remain exact. And these are the free modules. So, they will continue to be free.

So, therefore, when I tensor, what do I get? I get 0, $B \otimes P_n$ over A, $B \times P_0$ over A and then I will get here pB to 0. pB same as $p \otimes B$, $B \otimes p$. And these are now free modules, free B modules. So, these ideal in B has a finite free resolution and when each guy is the finite free model and Lemma says, in this case this ideal is principle. That was a Lemma. So, therefore by Lemma, pB is a principle ideal. P is not projective no, that we have not true. See the p, is not projective. And now step one, we are proved, see, this is projective by step one. And I remember, therefore, although I had to state the lemma for non-local ring, because this B is may not be local.

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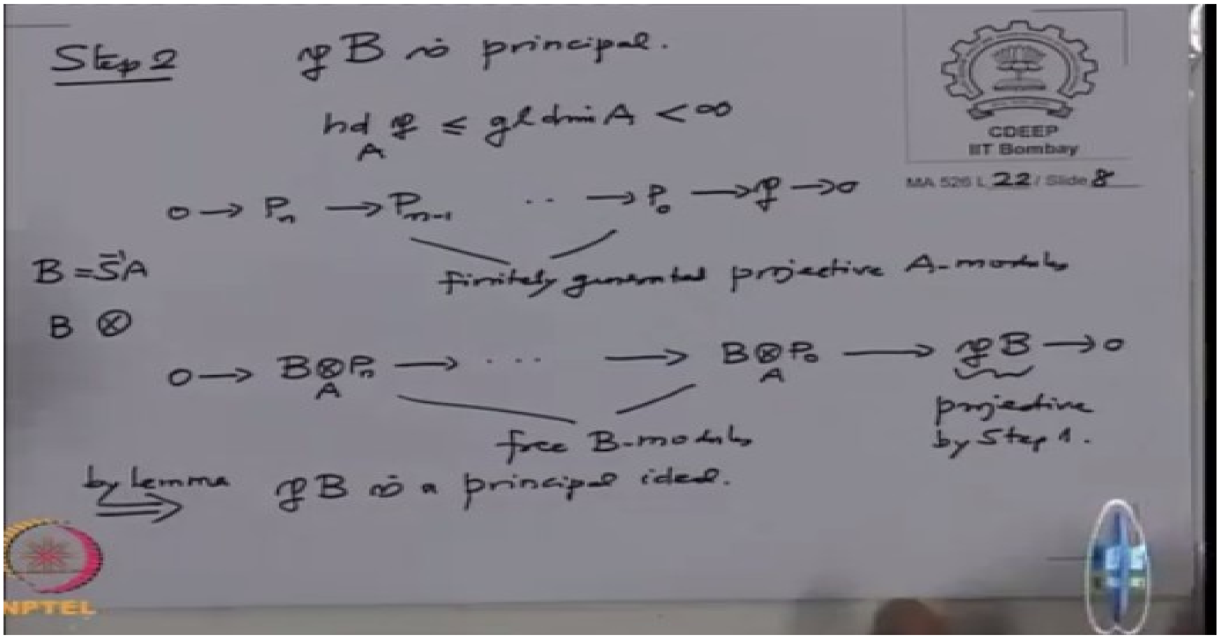

So, that wills step two. Now, step three, p is principle. And that is the end of the proof. So, we know by, step two, pB is principle. And we want to prove that it is principle. Okay. So this is what, this is S inverse of p. Where S is a multiplicative set generated by the t. And this one is principle, so, it will look like, some generator, x times B. But this x is element in B. I can choose this x element in actually in B. So, x is in B. Because this is extended ideal, I can choose the generator from the below. Anyway, if it is in B, then I can allow multiple by denominators, it's a unit, so ideal will not change. So, I can achieve what is this. Now, this in, x is in p, now we look at, this, all these powers of t, where tn intersection, this has to be 0, because this intersection is contained in intersection m power n. And Krull's intersection theorem says this is zero. So therefore, this is zero. That means x cannot be containing in all these ideals. So, there will be n, so that x is in At power n but not in the next. So choose n, with x belonging At power n, but not the next one.

That means what? That means this x, we can write it as somebody y times t power n with y in A. Then note this y cannot be in At. If y is in At, then x will be in the next power. So, y is not in At. All right. This x is in p, we know. That's how we've chosen x. If x is in p, then why t power n is in p, but t^n is not in p. Therefore y is in t, y is in p. So y belongs to p. Once y belongs to p, look at the ideal generated by y in B. yB, this is contained pB which is x times B, which is contained in y times B, because y divides x. This, so they all be equal. So, in particular pB equal to yB. So you see what did I do? Actually, I've chosen a good generator. To start with x was a generator, but now this generator is not in the ideal generator with t. Okay. Now I claim what? I claim, claim is p generated by $y \in A$. I know p generated by $y \in B$, but I want to prove that now p generated by $y \in A$. So one inclusion is obvious, because p is in y, y is clearly contained in p. I want to prove the converse. So let us take z here, $z \in p$. I know this z, z is here, therefore, it is multiple of y in B. So, that means this is y times somebody in B, any element in B will look like a by t power n. a is in A and n is some actual number. And what does this equation means? So this equation means, what does this equation means? That means, you cleared a denominator and you get y times A, so this will mean $t^n z$ which is $y a$, which belongs to, this belongs to, you get this equation. What was t? What was t? t was a… Remember what was t? And, I, we have noted that it generates a prime ideal that means it's a prime element. Right? And so side belongs to t, therefore, product belongs to the prime ideal. Therefore, one of them belong to the, one of them belong to that ideal generated by t. But who can that be? y cannot belong. Right? Therefore, A will belong. So, this will imply, a belongs to, actually keep doing it. So a t^n. But this will mean, this will mean, this a, $\frac{a}{t^n}$ actually belongs to A, because a has also power of t^n exactly. So cancel it there in the domain, so things are okay. So, therefore, what did I prove? This belongs to A, therefore that z actually belongs to ya. Because this is in a, so this z belong A multiple of y. So therefore, that is what we wanted to prove. We wanted to prove equality. We started $z \in p$ and we wanted to prove is it in A. So, that's what it is. Okay.

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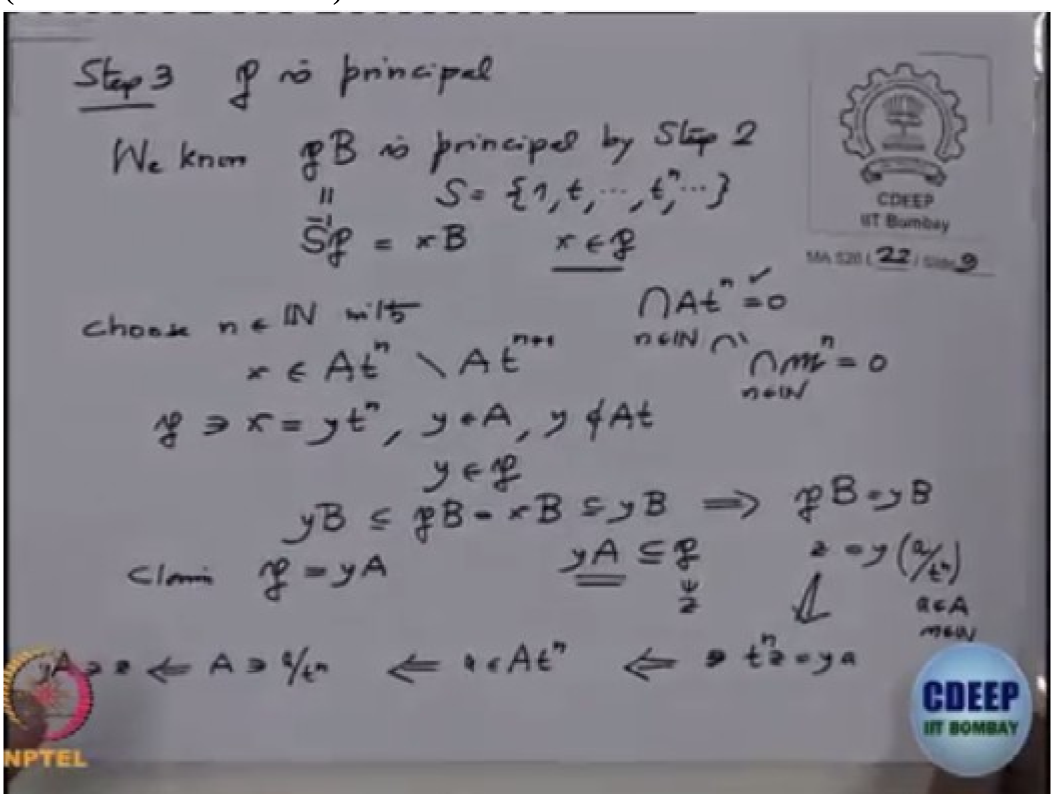

So that proves its principle and therefore, we're finished our proof that it's a UMT.