INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

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Lecture No. – 52

Homological characterization of Regular Local Rings (Contd)

Okay so now we come back to our proof of the theorem that is we want to prove that the ring is regular, so proof of theorem. So we have given that, okay so we have two parts, so we want to prove that, so A is regular let us assume first A is regular, then I want to prove that the global dimension of A is finite, this is what I want to prove.

Alright so A is regular means the maximal ideal is generated by correct number of elements, maximal ideal will be generated by $a_1, \dots, a_d$, where d is the dimension of the ring, and we have seen earlier A is regular means it's a domain, integral domain and therefore this $a_1, \dots, a_d$, these elements, this is called regular system of parameters, this is called a regular system of parameters.

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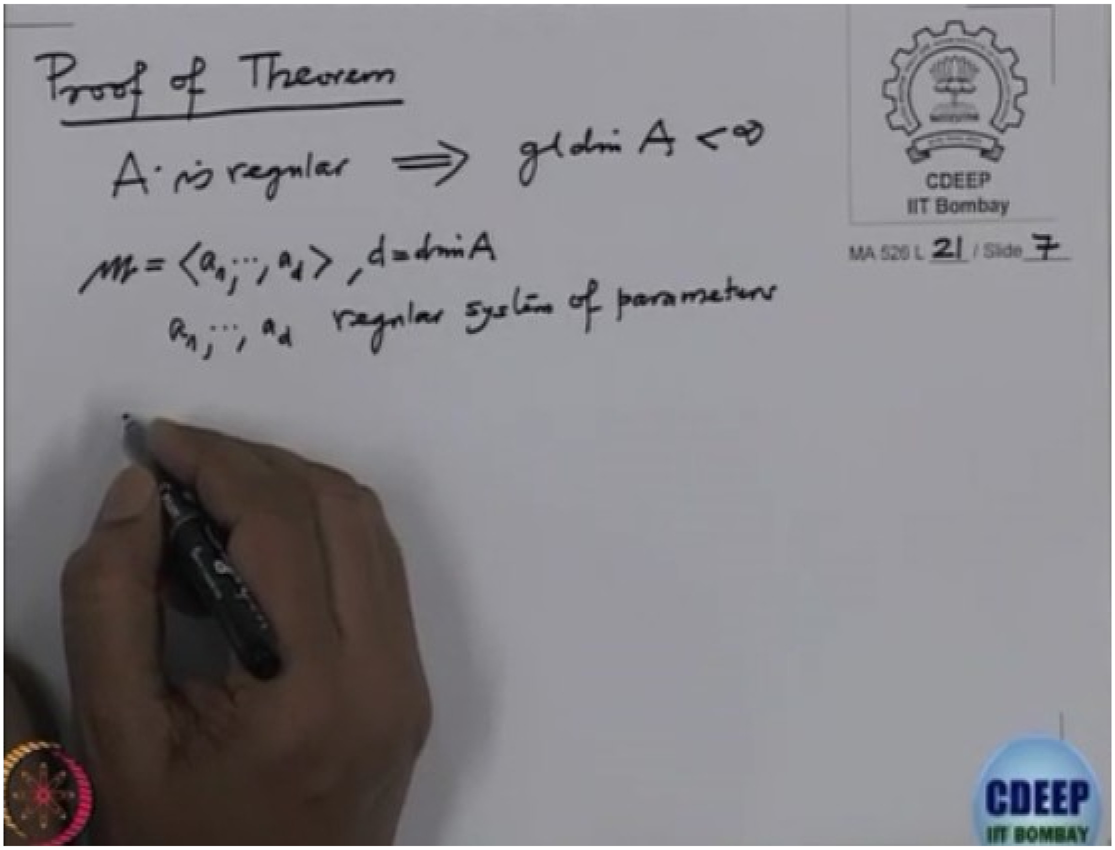

And note that this has a property that if I go mod $a_{1,} a_1$ is a nonzero divisor in A, also mod a_1, $\frac{A}{\langle a_1 \rangle}$, a_2 is a nonzero divisor, a_2 here is a nonzero divisor, in general A_i is a nonzero divisor in the residue class ring up to the earlier guy that is A_{i-1}, this is true for every I from 1 to d, such a thing is also called the regular sequence,

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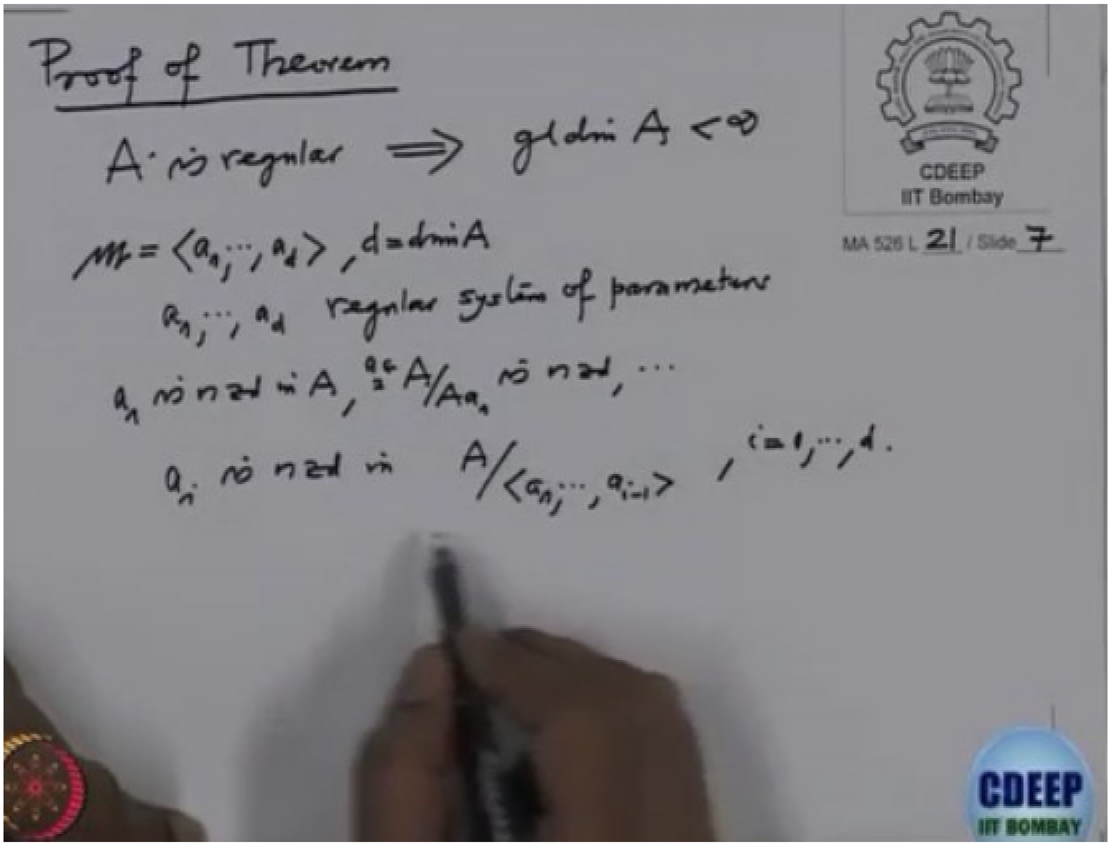

so every time the dimension drops exactly by one that is a reason that this has this property, AI's a nonzero divisor mod this, so in this situation now I want to prove that the global dimension of A is finite, so I want to prove that the homological dimension of the residue field is finite, this is what we want to prove.

But to prove this it's enough to prove that the homological dimension of the maximal ideal is finite, because this m and $\frac{A}{m}$ they come in a short exact sequence when the middle term is A, okay.

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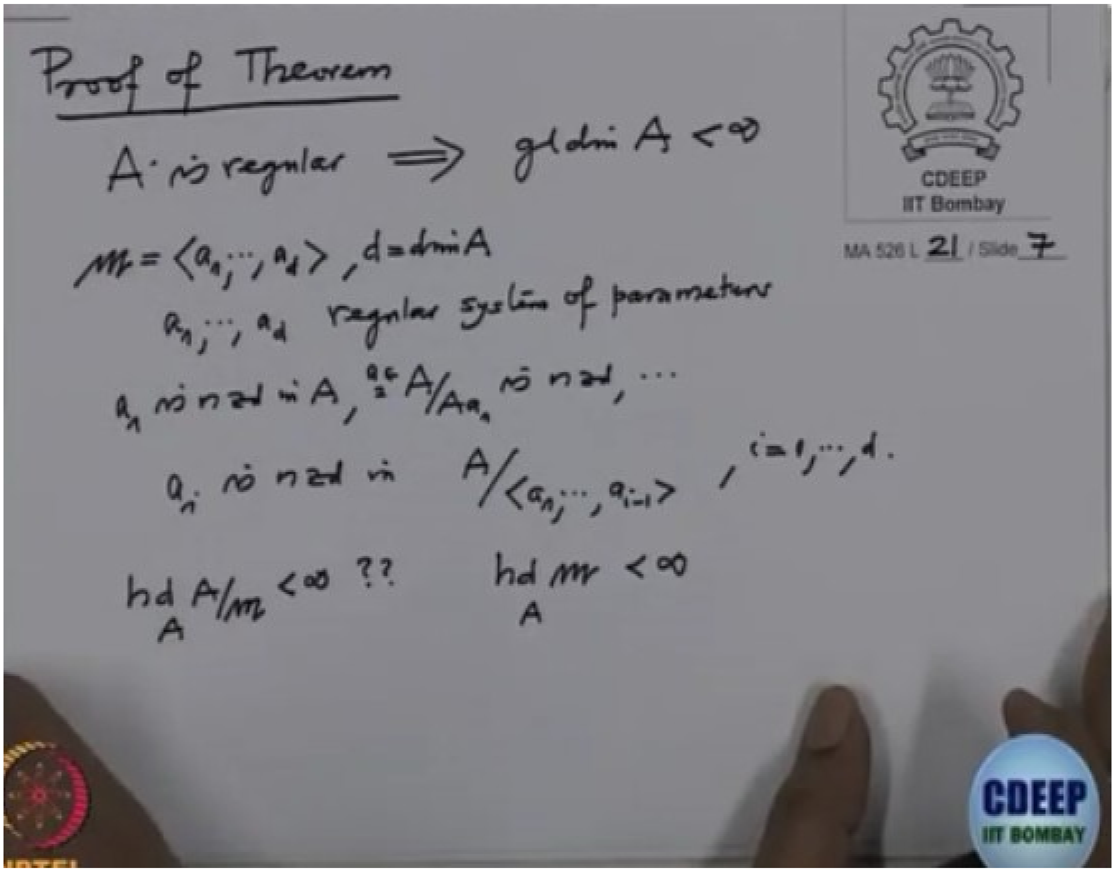

So what we just now proved in the lemma, you see here if you have a nonzero module and A is a nonzero element then the homological dimension of mod A is one more than the homological dimension of the module, and this is the respective of whether m has a homological dimension finite or not, this equality always holds for you and for infinite, so repeated application of this, it's enough to note that at 1 it happens, and that is so, so to prove this is finite you prove this is finite, and prove one more down infinite and then its repeated application of this shows that homological dimension of m, $\frac{A}{m}$ is actually R, it's actually what is the number d, this is d because you prove that homological dimension of $\frac{m}{mA}$ is finite that is where we have checked this corollary here, yes, this one, if homological dimension of $\frac{A}{mA}$ is finite, so repeated application will tell you homological dimension of $\frac{A}{m}$ is actually d, because so that already proves that the global dimension is actually this d,

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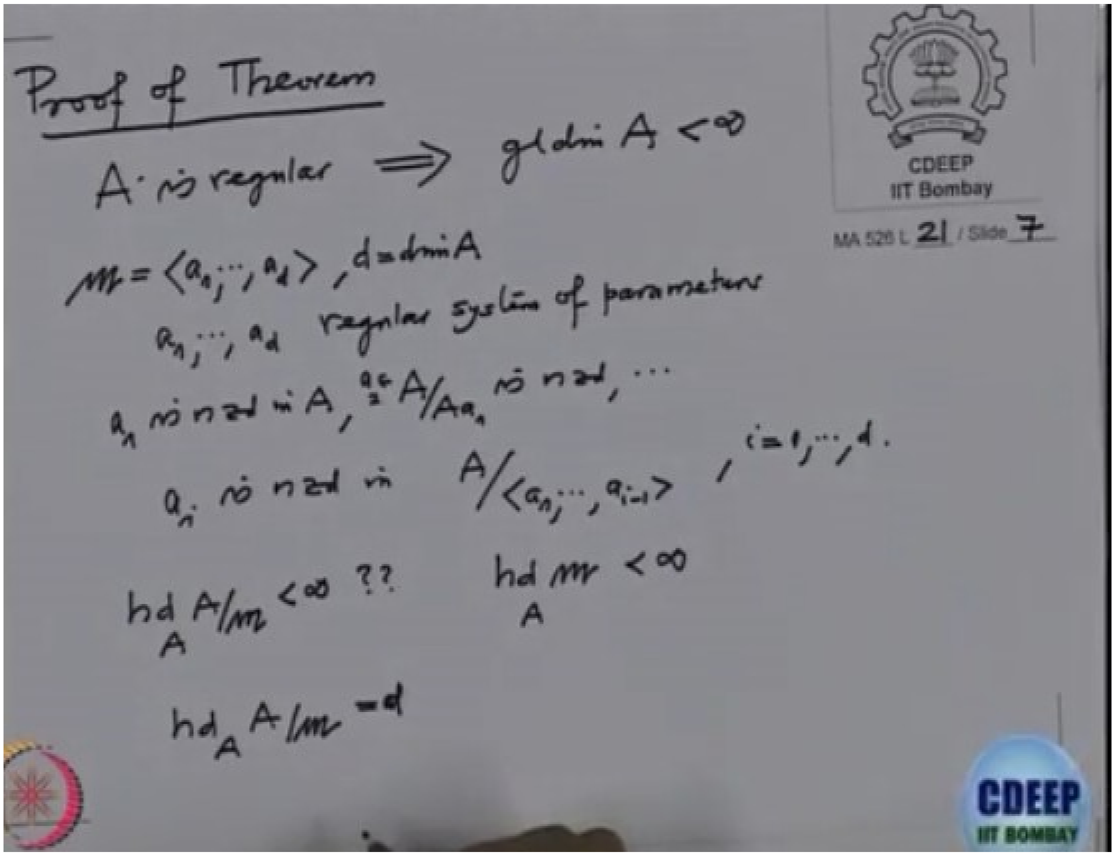

so in one stroke, in one stroke it proves the global dimension is finite and equal to also Krull dimension, so this is GL dim A, so this implication is, to prove there.

The next one, we want to prove the other converse that means we want to prove that if the global dimension of A is finite, then m is generated by regular sequence, m is generated by, if I prove that m is generated by $a_1, \ldots, a_r$, with the property for each i, a_i is a nonzero divisor in A mod earlier elements, ideal generated by the earlier elements. If I prove this m is generated by such a sequence of elements $a_1, \ldots, a_r$ so that a_i is a nonzero divisor mod earlier one, so then what will happen? That if I exhaust all the element, the homological, the dimension of the ring will exactly drop by r.

Okay let us prove this first and then we will tug it up, let us prove this, okay, so this two,
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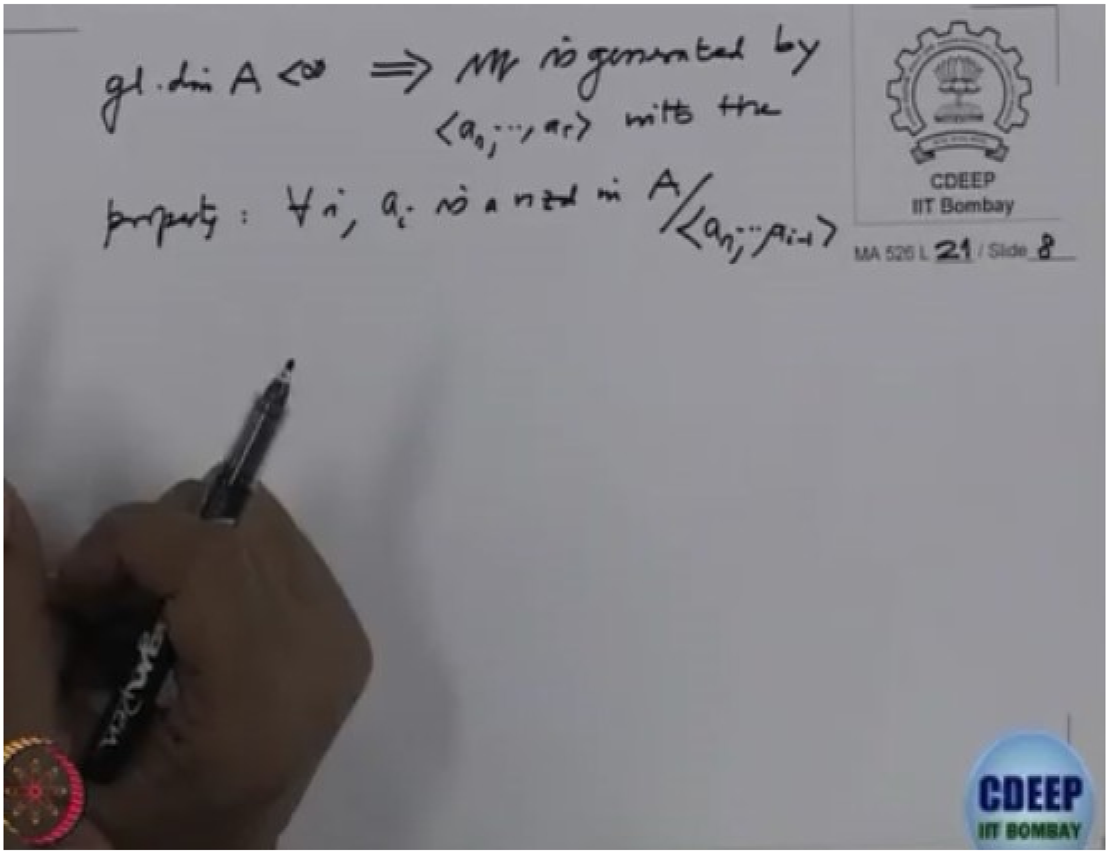

so I will prove that actually this r is dimension of A, then also we are done now, this is how what I will prove, this r is nothing but, take r is $\mu(M)$, which is dimension of K, K is residue field, $\frac{m}{m^2}$.

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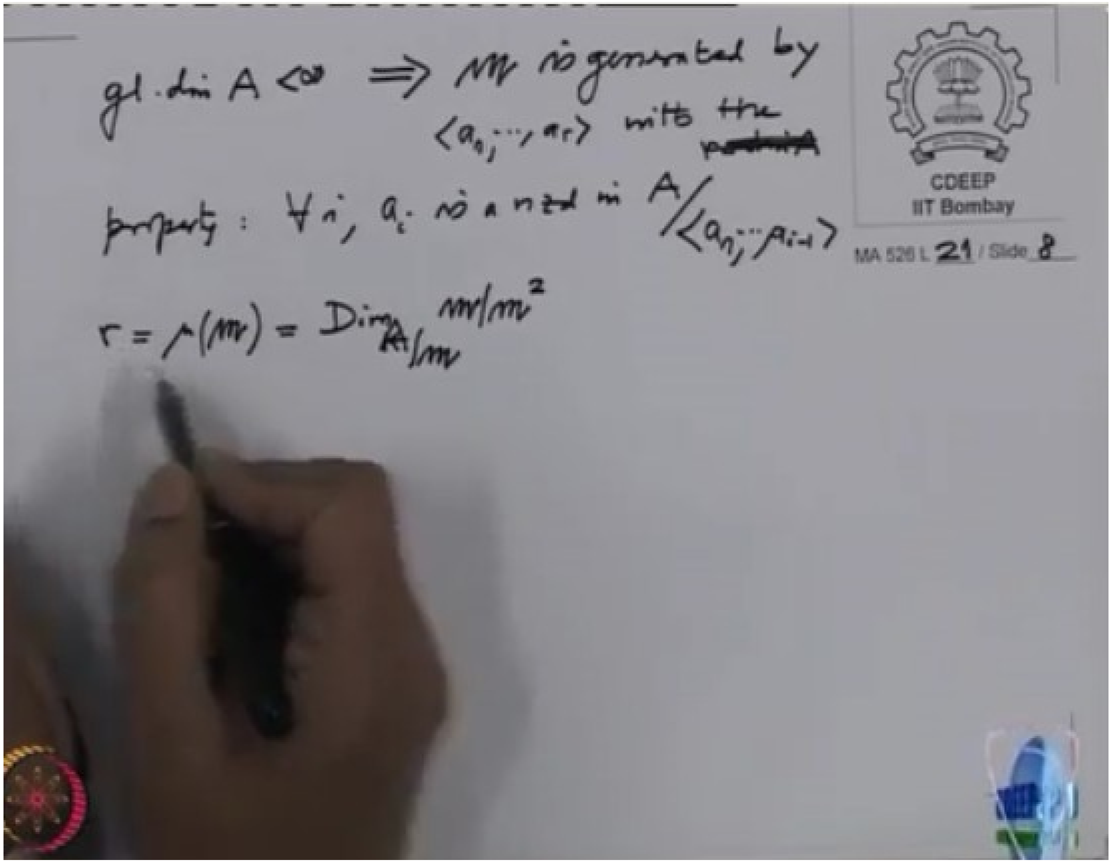

And I'm going to prove that by induction on r that m is generated by a sequence of r elements where each a_i 's are nonzero divisor mod the earlier ones, this is what I want to prove, so start with m, I know m equal to minimal number of generators, m is dimension of these vector space, so any generating set, minimal generating set of m will have r elements, this is by Nakayama lemma, and I want to choose such that the first one is a nonzero divisor, next one is a nonzero divisor mod the first one and so on, this is what I want to choose. And that I want to do it by induction on r, so r = 0 means this vector space has a dimension 0 means m is m^2 , but that will mean m is 0, so then there is nothing to prove, okay.

So suppose now r is positive, and what do you want to choose? I want to start with one guy, so that means I want to produce a nonzero divisor in m which is not in m^2 , then it will be part of a generating set for m, so look at $m \setminus m^2$ and I am looking for an element a_1 here which is a nonzero divisor, this is what I'm looking for.

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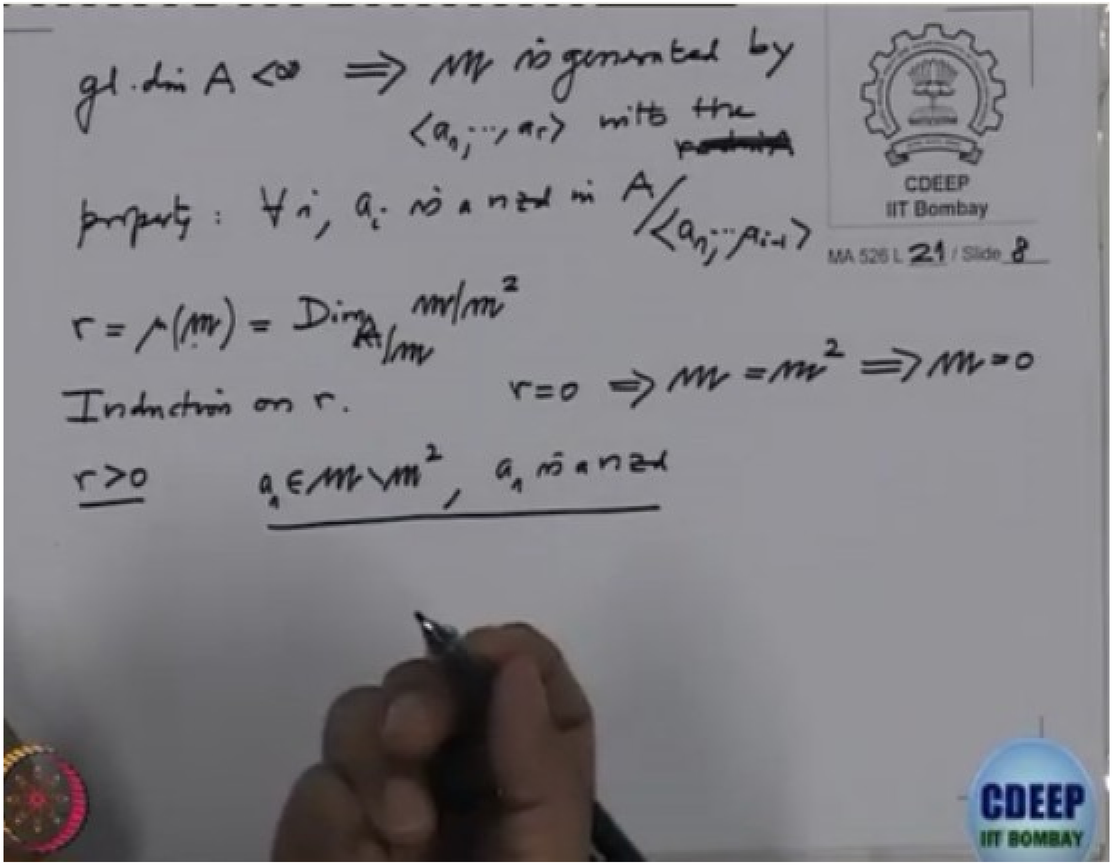

Okay so if everybody is a zero divisor, if that's what I noted that, no if everybody element is a zero divisor, if every element in $m \setminus m^2$ is a zero divisor then every module has a homological dimension zero, that's what we proved, but in particular it will mean that this module, the residue field, this will have homological dimension 0, but that is, that will mean that $\frac{A}{m}$ is free, A module, that is not possible, the residue module can never be a free module because here every element in unrelated by m,

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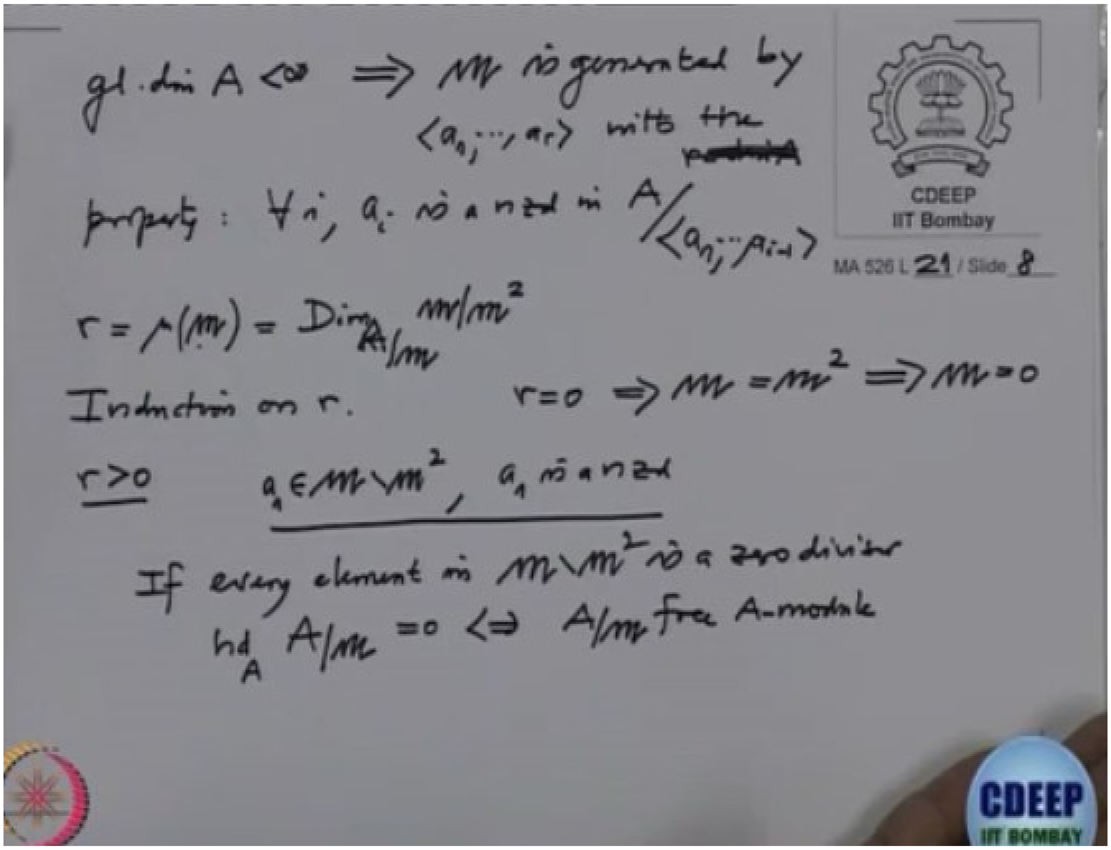

so it can't have a basis, so this is not possible, that means there is an element in $m\backslash m^2$ which is a nonzero divisor, so we started, we choose that a_1 we started, so this we want to prove it by induction now, so if r is positive then we started, we can choose a_1, so there exists a_1 in $m\backslash m^2$ which is a nonzero divisor in A, that is not possible.

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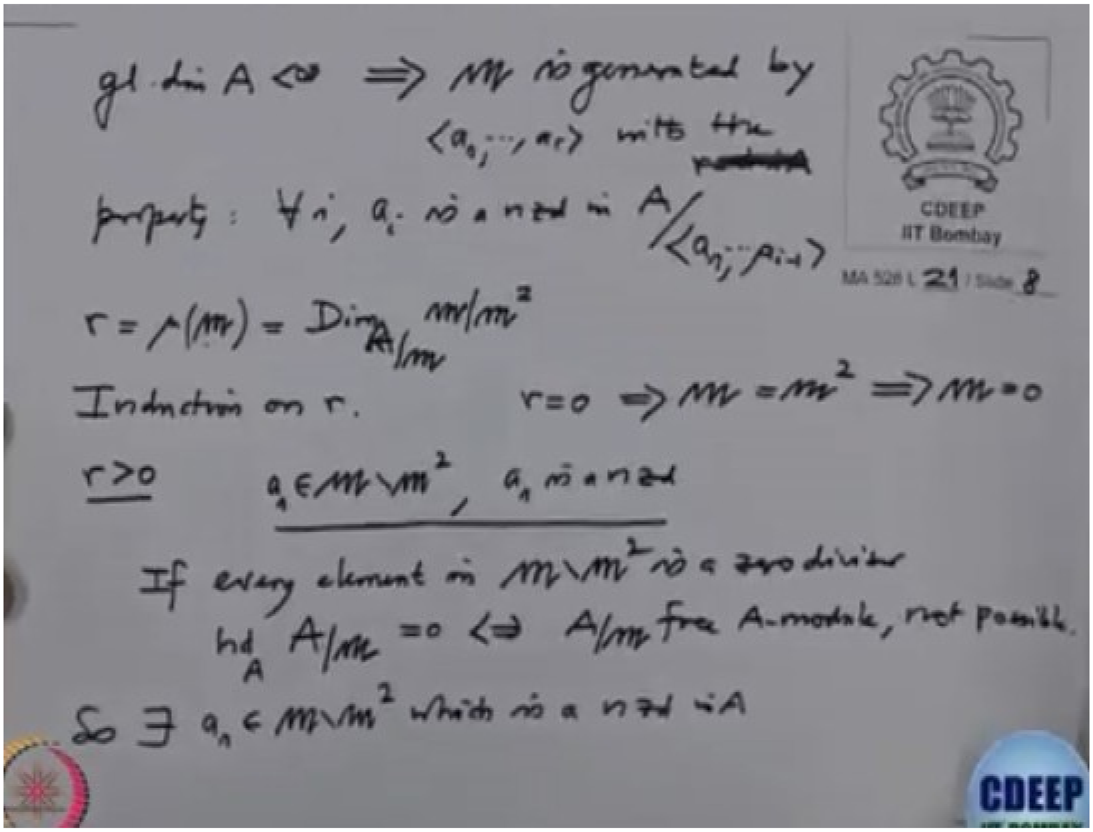

Now you go mod that, now look at the residue class ring $\frac{A}{aA}$, what is that we want to prove that? We want to prove that, because now we have chosen a nonzero divisor which is not in $m \setminus m^2$, therefore this ring will have a finite global dimension, GL dim A/a is also finite, because we are assuming global dimension of A is finite and A is, a_1 I should write, a_1 is an element in $m \setminus m^2$ which is a nonzero divisor and that assumption we have checked that if I go mod a_1 then also you get a finite global dimension for the residue class ring.

Now you do it by induction, now you apply to this ring $\frac{A}{\langle a_1 \rangle}$ and the maximal ideal is $\frac{m}{aA}$, this is a maximal ideal a_1 , and now you choose here a generating set $a_2, \ldots, a_r$ so that a_2 is a nonzero divisor in this ring and so on, the same property so choose $a_2, \ldots, a_r$ in m such that $\overline{a_2}$ is a nonzero divisor in $\frac{A}{\langle a_1 \rangle}$ and so on, so $\overline{a_i}$ is in this ring, is a nonzero divisor in $\frac{\overline{A}}{\langle a_1, \ldots, a_{i-1} \rangle}$ all this are bars, so just lifting up, we'll get what we want, so this implies there exists $a_1, \ldots, a_r$ in m generating set such that a_i is a nonzero divisor in $\frac{A}{\langle a_1, \ldots, a_{i-1} \rangle}$, so this means A is regular, so therefore A is regular.

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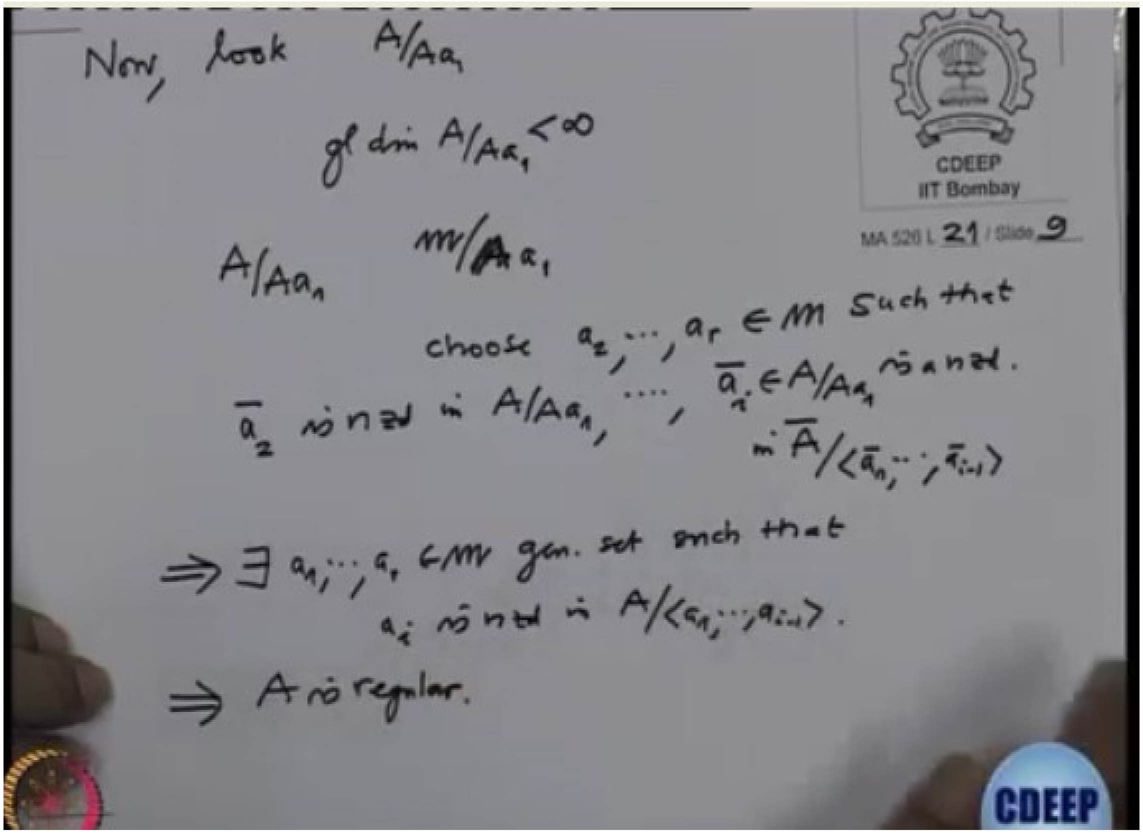

Okay so the most important thing why we proved this theorem, see if you remember we already knew that if A is regular, then the polynomial ring is regular in finitely many variables, or even the powers ring is also regular, but why, what we could not conclude is if A is regular and if P where a maximal ideal, P where in any prime ideal, then A localize at P is also regular, this we could not conclude, and why we wanted that? Because we wanted to define regular locals, so now don’t assume the ring is regular, so A is arbitrary ring and then we wanted to define regular locals to be all those prime ideals P in the spectrum such that AP is regular.

So for example what will be the regular locus of a local ring? Regular locus of A where A is regular, regular local, it should be the whole spectrum this was expected, right, so that means not only the maximal ideal is regular but the localization is also regular, we have not proved this,
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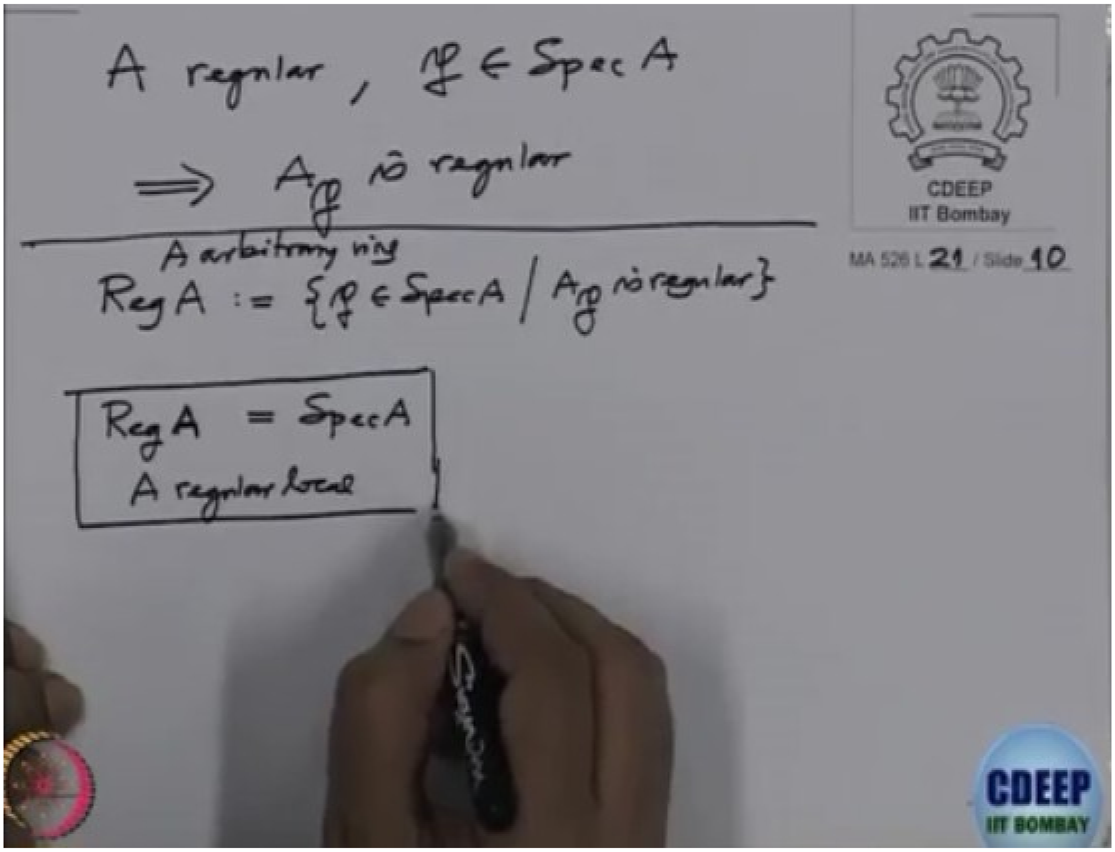

but okay so for this reason we wanted to prove that if A is regular local ring, then any localization of that is also regular local, so now that follows from the corollary, so let me write the corollary and then we will get back to this.

Okay A regular local, and P any prime ideal, then A localize at P is also regular local, okay, so this is much easier now because if you want to test some ring is regular local I have to test that the residue field of that ring has finite homological dimension,
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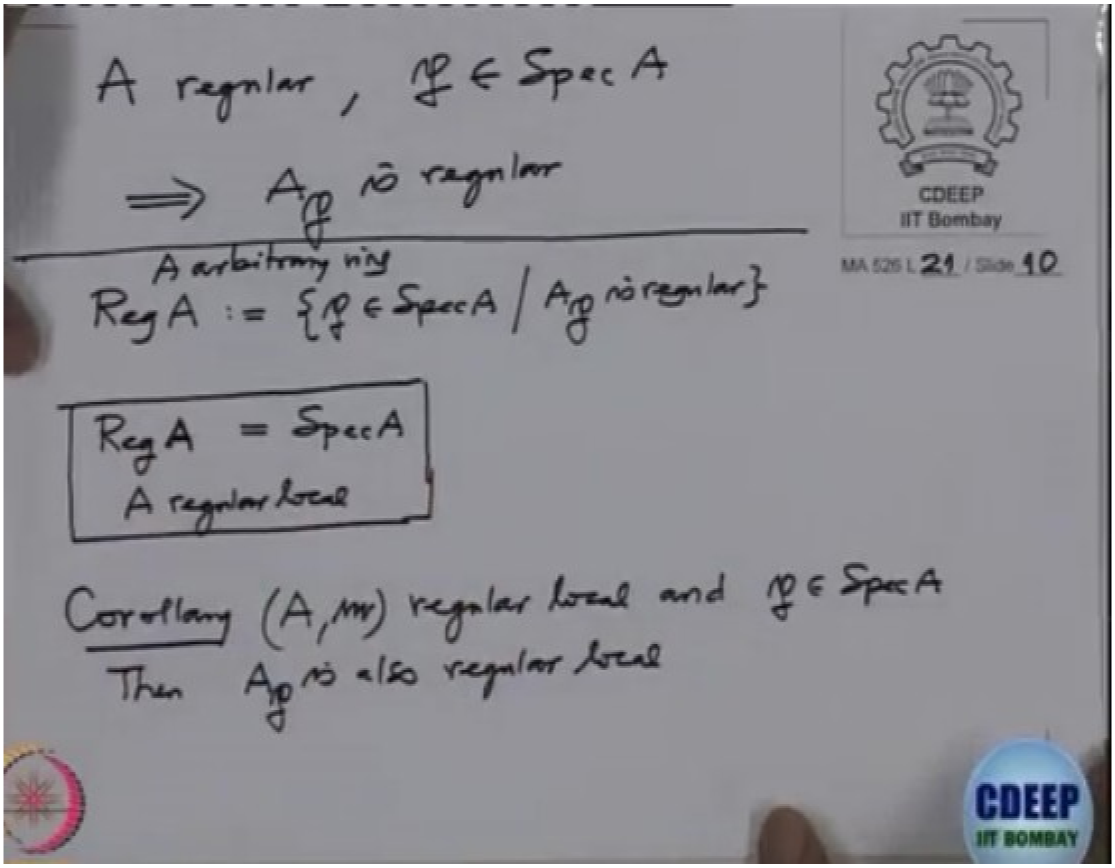

so we will show that the global dimension of the local ring, A localize at P is small or equal to global dimension of A, if you show this by assumption this is finite, this is by assumption because A is regular therefore the global dimension is finite and if I show this then this will also be finite, okay so that means what do you want to show? That means I want to show that alright.

So let us look at the module $\frac{A}{P}$, $\frac{A}{P}$ is a module over A and A is regular ring, A is regular local so therefore $\frac{A}{P}$ will have finite projective resolution, we know that every module over a regular local ring is of finite homological dimension and that is equivalent to saying that module has a finite projective resolution that means a projective resolution will contain only finitely many elements, so that is 0 here, then $F_0, \dots, F_n$ and then it has become 0, all these F_i 's are free module, see because over a local ring projective and free are same, I'm writing them as F, so with there exist a resolution as A modules for the A module $\frac{A}{P}$.

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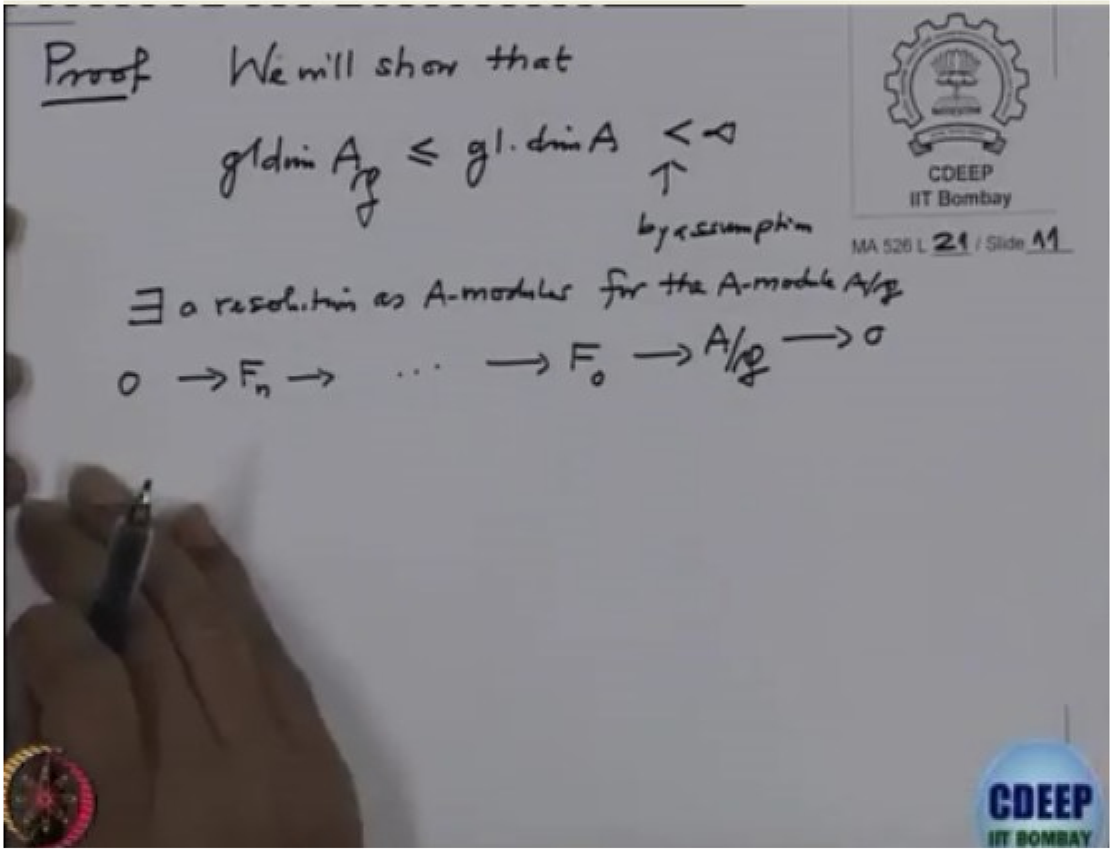

And now let us localize this, this resolution, this long sequence where each terms are free and not this and I want to localize this, if I localize it then localization is exact, so this will remain exact, so that means what? I'm tensoring this sequence with A localize at P dash this, so I'll get 0, F_n tensor over A localize at P and so on, $F_0 \otimes A_P$ over A, and then I'll get this tensor with A localize at P is nothing but the residue field of A_P , this is the residue field of A localize at P, and I have this is, all this terms are free, this are free over, this are free AP modules, and therefore this module as A_P module will have homological dimension in fact less equal to n, so therefore homological dimension of A localize at P of the residue field this is less equal to n, and this n was what? n was this homological dimension of this, so therefore global dimension, so therefore global dimension of A localize at P equal to homological dimension of this is less equal to n, which is less equal to global dimension of A, because global dimension is by definition is supremum,

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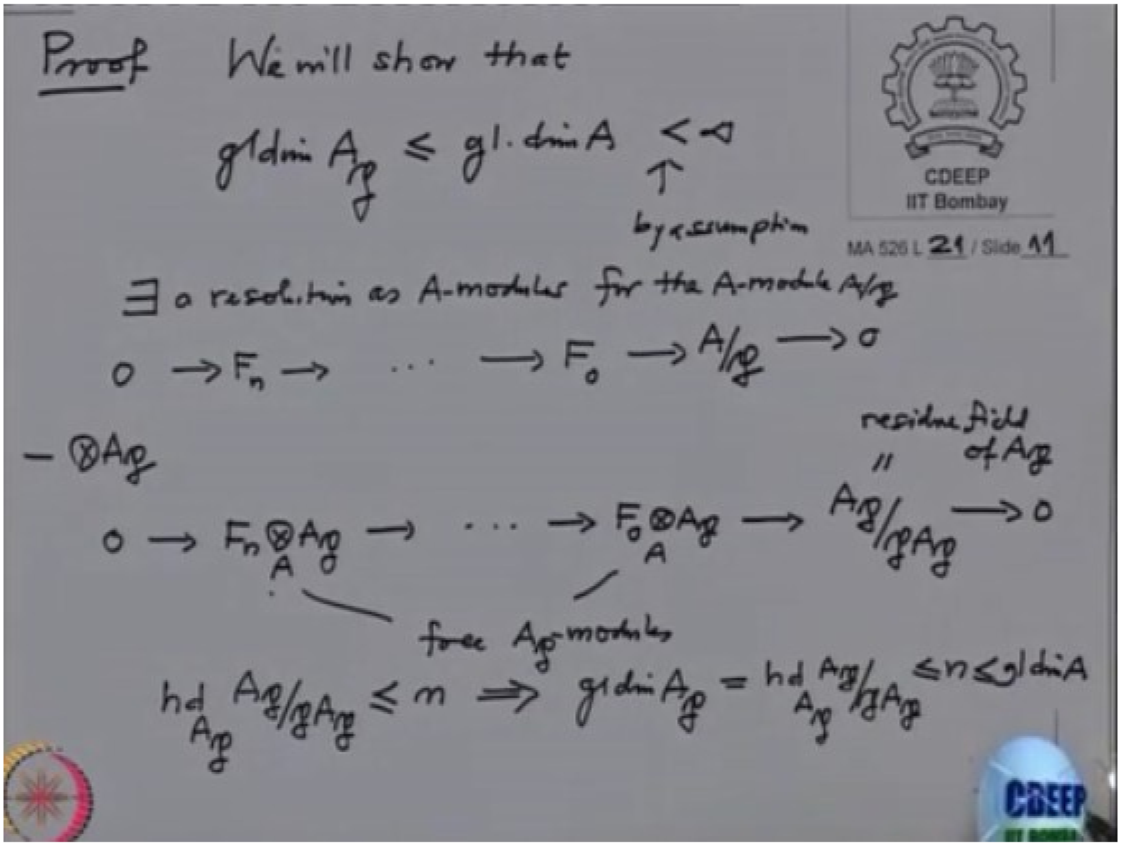

so that proves this.

So now we have proved that, so arbitrary, so if you take any ring, arbitrary ring, arbitrary commutative ring then we have this topological space spec, and I want to define a set reg A or equivalently sing A, okay so this will be compliment of that, and this is by definition all those prime ideals such that A_P is regular.

And now so I will question, what kind of this sets are this? They are in a topological space, so because they are compliment of each other, whether it is open or closed and so on, things like that, so one would like to prove that the singular locus is a closed subset in spec of A, but unfortunately this is not true,
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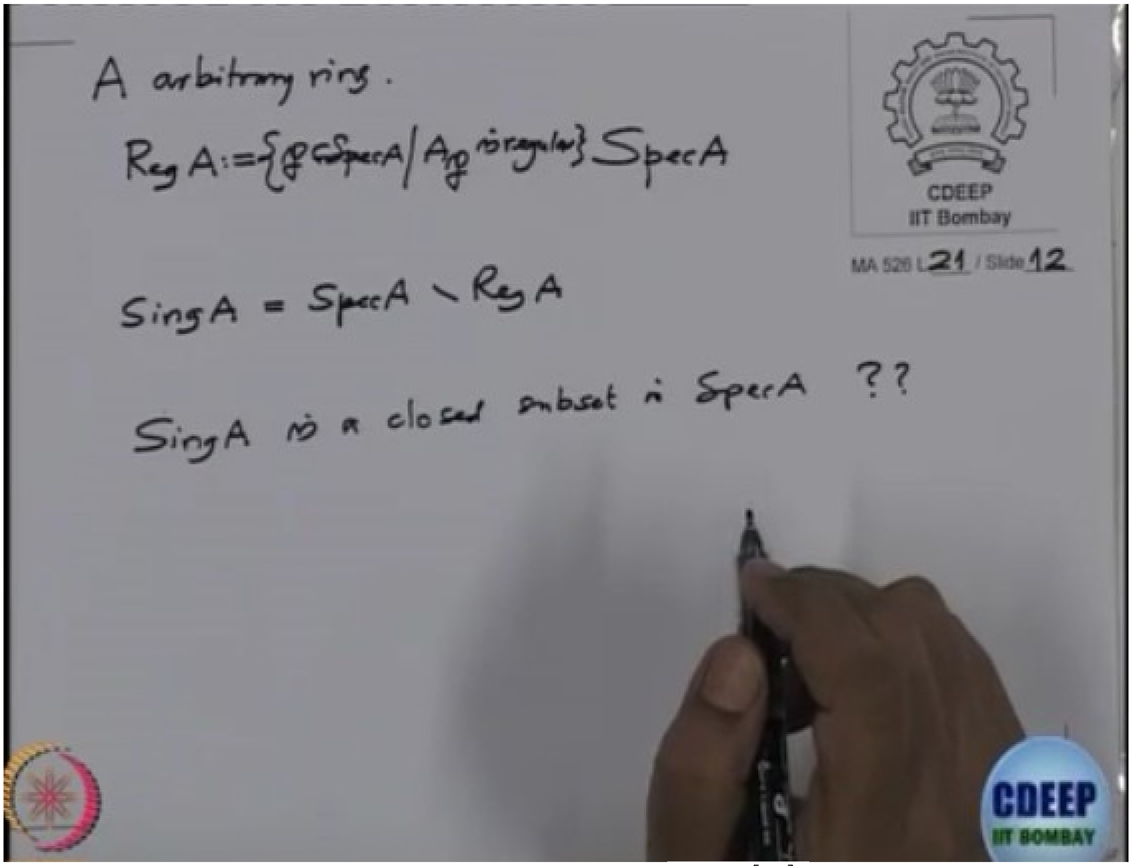

let us see one simple example, so let us take an example $A=\frac{K[X]}{\langle X^2\rangle}$, so it's a local ring, is it regular? It's not regular because it's not even a domain, see if it were regular it will be integral domain, but it is not even a regular, so and what is the spectrum? It does only one prime ideal or maximal ideal so this is generated by X, small x, and the singular locus, this is not very good example, but it's extreme example, so the singular locus is, this is the singular locus, so it's not a proper subset in this case, so one would like that actually this question one can improve is it a closed set, and not only closed set, it should be thin, thin in the sense it should have less number of elements so that you have more elements outside that, or in other words this compliment should be dense, so you have a topological space and this, whether it is closed and compliment, whether is it dense or not?

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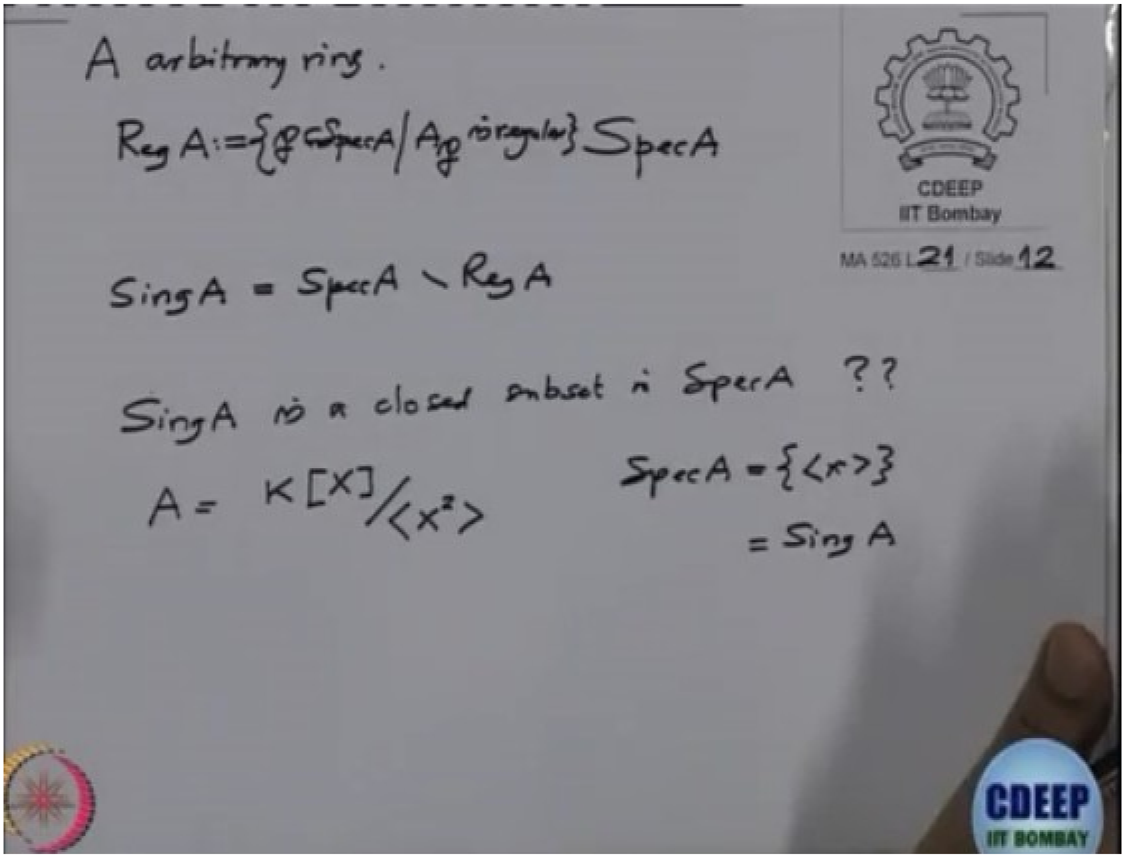

That is the better question, so that is, first thing we need to assume as you have seen in the examples earlier that regular ring cannot have more than two minimal primes, right, so the irreducible components should not intersect, but in affine, if you assume good situation in the sense that if you assume that A is finite, A is reduce and finite type K algebra, where K is a field, then this is true that the singular locus is closed and the regular locus is actually indeed a open dense of set, than this prove I have written in the third updated assignment, so please read it there.

So the one I has abstractive this ring, this ring which has this property singular locus is closed and regular locus has dense open, those rings are called excellent rings, and they have nice theorem about excellent rings at, for example rings of integers is excellent,

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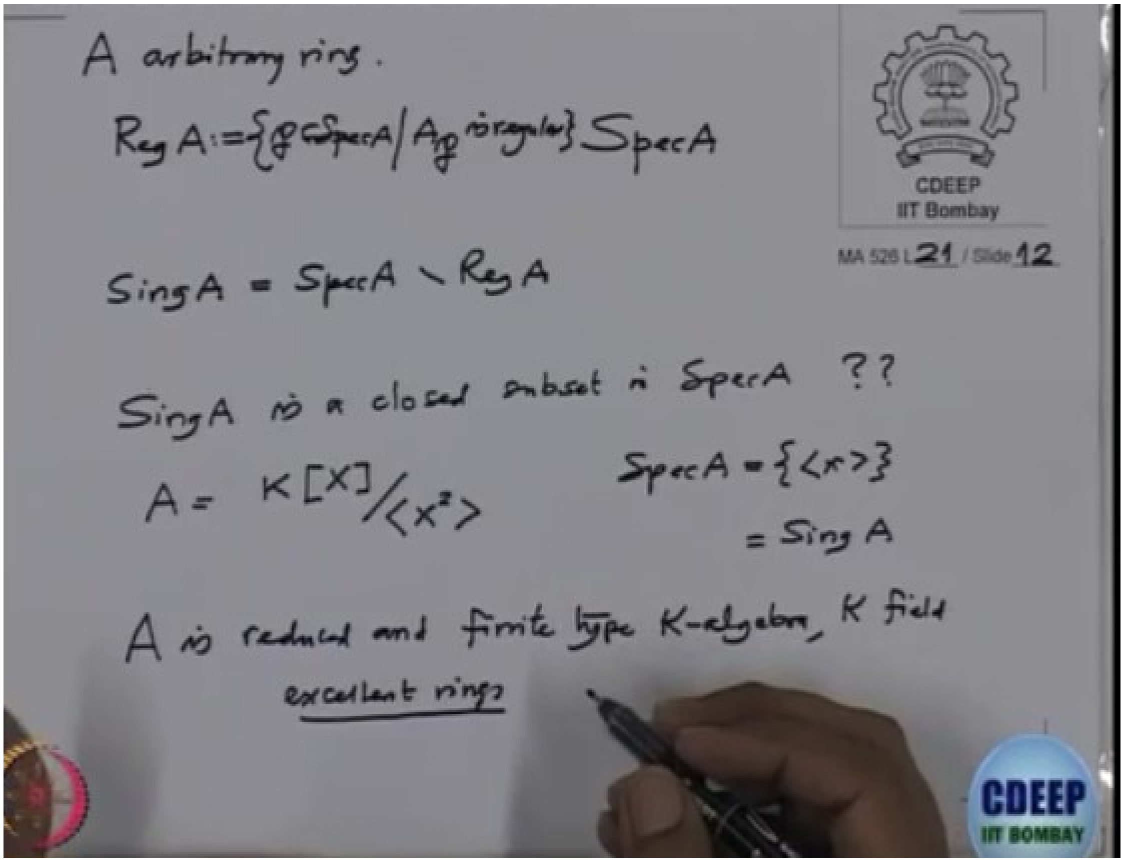

polynomial ring over excellent is excellent, finite type algebras over excellent are excellent and so on.

One more thing just I want to mention here, you might have studied probably if you have a field K, then GL_n , this is subset of K^{n^2} , and let us take for a simplicity K is the field of either real numbers or complex numbers, and then this, on this you put a usual topology, then this set is actually open, open is not so bad to prove, open is much easier to prove,

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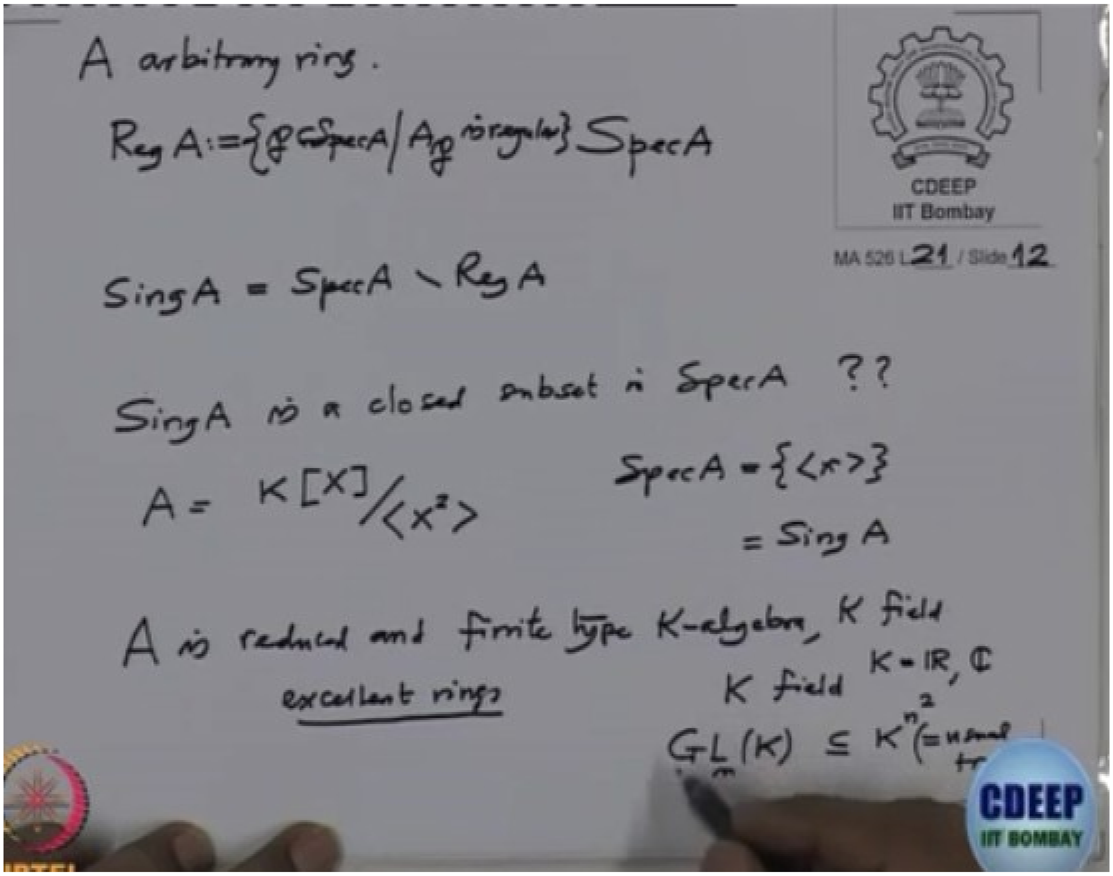

but it is dense also, we try to prove that this set is dense, this is very useful because suppose we want to prove some formulas for matrices, for example if you want to prove that if I have two matrices A and B, and I want to prove that a characteristic polynomial of the product equal to characteristic polynomial of the other way BA such assertion, suppose you want to prove and you would have seen there one reduce it to the case of, one of them is invertible, and then proves it, because then one will be the conjugate of the other end and so on now, so if you want to prove some formula for matrices or such thing, then one could assume that they're actually in GL_n , and then prove there by using, so if you have a continuous formula which is true for GL_n , then the same formula should be true for arbitrary MN, because the set is dense and the formula is continuous so you can always take the limits and prove it, so that is very good for guessing things for in general matrices, what could be the formula, but it has to be continuous formula.

But what does continuous means? It should involve only the plus multiplication and so on, and those operations are continuous, that's it, so we'll stop.

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