Lecture - 51

Homological characterization of

Regular Local Ring(RLR)

(Contd)

Gyanam Paramam Dhyeyam: Knowledge is supreme.

We will continue our investigation about homological characterizations of Regular Local Rings. So the main theorem we want to prove with the following. So theorem, we were preparing to prove this theorem. So let (A, m), be a noetherian local ring. Then, global dimension of A is finite, if only if A is regular. This is what we want to prove today. And remember that the global dimension, we have analyzed. So we approve earlier, global dimension also a local ring. Anyway, global dimension also local ring. Anyway, global dimension by definition it was. This was the definition. Supremum of homological dimensions of modules over A. Supremum is over modules in the category A mod. I also remarked in between that it's enough to consider finitely generated A modules in this. And for the local ring, it's an advantage that this is same as homological dimension of the residue field. So you just have to compute homological dimension of this module. Okay, so this we will use today. So, well, remember last time we proved Lemma which I will use it today again. So I'll just recall the lemma we proved last time. So this Lemma was. So always today is (A, m) noetherian local, if you take an element outside m^2. If A is an element in m, which is not in m^2. Then what we proved is then the short exact sequence of $\frac{A}{a}$ modules. That is $\frac{m}{mA}$ which is clearly $\frac{m}{A}$ model. $\frac{m}{Aa}$. Which is also clearly $\frac{A}{a}$ module and the kernel here is by ma. This is clearly short exact sequence but Lemma says it splits is a split exact sequence. That means there is a $\frac{A}{a}$ module morphism such that, if I go like this it's identity on this. That is the split, meaning of split. Or equivalently the middle one the direct some of the outer two. So immediate corollary into this. Same (A, m) noetherian local. And I assume that $ab \in \frac{m}{m^2}$ which is not a zero divisor and a is not a zero divisor. In A, then, okay if with this situation, if global dimension of A is finite then global dimension of the residual ring is by is also finite. So proof. This will follow from this. So what we want to show first. You want to show the global dimension of this local ring is finite. That means you want to that the residue field away finite homological dimension over this ring. And what we've given is, the residue field of A, is a finite homological dimension over A. So, given, homological dimension of the residue field of this as A module is finite, then we want to show that homological dimension almost. See the residue field is same. So let's call ring as $\frac{\overline{A}}{\overline{M}}$. As $\overline{A}$ module is finite. This is what we wanted to show. To show that.

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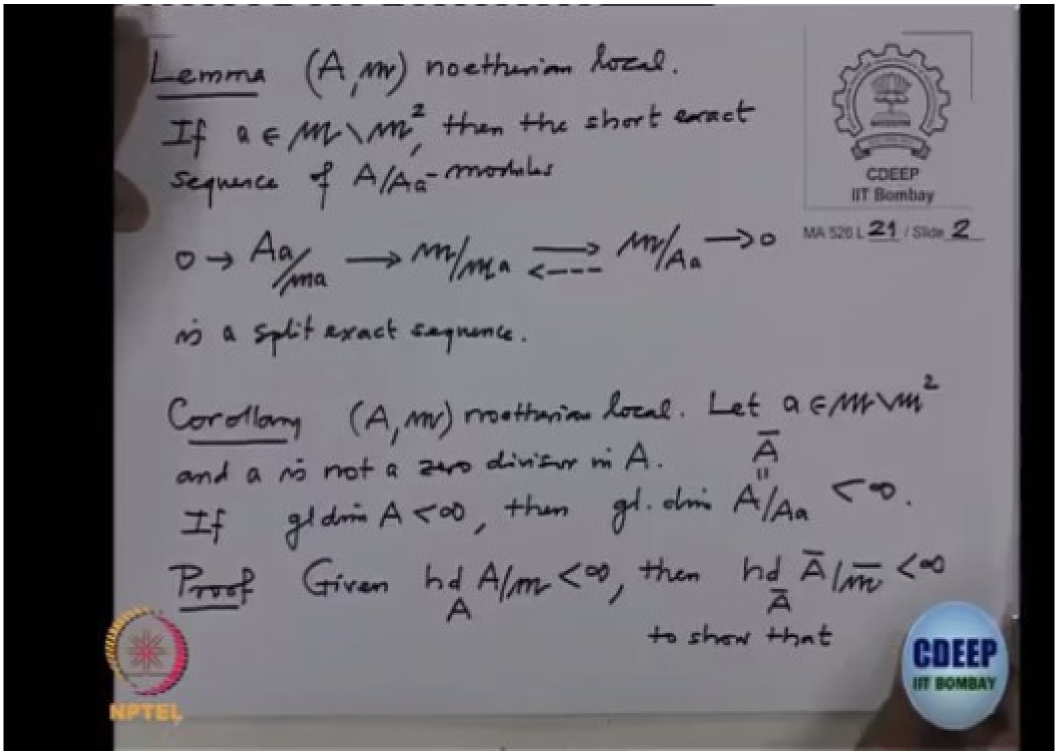

Yes, okay. So how do we show that, in any cause we have an exact sequence like this? 0 to $\frac{m}{Aa}$ and $\frac{A}{Aa}$ and to this residual field k. K is the residue field of $\frac{A}{m}$ which is also same thing as residue field of $\frac{\overline{A}}{\overline{M}}$. This is $\overline{A}$, this is $\overline{M}$. So to prove the corollary, it is sufficient to prove that how I want to show this homological dimension is finite. As $\frac{A}{a}$ module. So, in the earlier Lemma, remember we have the $\frac{m}{Aa}$ here. This is $\overline{M}$. And the speed exact sequence was $\frac{m}{mA}, \frac{A}{mA}$. This was speed exact sequence from the earlier Lemma. And we want to show this guy as homological dimension finite over $\frac{A}{a}$. Because if you show this as final homological dimension then this is homological dimension zero over $\frac{A}{a}$. So therefore this will have finite homological dimension. Is it clear? So enough to show that, homological dimension of $\overline{A}$ over, of $\frac{\overline{A}}{\overline{M}}$ is finite. This is finite and this will be homological dimension finite from this exact sequence. Okay and to show this is finite homological dimension and this exact sequence splits. So therefore. Okay, so remember that because global dimension local of A is finite that is homological dimension of A by m is finite. So that is equivalently to saying, so that is homological dimension of M is finite. And some lecture before we have proved that if homological dimension it was module is finite and if you have a non-zero divisor for a ring as well as the module then the homological dimension of $\frac{m}{ma}$ is also finite. So that implies homological dimension of $\frac{m}{A}$ is finite over $\frac{A}{a}$. This is the Lemma we proved sometime back. Now may be one or two lectures, when we studied homological dimension. I should say here, not $\frac{m}{A}$, but $\frac{m}{mA}$. So what result do we have using it here? If I have a homological dimension of a module is finite and A is a non-zero divisor. For A, as well as m. Then homological dimension of m by m times say. As it $\frac{A}{a}$ module also finite. As strictly speaking, A should write on the other side and this means, this is ideal A dimension. So this is what we have proves earlier. So because now we know global dimensionally is finite that means residue field has finite homological dimension over A, that means homological dimension of the ideal m over A is finite. But now the element A we have in the assertion is in a which is in A, a is in $m \setminus m^2$. And if I prove that it is a non-zero divisor for. Now the module is m and module is ideal m and if I have to check this implication that means if I have to this assertion I will use have to justify that A is non-zero divisor in a ring. That's enough, because if it's non-zero divisor in a ring it is non-zero divisor for a module m also. So, we will check that in a minute but then we would get homological dimension of $\frac{m}{mA}$ is also finite. So look at this exact sequence wrong. This one, you want to prove homological dimension is finite. This one has homological dimension finite. And this one is a split exact sequence. And therefore then, we will finish what we want to get. Because direct sum of A and of homological dimension normally comes finite rings.

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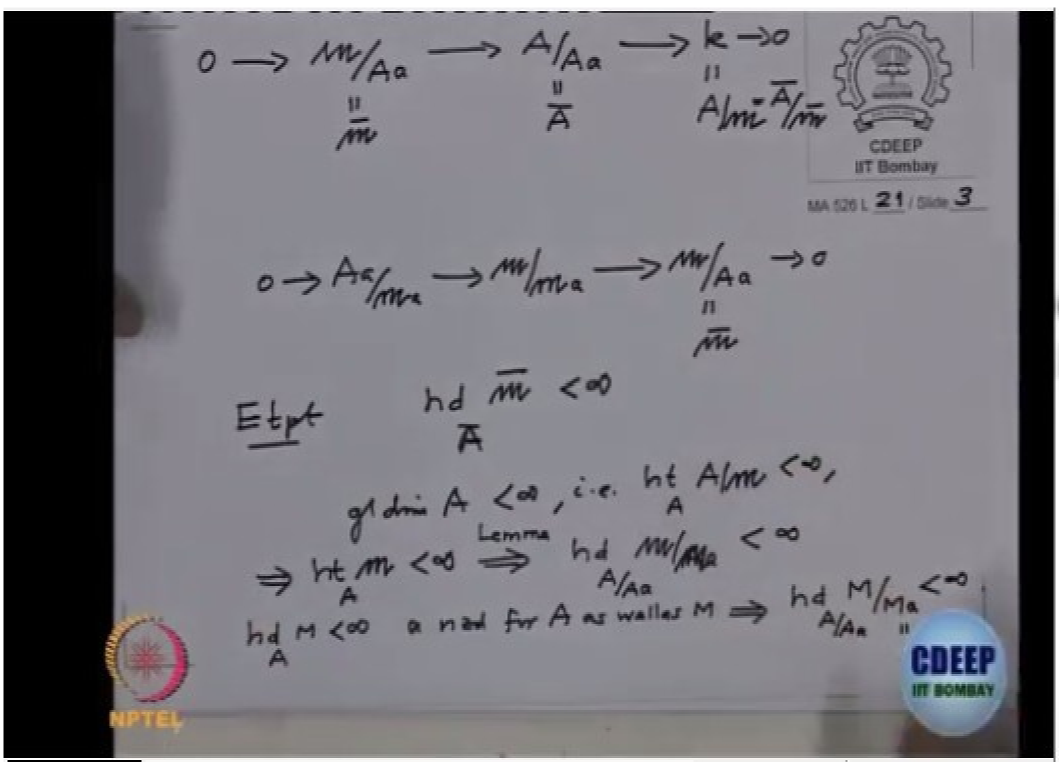

So we know only have to check that a is a non-zero divisor. So we need to check that if I have an element a which is in $m \setminus m^2$. Then a is a non-zero divisor in A. No, no. Corollary was… Yes, yes. Sorry this was already I have assumed it is a non-zero divisor. So we don't need to check. This is given. I will check when I need it A under some assumption I will have to. It is true, but with some assumptions. That's why I've assumed it here. So I will come back to this point. Good, you reminded me. Okay. So that proves, what did you prove therefore corollary let us recall. If I have any noetherian local ring, a is in $m \setminus m^2$ which is a non- zero divisor then with the homological dimension of A is finite then homological dimension module that A is also finite. This what we have proved. Okay. So, again this also, anyway it maybe repetition but doesn't matter. Lemma if I had a non- zero module m and as usual over A is local and a is in m, which is a non- zero divisor for m. Then, homological dimension of m by am equal to homological dimension of Am plus 1. In general, both sides maybe infinite. So this, if you like this also follows from this. Right. But let me prove it completeness it is not so difficult. So that will also give us a practice for computing homological dimension. So look at the exact sequence 0, m, m, to $\frac{m}{Am}$ to 0. And this map is a multiplication by m. Multiplication by A. And fact that it is a non- zero divisor means this map is injective. This is the kernel. So now to conclude homological dimension, we have two apply to Tor, we have to apply tensor product in the residue field and conclude the Tors. So when you write a long exact sequence, what do you get, you get Tor_{n+1}(M, residue field). Let's call residue field to be K. Then this is the Tor_{n+1} I started here. $\left(\frac{m}{am}, K\right)$ then the connecting morphism will come here. $Tor_n(M, K)$. See I started here. M plus 1, then into M to this, then $Tor_n(M, K)$ and $Tor_n\left(\frac{M}{Am}, k\right)$. This map is a multiplication A again. This is a connecting Homomorphism. Okay, what do we know now. This is 2 for. Okay, I want to check now is, this is zero here. I can put zero here because what happens with and what is this map? This map is a multiplication by A. So when I take, take this map, I want to put. So this map is a multiplication by A, but on the other hand it is the factor is K is $\frac{A}{m}$, so actually it will be zero map. Because when you take the definition of the connecting Morphism it will come from denser product that the other factor is $\frac{A}{m}$. so A is m therefore that will become zero there. So that means, I can put zero here and forget this part that means such a sequence is exact but that will mean that what do you want to prove that, you want to prove that homological dimension is one more. So this, this one. So already if this, for large, for large m it is zero then, if this is zero then this is also be zero. So that is how is one homological dimension.

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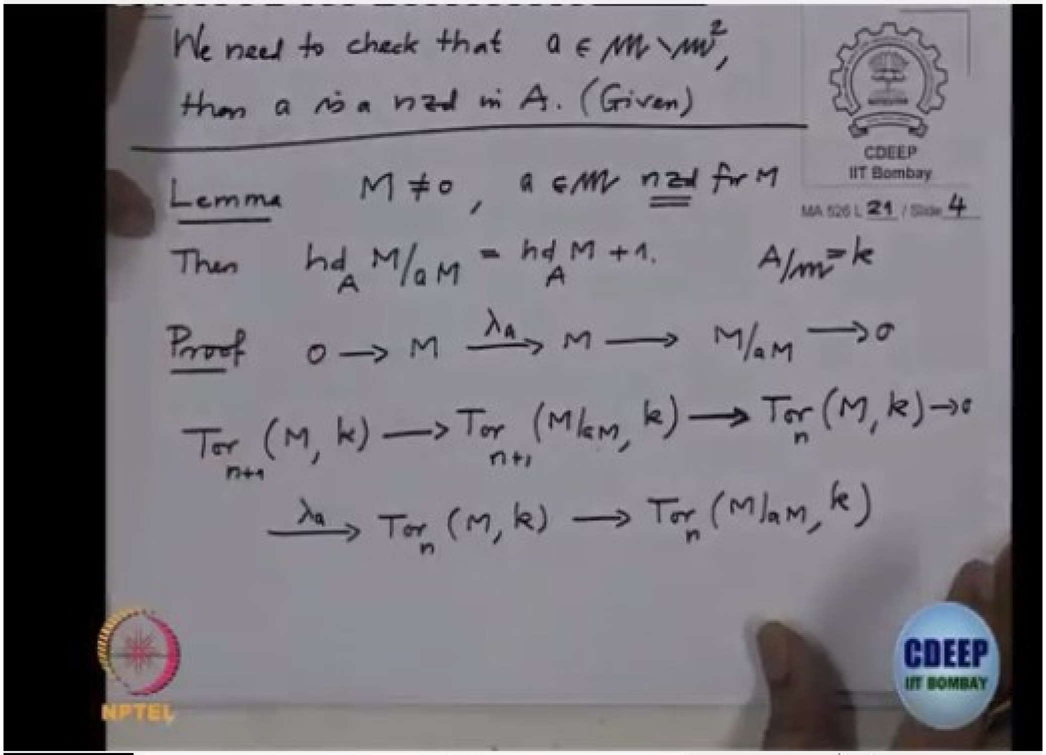
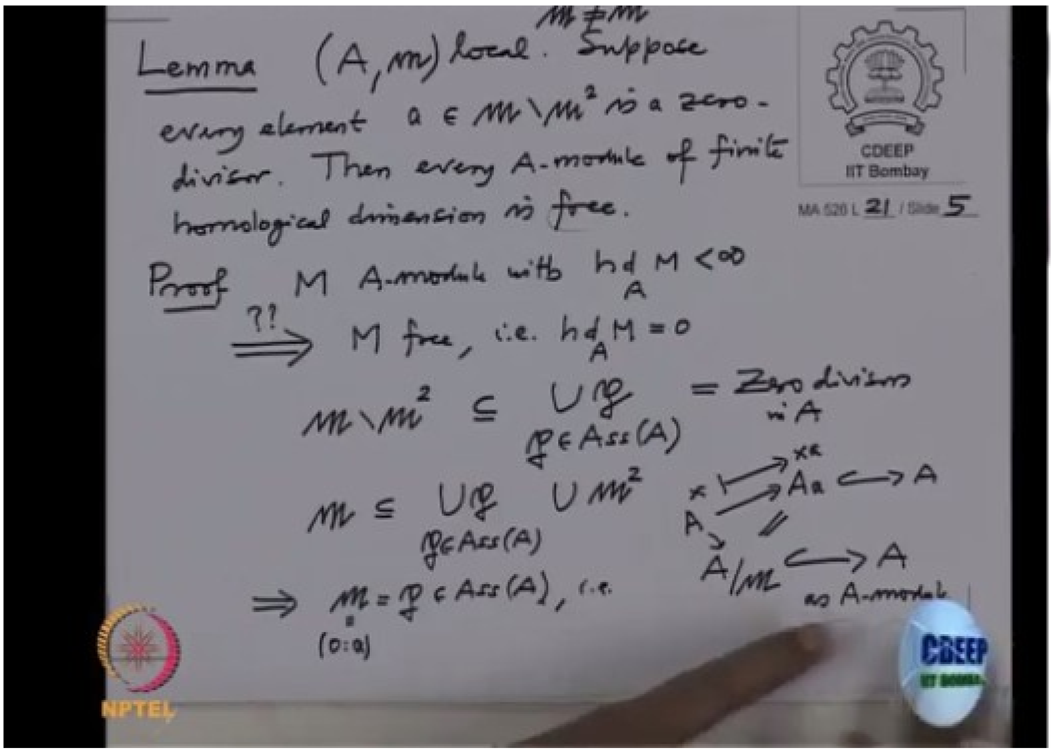


All right. Now that address about $\frac{A}{m^2}$. So that Lemma is following length. So A noetherian local and suppose, suppose every element $a \in m \setminus m^2$, is a zero divisor. Then, every A module of finite homological dimension is free. Proof, see where in the Lemma we say if $a \in m \setminus m^2$. If it's a non-zero divisor, something else happens. This Lemma says if all elements are zero divisors, then module we have actually finite not, if it is a finite multi dimension, then it has to be zero. Free means homological dimension zero. So and we can always choose, if all elements are zero divisors and things are easier, otherwise you can choose a non-zero divisor and apply those Lemmas. Okay, so why is that? So, what we want to prove? We want to prove that if M is a module, with finite or multiple dimension then we want to prove M is free. That means and for local ring, so free is equivalent to same projective but that is equivalent to homological dimension of M with zero actually. This is what we want to prove. We want to prove it is free or you want to prove it is projective. That's equivalent for a local ring. And what is given? We have given that all elements of m minus m square are zero divisors.

So that means we have given that M minus m square. This is containing the union of associated primes.

Zero divisors are precisely the unions of the associated primes. So this AD is precisely the zero divisors in A. But this will mean that m is containing this union, union m^2. And of course, we're assuming A is not m^2. A is a local ring with the properties that m is not equal to m^2. Because if anywhere m^2 then by Nakayama Lemma actually m will be zero and we will field case that's all. So, now here by prime avoidance because m is m^2, by prime avoidance this will imply that m is actually associated prime. By prime avoidance. Because m is not continued in m^2. So that means this is so that is this, m, the residue field $\frac{A}{m}$, will contain inside A as A module. Somebody associated prime ideal means this is annihalator somebody. Is an annihalator some element A, but then $\frac{A}{a}$, this is contain inside A, but $\frac{A}{a}$ is $\frac{A}{m}$. $\frac{A}{a}$ is $\frac{A}{m}$ because, you see, you take the map from A to A to ay m, any x going to xa, and the kernel this map is precisely M. And therefore this map is factor through this and it's subject to already, so that was this will be equal. So therefore, associated prime ideal means, copy of the $\frac{A}{P}$ is containing a. (Refer Slide Time: 24: 43)



But I'm applying it for M. so residue field goes inside m. then it goes inside A. Okay, so now take any module M. Let M any module with homological dimension actually m equal to n which is finite I want to prove M is projective. What happens if M is minus 1? If $n = -1$, that means actually m is actually zero. So you can assume M is bigger equal to zero. And look at the exact sequence like that. Now, you see A and copy of the residue field is inside A, and then we look at the cokernel of this. So that means A by K. This is only A module sequence. And I'm going to use this sequence to compute homological the Tors. So that mean I'm going to apply the functor M, tensor - to this sequence and write the homologies. So we start again with Tor, it's a homological dimension is in. So $Tor_{n+1}(mA, K)$, that I started here, then $Tor_n(M, A)$. This is exact, but this is zero and for us, this is non-zero. Because of homological dimension that means Tor_n and M k is non-zero. So that is non-zero and this is zero. Therefore it will imply this is non-zero. Because this goes inside injectively here. And this is zero because the earlier when is $Tor_n(M, A)$. What is earlier here? $Tor_{n+1}(M, A)$. All right, so that is zero. So therefore, what we got is, this is non-zero, that means what does this mean? What does the $Tor_n(M, A)$, non-zero means this will mean, this implies n is actually zero. Because what is $Tor_n(M, A)$? A Tor will commute, so I could have done the other way. So this is same as $Tor_n(A, M)$ Isomorphic. But A is projective. So that would Tors, higher Tors are zero. If the first factor in the, when you compute theTor, in the first factor is flat then the sequence will remain exactly after Tensoring. Therefore, no homology. So this is zero. That was the reason here also $Tor_n(M, A)$is zero. But here, n cannot be more than zero. The ring is under, the ring A module is always flat. It's always flat. So therefore n is zero, but M zero means homological dimension zero m is project. So homological dimension of M is zero. So therefore M is projective. Now therefore M is free. So that proves the length. So that is the advantage with working with Tor because Tor has a propererty it's Tor_{mn} is isomorphic to Tor_{nm}. That's not two for X. And both the variables Tor is co variant. And Ext is co variant in one variable and contra variant in another variable. So those, that is why working with the Ext is more difficult but all the variant working with x is better results.

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