INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

IIT BOMBAY

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

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Lecture No. – 50

Homological characterization of Regular Local Ring (RLR)

Okay now we'll recall, we have already defined what is the global dimension right, it is supremum of all homological dimensions.

So now we'll prove about global dimension, so this is the theorem, so (A, m) local, noetherian local always and n is any integer, positive integer and the following are equivalent. One, global dimension of the ring is less equal to n, two, $Tor_j(M, N)$ is 0 for all A modules M and N, and for all integers big or equal n+1. And three, $Tor_{n+1}(K, K)$ is 0, where K is a residue field.
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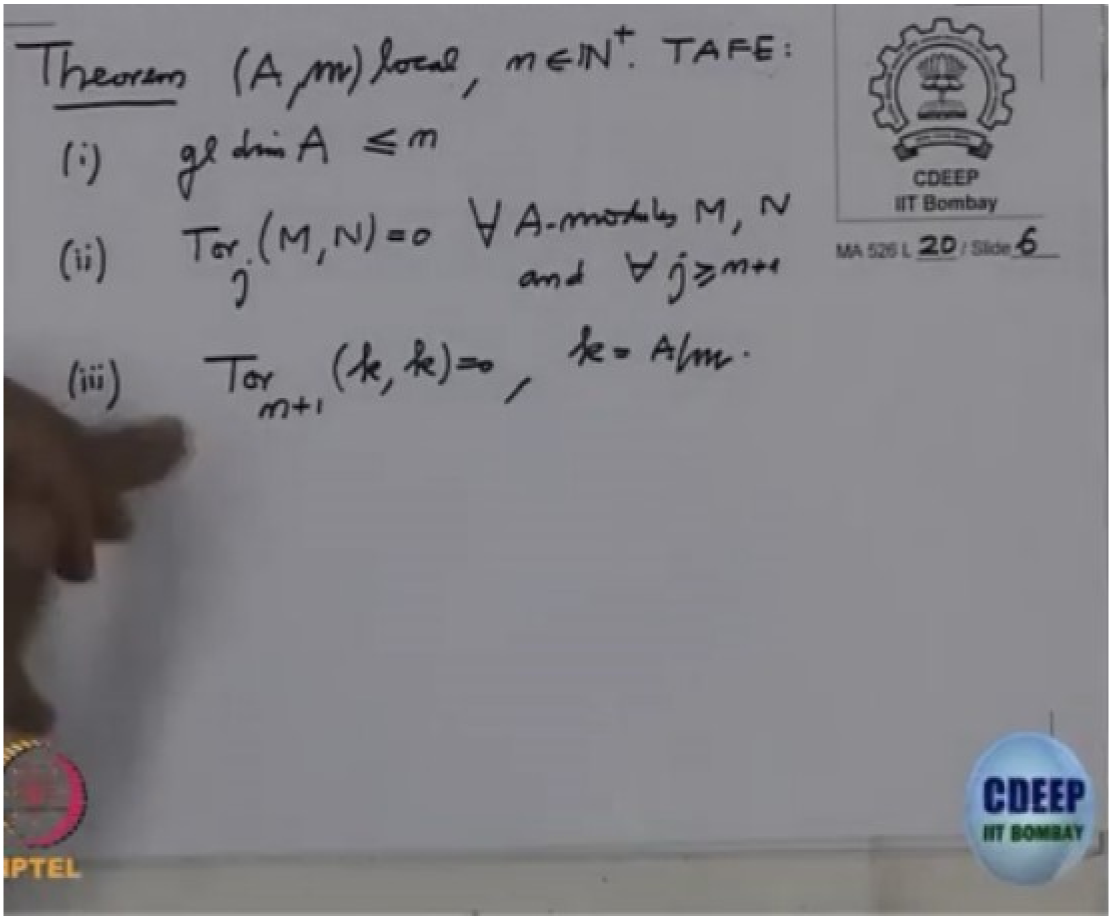

So you see this, only we have to check only one Tor_{n+1} that is the advantage, alright.

So proof, see the global dimension A is by definition sup over all modules, now the homological dimension of M this was our definition, so for example if I want to prove 1 implies 2, we have given that this supremum is less equal to n, that means homological dimension of each module is less equal to n,
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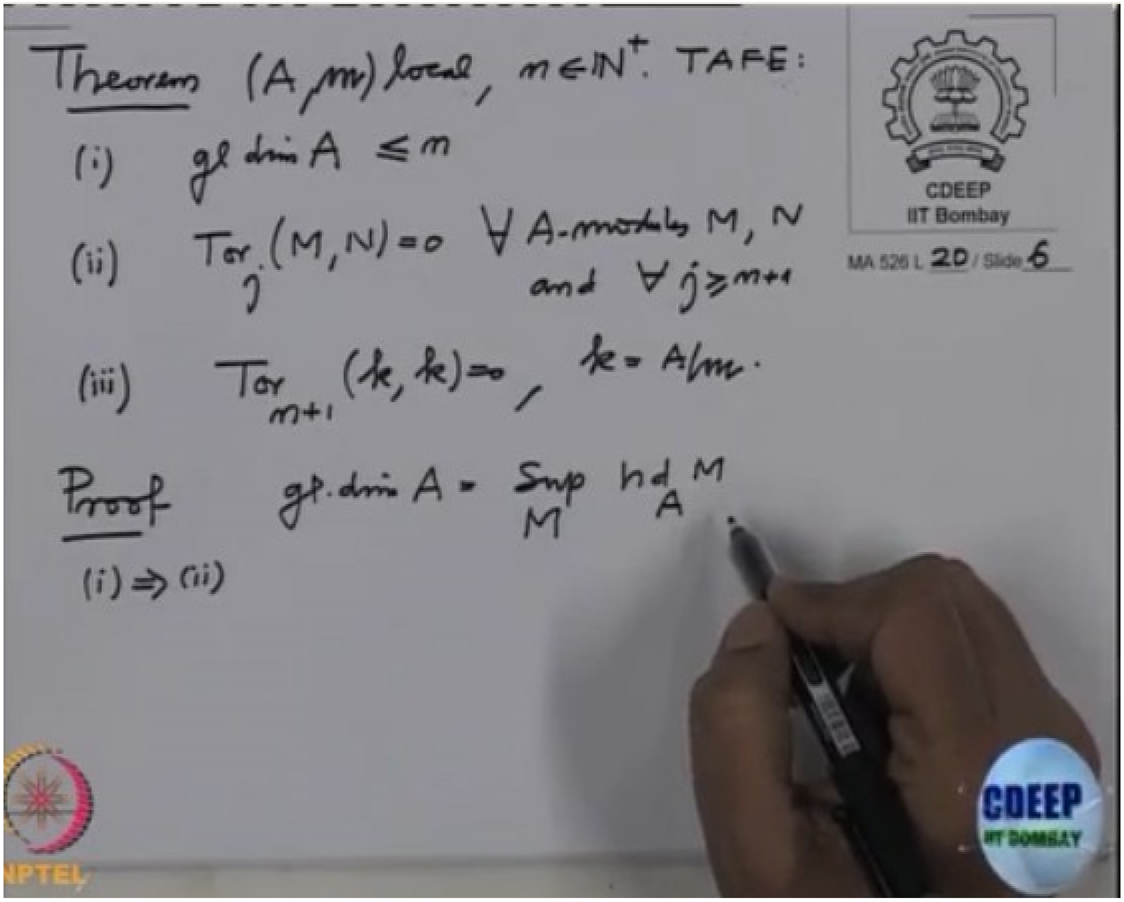

but then earlier proposition we know if I fix this module M, and is the homological dimension of M is less equal to n, then for any module N and for any integer j big or equal to n+1 tor is 0, so that is what precisely this 2 is, so this follows or immediate from 2 implies 3 of the, no this is immediate from 1 implies 2 of the earlier proposition, okay, that is 1 implies 2.

And 2 implies 3 is trivial because here it is j = n+1, so take j = n+1 and the module M equal to module N equal to the residue field, so the real proof 3 implies 1, okay.
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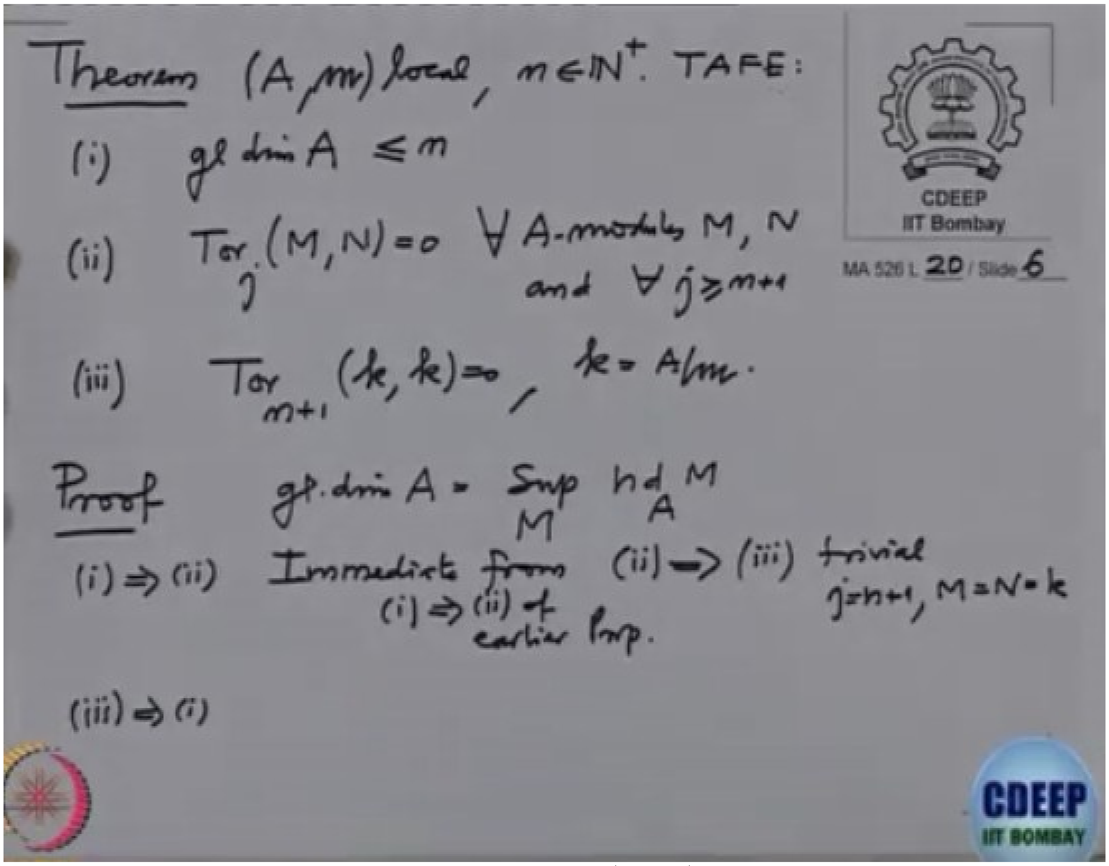

So suppose 3 hold that means I've given the $Tor_{n+1}(K,K)$ is 0 and from here I want to conclude homological dimension of every module is less equal to N, then the supremum will be less equal to N, so enough to prove, so I have shown suppose that $Tor_{n+1}(K,K)$ is 0, and what do you want to prove? To prove homological dimension of M is less equal to n for every A module M finitely generated, alright.

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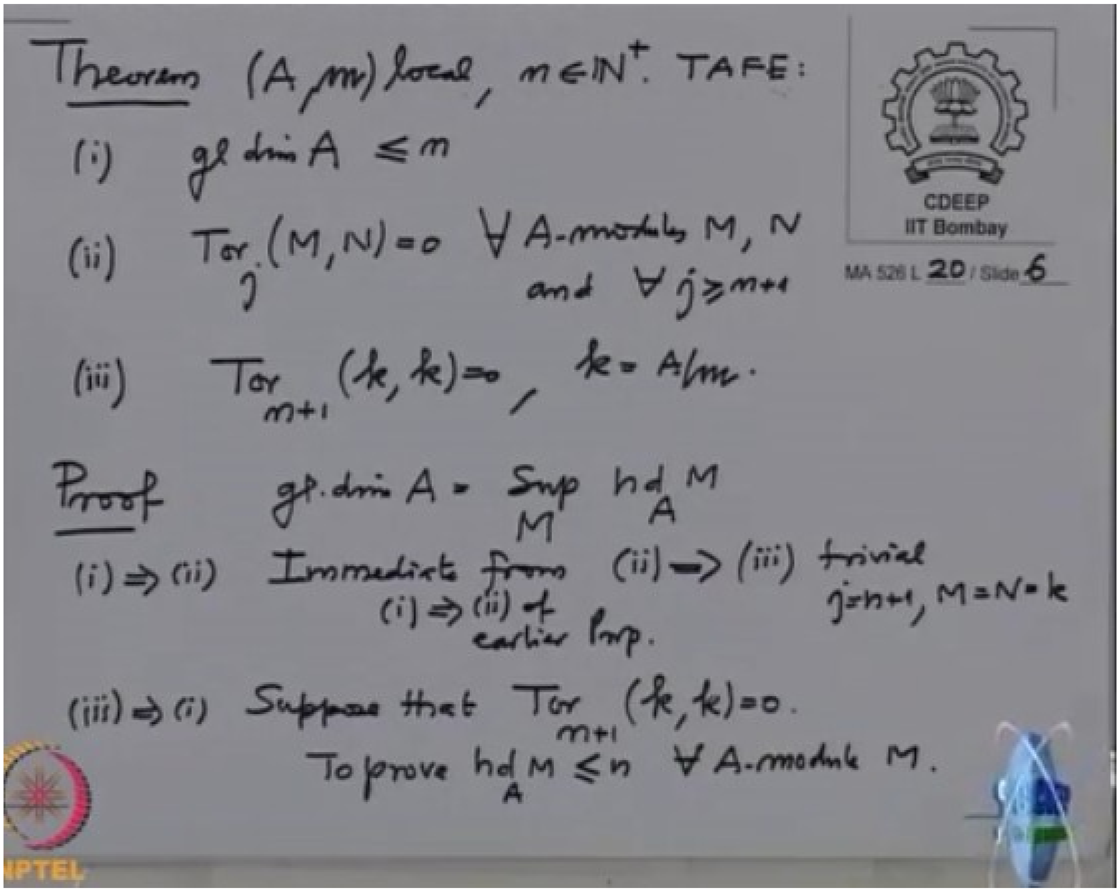

So tor is 0, so look at we want to prove that homological dimension, so homological dimension of M is less equal to N, we know this from the earlier proposition this is equivalent to saying tor N+1(M,K) is 0 this was earlier proposition, right, but we know that the tor commutes again integer in the variables, so because the tensor product is commutative, M tensor N is isomorphic to N tensor M, so this is same as tor N+1(K,M) and this one is 0, so earlier proposition says that, third condition says if tor N+1(M,K) is 0 then homological dimension is less equal to N, but you see we have this, the other way, anyway this is 0, so okay, so therefore it follows that homological dimension is less equal to N, this we have proved it for finitely generated modules, so okay, so in the definition itself one possibility is in the definition itself we say that global dimension is by definition supremum of homological dimensions of M, where M is finitely generated module and you can always reduce to finitely generated K's because one possibility is because the ring is noetherian and, okay, so we will discuss this in tutorial, how to reduce to the finite dimension, finite generally generated case.

So here right now we are taking this supremum more finitely M finitely generated A module,
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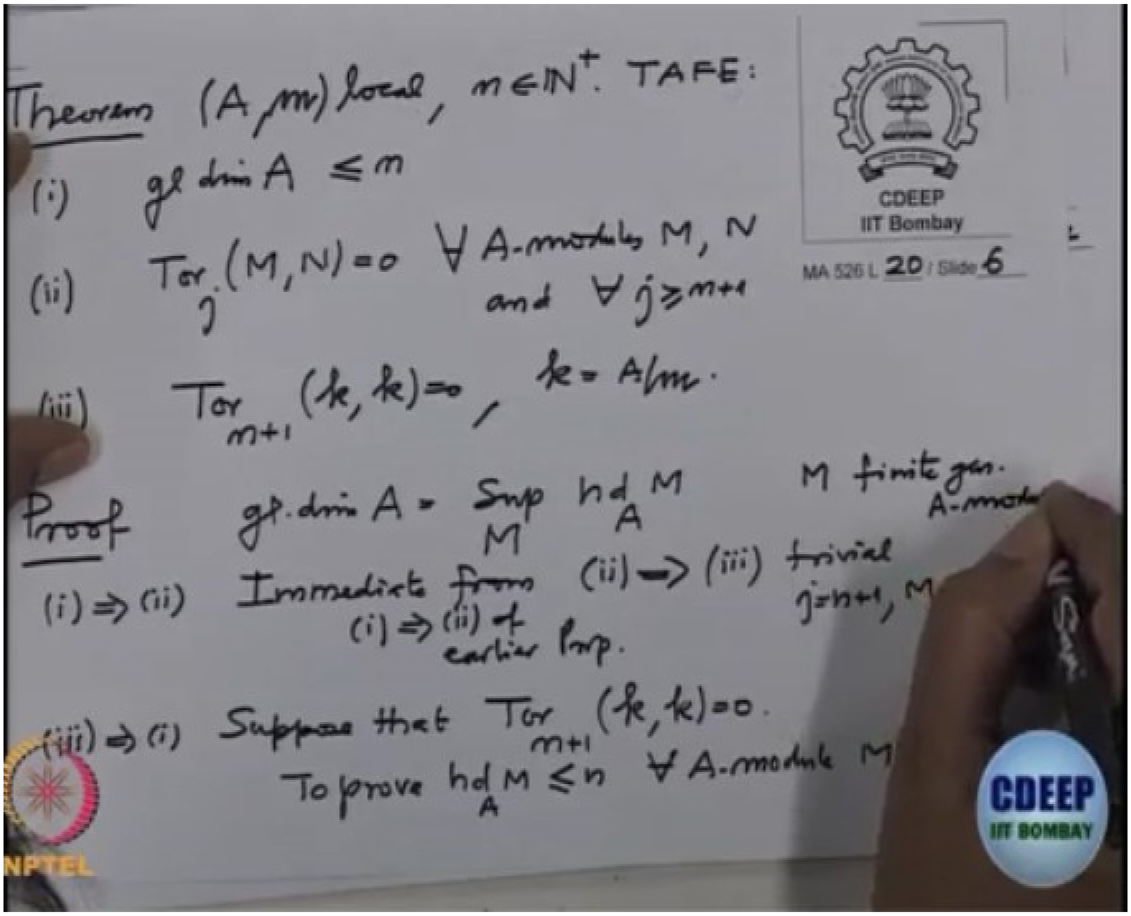

so because all our proofs will neatly work over finite, work for finitely generated modules or noetherian rings, alright. So one corollary I want to note is so one of the important corollary, which one you are saying? Oh this one, see in the earlier proposition we have proved that tor homological dimension is less equal to n means equivalent to $Tor_{n+1}(M,K)$ is 0 so this is what we have proved, this we have proved this if and only if you have proved in proposition, earlier proposition, and this one is isomorphic to this, so therefore $Tor_{n+1}(K,M)$ is 0.

Now we are proving what? We are proving 3 implies 1, right? We are proving 3 implies 1, right, okay so, so 3 says $Tor_{n+1}(K,K)$ is 0 right, and $Tor_{n+1}(K,K)$ is 0 and earlier proposition says you get hom on a homological dimension, I apply the earlier proposition to, no, so we want to prove homological dimension less equal to N, so therefore this is enough to prove this is 0, therefore enough to prove this is 0, okay, so if you want to prove this is 0, earlier one, no earlier proposition we proved homological dimension is less equal to n is equivalent to $Tor_{n+1}(M,K)$ is 0, applied to K, yes, alright, so corollary I wanted to note was if you have noetherian local ring (A,m) then the global dimension of A is precisely the homological dimension of the residue field as A-module, so we only have to check whether, what is the homological dimension of M, $\frac{A}{m}$,

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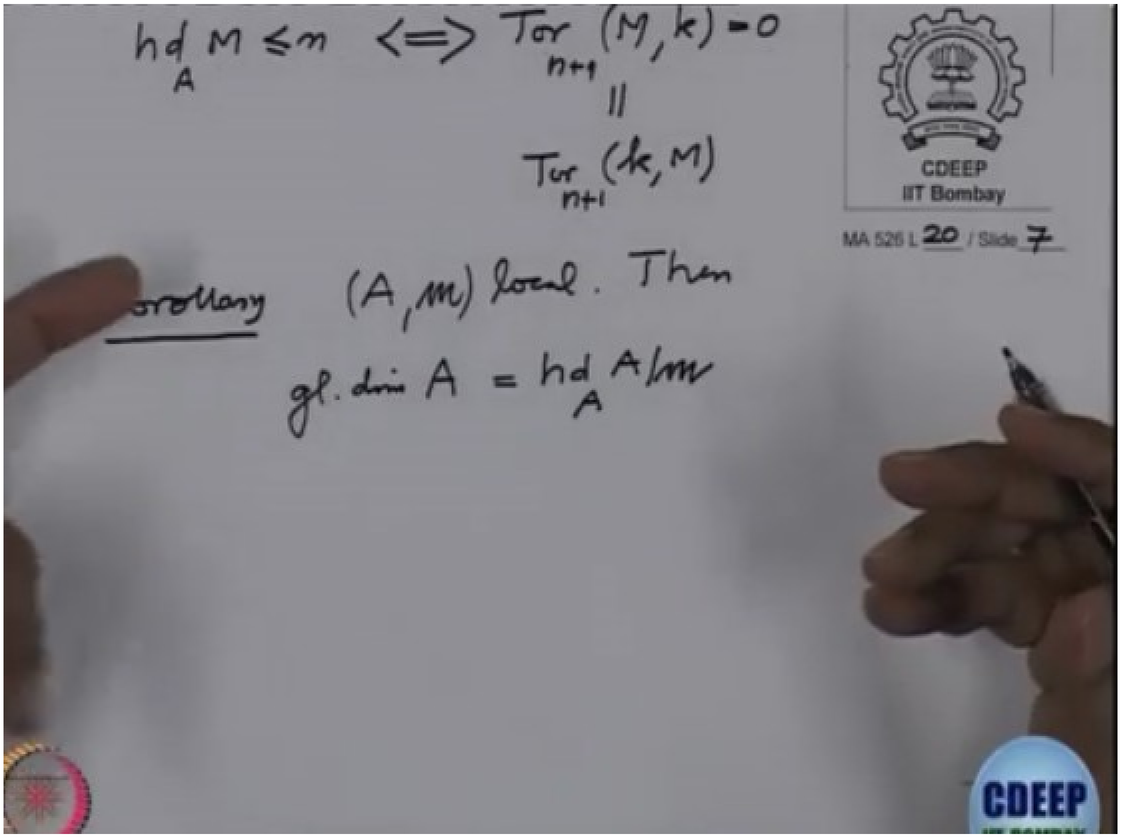

that means we want to take a projective resolution of $\frac{A}{m}$ and see how long the projection resolution goes, if the length n there is no homological dimension is n, if the length is not finite that means global dimension is not finite, and this is very easy to check.

Okay now this is what, now we want to give a homological characterization of, this is a homological characterization of regular local rings, remember earlier we had two characterization of regular local rings,

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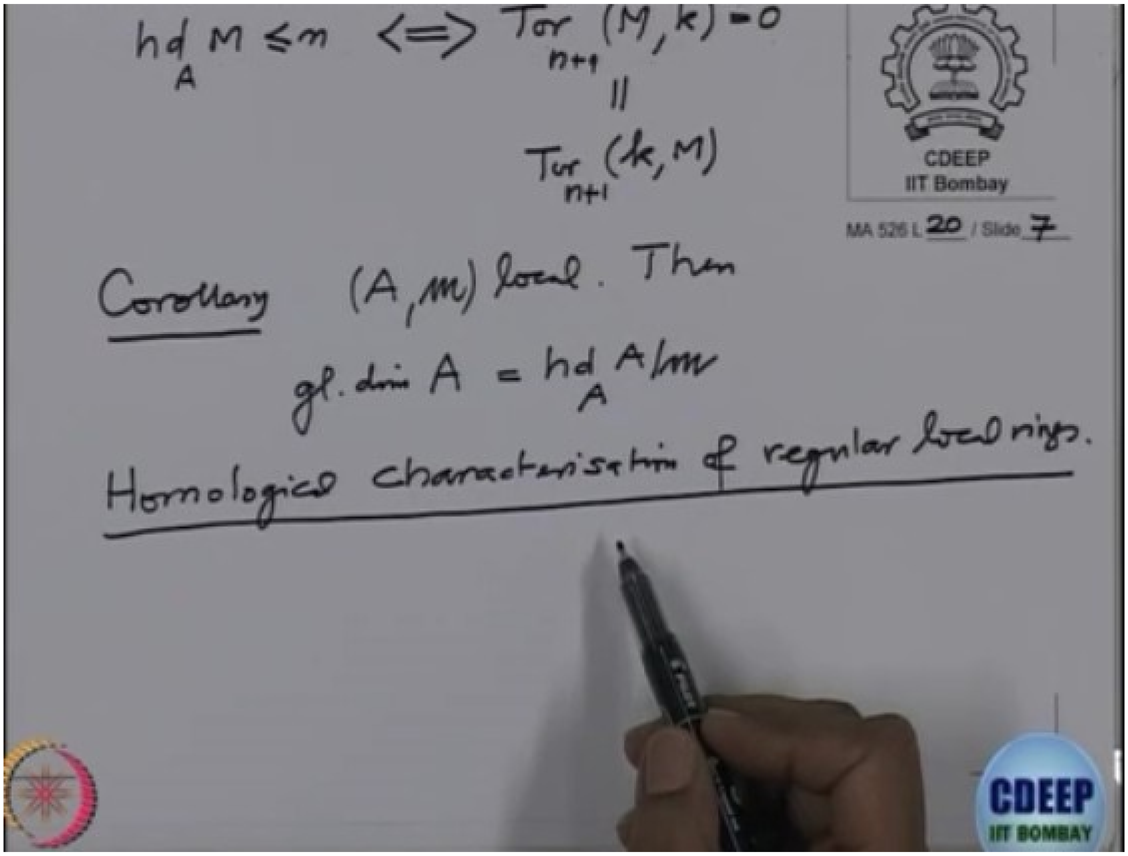

one of them is the associated gathered ring is a polynomial algebra that is one, this is very important characterization, because this is, this says that the tangent cone is affine space, and the second one was Jacobian categorian this was defined Jacobian matrix using the presentation of, and that we have proved it only for affine algebras, so we have taken affine algebra, then we have return a affine algebra as a quotient of a polynomial algebra and then in terms of those equations we defined a matrix by using that differentiation and then we found a upper bound for the rank of this matrix, and when the upper bound is attained that is precisely what I'm saying the ring is regular at that point, and that localization this was it.

Now this one is more intrinsic, but of course the cost we are paying is the homological cost I mean so this doesn't, so the earlier two depended on the presentation, and one has to check that the if you write your homological, if you write your affine algebra by different presentation, different maybe somebody takes a different polynomial algebra and then the equations will change and the matrix will change, so all this one has to check that they're equivalent, but if you do this way it is intrinsic. Okay so the theorem 1 wants to prove is (A, m) local, then A is regular, if and only if global dimension of A is finite, or this is if and only if homological dimension of the residue field as A module is finite, that means $\frac{A}{m}$ has a finite projective resolution, okay.

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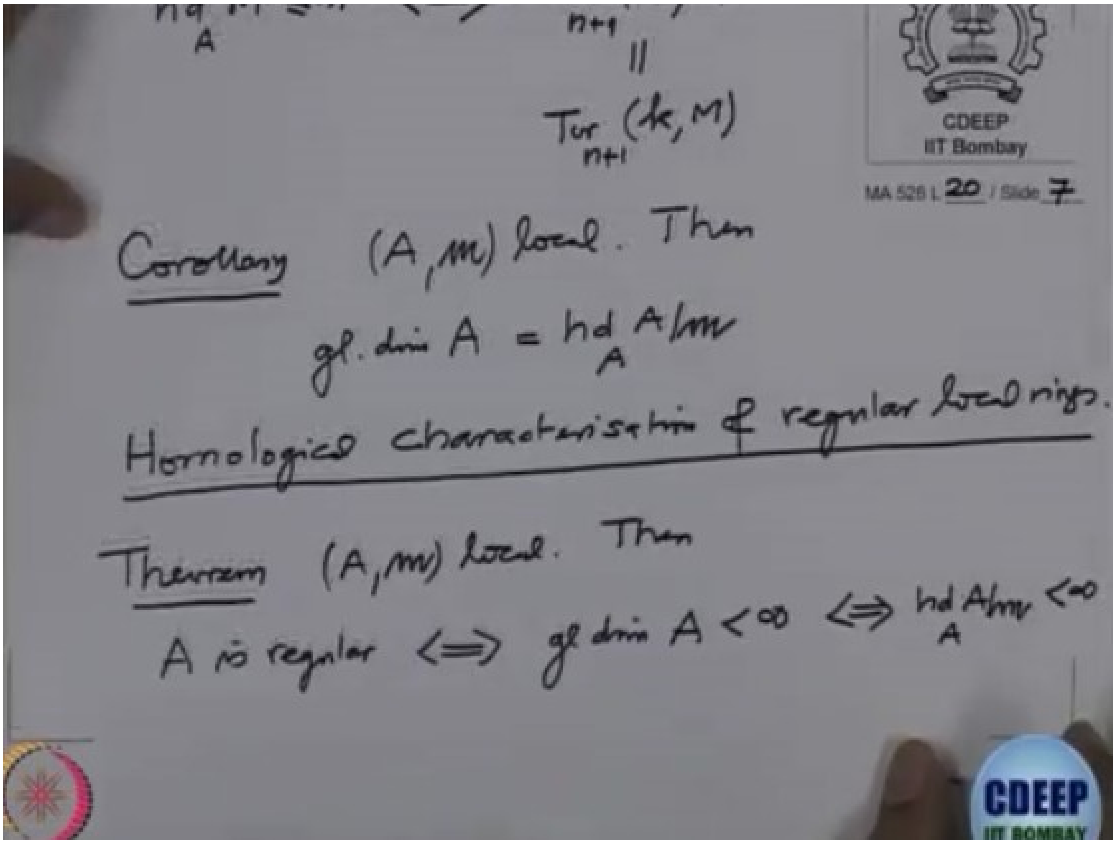

And after we prove this then we will worry about what is this then, whether it is a Krull dimension or not, okay. So everything we want to do for the residue field, projective resolution, homological dimension and everything, so I'm writing couple of lemmas for this, preparing this theorem, so lemma 1 if I have, so our local ring is fixed, so suppose I have an element A in $M \setminus M^2$, so this means, this I can extent when I go mod M square it is nonzero, nonzero element in a vector space $\frac{M}{M^2}$, $\overline{A}$ this is nonzero here, so I can extent it to a basis, and a basis when I pull it back it will be a minimal set of generators for the maximal ideal, minimal number, so that means any such A I can extend it to minimal set of generators for the maximal ideal.

See this is not usually true for, if your ring is not local, it is not true we cannot extend to a minimal set of generators in general, okay. So then and we have a natural exact sequence from here, see M, see this M will contain the ideal generated by A so we have this quotient and from here M mod M times A to here there is a natural map, see this ideal is bigger than this, so this is subjective map here surjective map here, and what is the kernel? Kernel is this principal ideal mod this ideal M tensile, so this sequence is exact,

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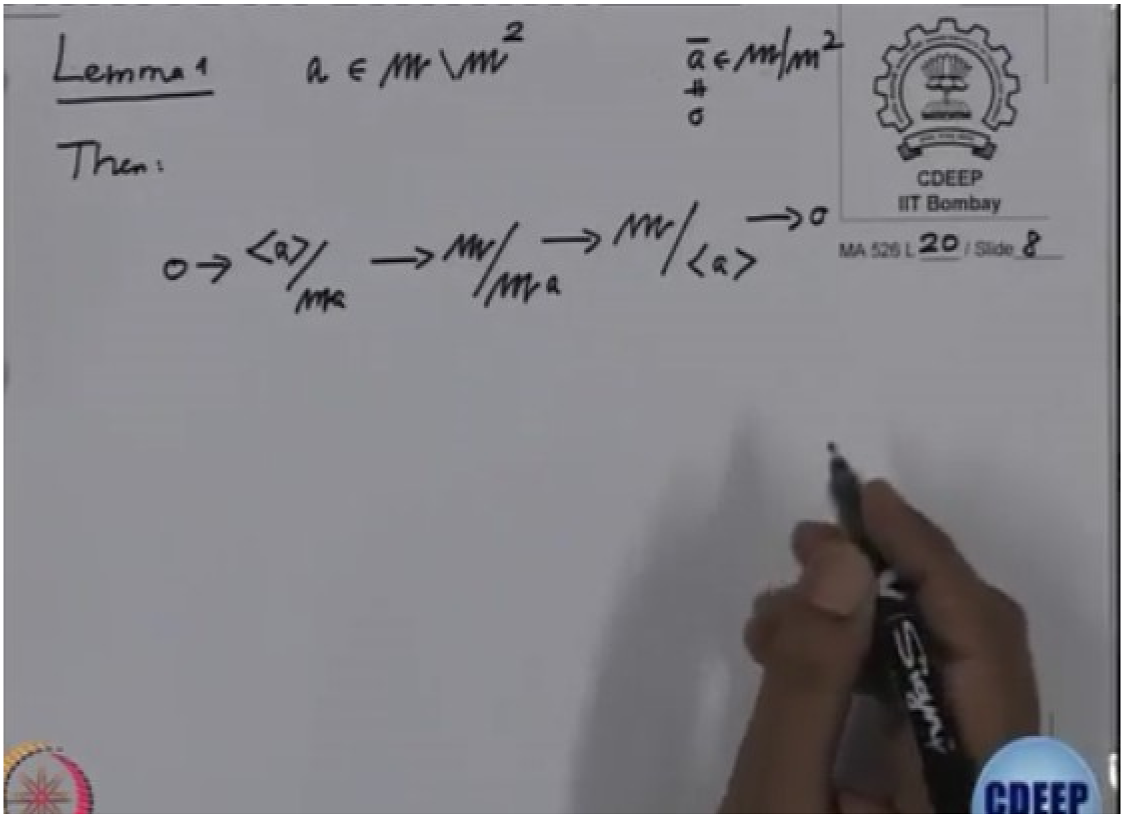

this is exact but I want to say more, this is a sequence of, the sequence, short exact sequence of A modulo a ideals, A modules, all this K's are $\frac{A}{a}$ modules, because all of them are related by A, alright.

So this one splits, that means there is a map here in this direction so that when I go like this I get identity on this, that is a meaning of its splits, so such a map is called a section of this, okay. So I want to prove that splitness, so proof so let us call this vector space $\frac{M}{M^2}$ the dimension of this vector space, let us call it D,

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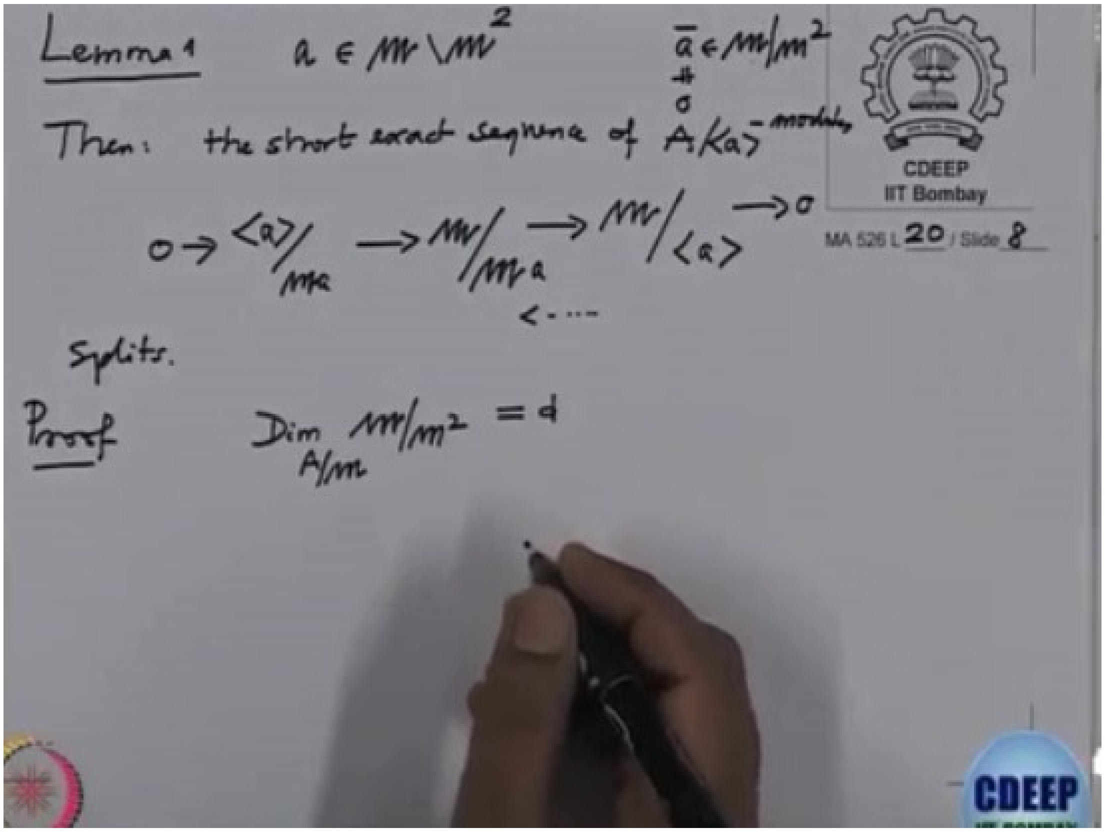

so and as I said a is not here means there exist a_1 which is $a, a_2, ... a_d \in M$ such that there images in $\overline{a_1}, \ldots, \overline{a_d}$ in $\frac{M}{M^2}$ is a basis, is $\frac{A}{M}$ basis of $\frac{M}{M^2}$.

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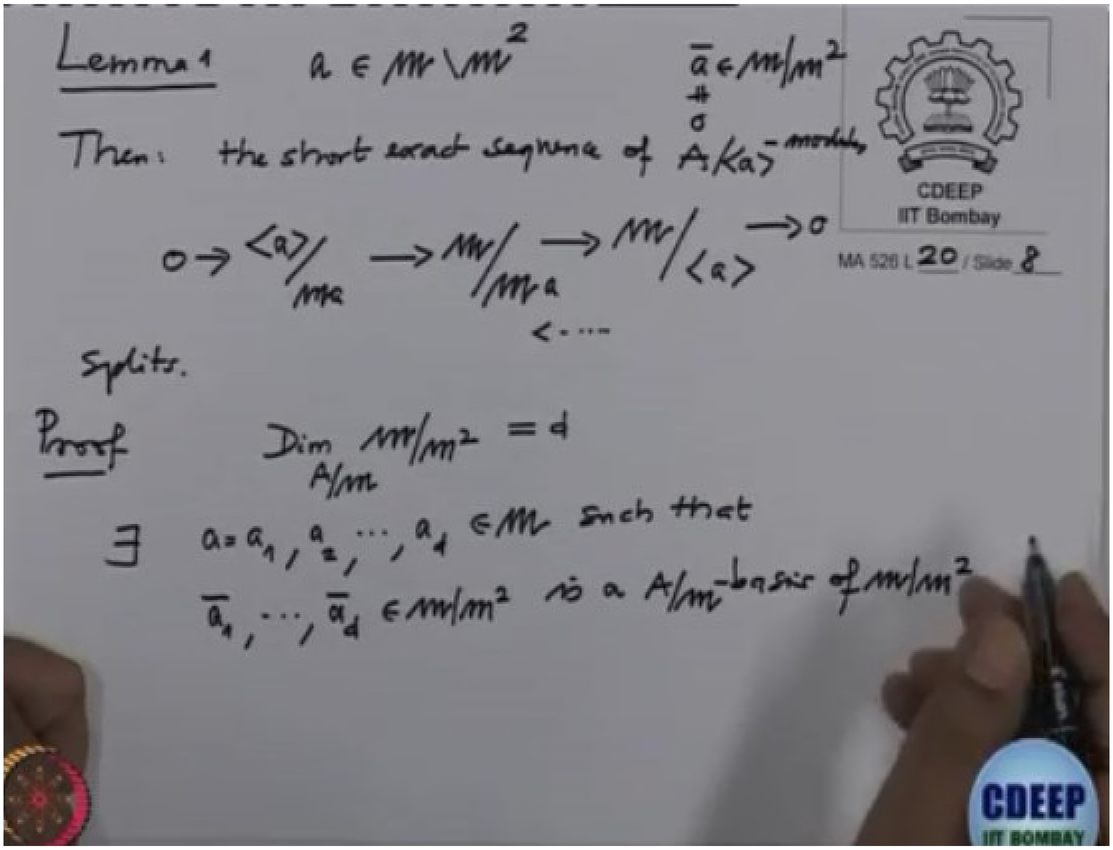

So let us take A to be ideal generated by all this guess, have to lead the first one, so $a_2, \ldots, a_d$, this is the ideal generated by, this obviously contain in M and M is generated by one more element, alright. What do we want to check? We want to define a map actually, we want to define, what do you want to check? We want to say that this is a splitting that means we want to define a map from this to this so that this composition is identity, okay.

So let us look at B in M such that B times A in the ideal ring,
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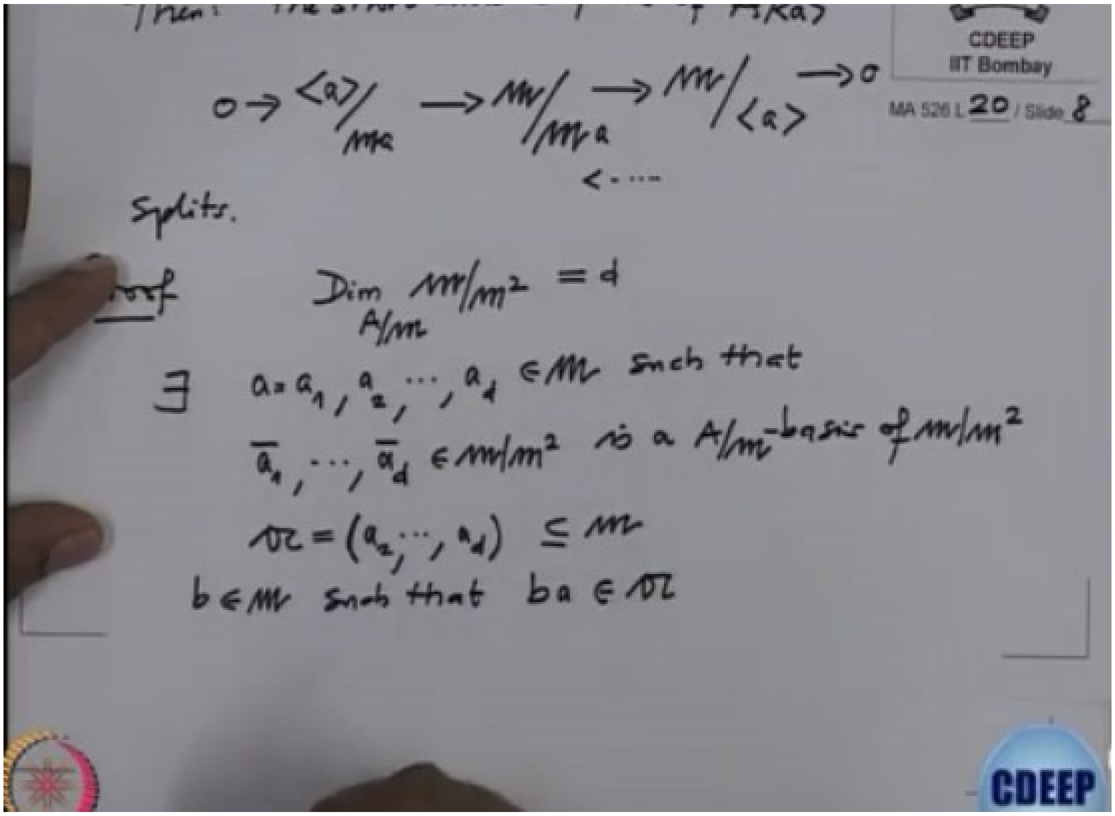

yes, so what is this? See M modulo ideal generated by A is like $a_2, \ldots, a_d$, because I have killed that A, and then I'm trying to define a map so I want to analyze the situation when B is in M, so that B times A is in A, then what happens?

Okay, first of all let us look at A intersection A times I, I say this is clearly contained in A intersection M times A, one inclusion is obvious A intersection M times A, this is clearly contained in A intersection A times A, this ideal generated by A, obviously this is smaller ideal, this is bigger ideal so this is clear.

So this one we have proved it because if you take B so that, okay, so if you take any element here if the multiple of A is in A, then B has to be in M, you take an n element here it will be a multiple of A that is call it BA, if it is in A that B has to be in M that's what we want to prove, but that is clear, because of the minimality, so this is clear, this is clear therefore what we got is they are equal, so A intersection A times A this is same as A intersection M times A.

So now you look at $\frac{A+ma}{ma}$ this is isomorphic by usual standard, isomorphism theorems $\frac{A}{A \cap ma}$, but this is same as this, so this is isomorphic to $\frac{A}{A \cap Aa}$, but again get back, this is isomorphic to $A + \frac{A}{a}$, but this is same as M times, $\frac{M}{A}$, so what we got this isomorphic M times A, so it shows that means I have a map from $\frac{M}{A}$ to, M mod, this is correct M mod

$\frac{M}{A}$, M mod $\frac{M}{A}$ to M mod $\frac{A}{a}$ I have a natural map, which maps any coset here, so let me, it's dangerous to write like that, so this one, this maps,

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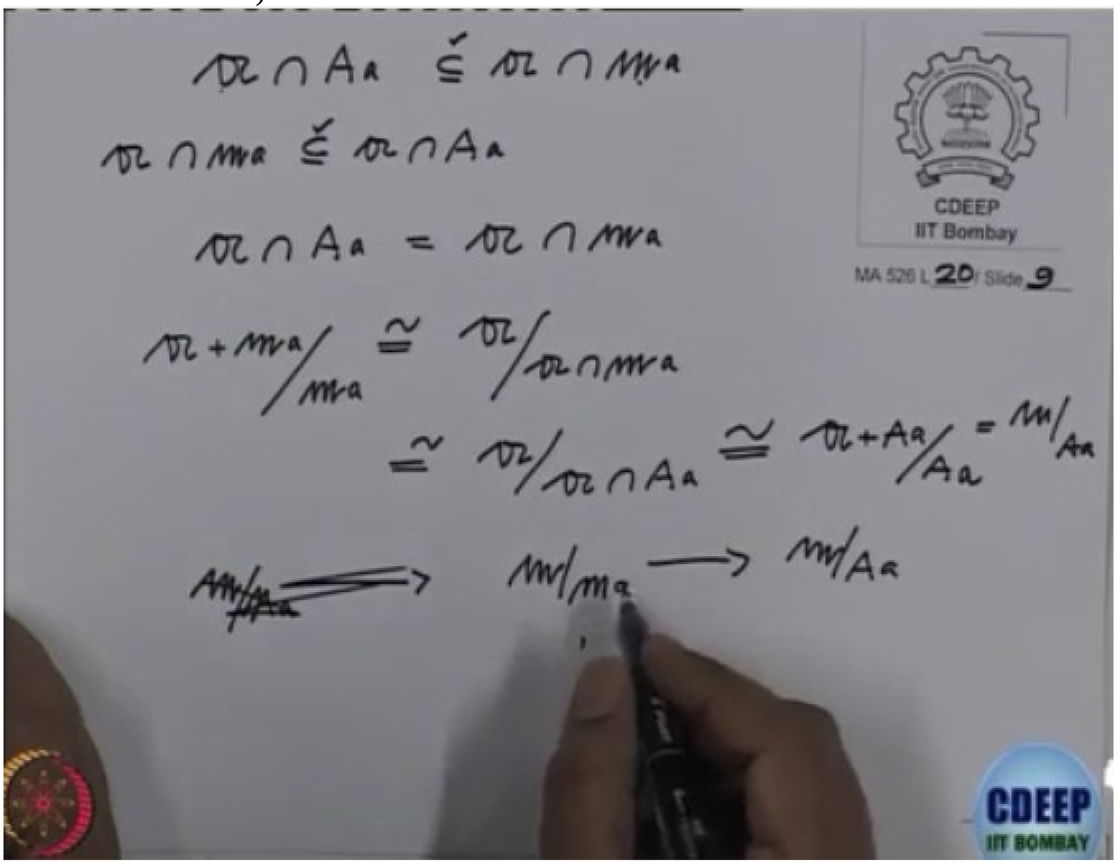

this one to isomorphically on to this, so here we have this $A+\frac{m}{a}$, mod $\frac{m}{A}$, this should map, we have checked that this one is isomorphic to this one, so this ones are isomorphic, this ones are isomorphic this is same, so I'll map this isomorphically on to this so that will give me the section, this is the section,

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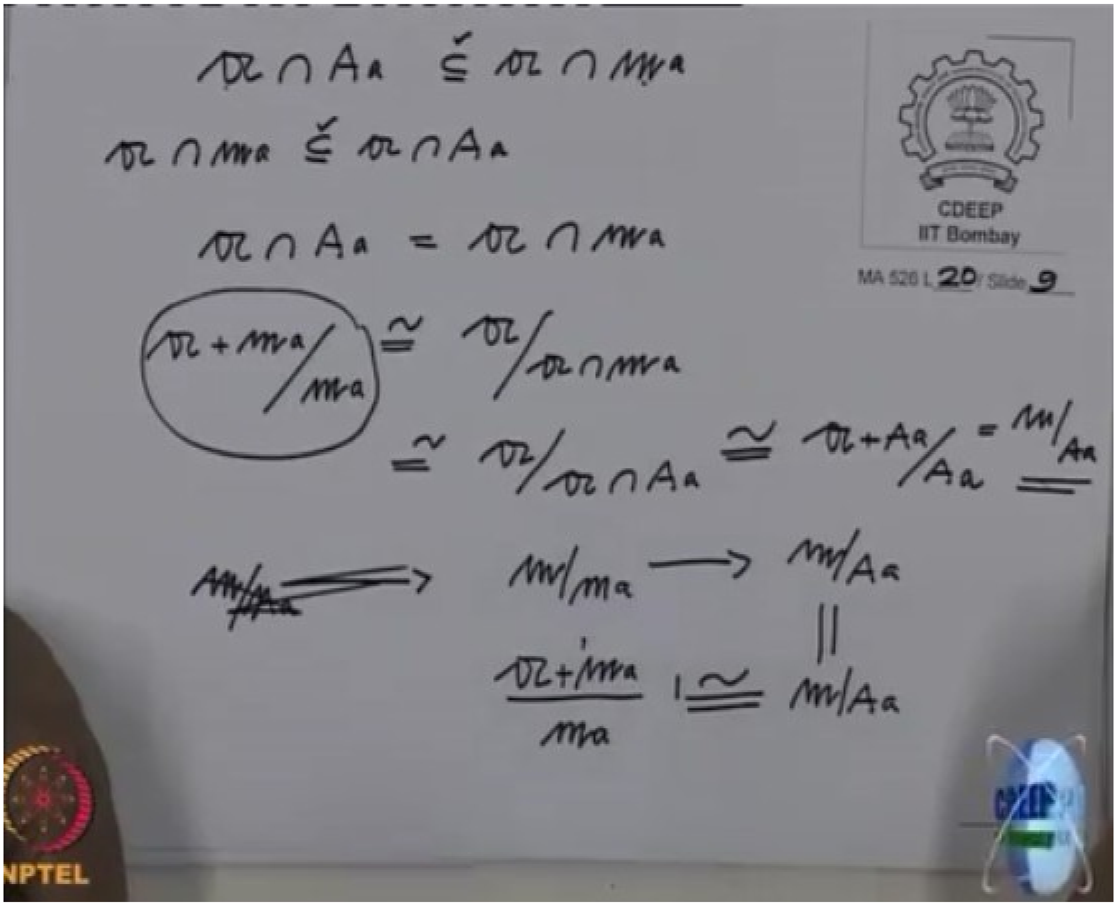

yeah I wanted a map from here to here, here to here, but this one is isomorphic to this, so I have defined this way, this isomorphism will define this way and further inclusion and that will give the split, okay.

So now still we have couple of minutes, let us write one corollary, corollary is again A noetherian local, and assume that the global dimension is finite, and let us take if A is an element in $M \setminus M^2$, and also assume that A is a nonzero divisor in A, then the global dimension modulo that small a is also finite.

So proof, note that this, the residue field of this $\frac{A}{aA}$ that is, this is also $\frac{A}{m}$, so the residue fields are same, and to check that this global dimension is finite I have to check that the homological dimension of the residue field as a module over $\frac{A}{a}$ is finite, okay,

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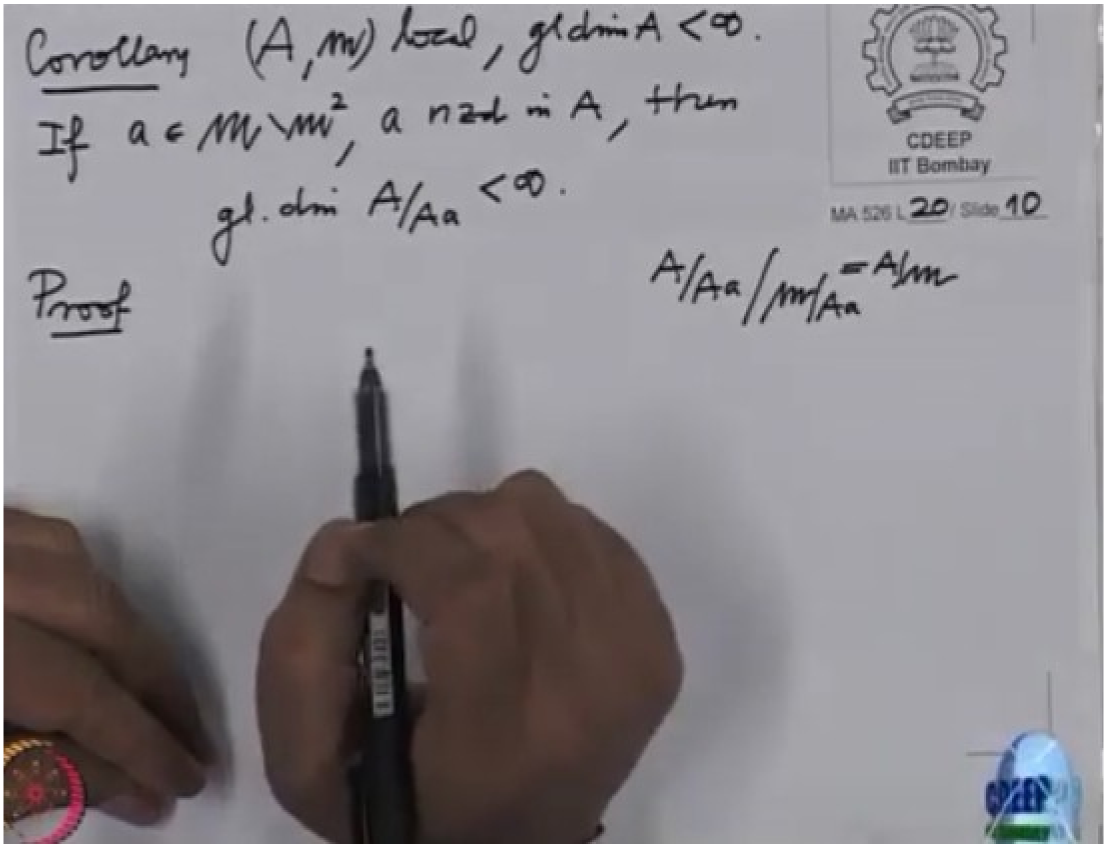

for that so to do that I have to take the sequence of $\frac{A}{a}$ modules and compute the tor or whatever, right.

So look at I have K here, the K is a residue field and then I have a elements here $\frac{A}{aA}$, this is the residue field of this, and what is the kernel? Kernel is $\frac{m}{aA}$, now this is exact sequence of $\frac{A}{a}$ modules, and I'm going to use this exact sequence to compute the homological dimension of this residue field, and if I check it is finite then we are done, alright. So but you remember now so global dimension of $\frac{A}{aA}$, this is homological dimension as a module over $\frac{A}{aA}$ is of the residue field, so I have to compute this, but we have proved earlier that is, okay so this one, so it's enough to prove that the homological dimension of this is finite, because this is a short exact sequence and we've seen earlier, how homological dimensions are related to this, I think this is one more than this, so you know to prove that, to prove that homological dimension of $\frac{m}{aA}$ as $\frac{A}{aA}$ module is finite,

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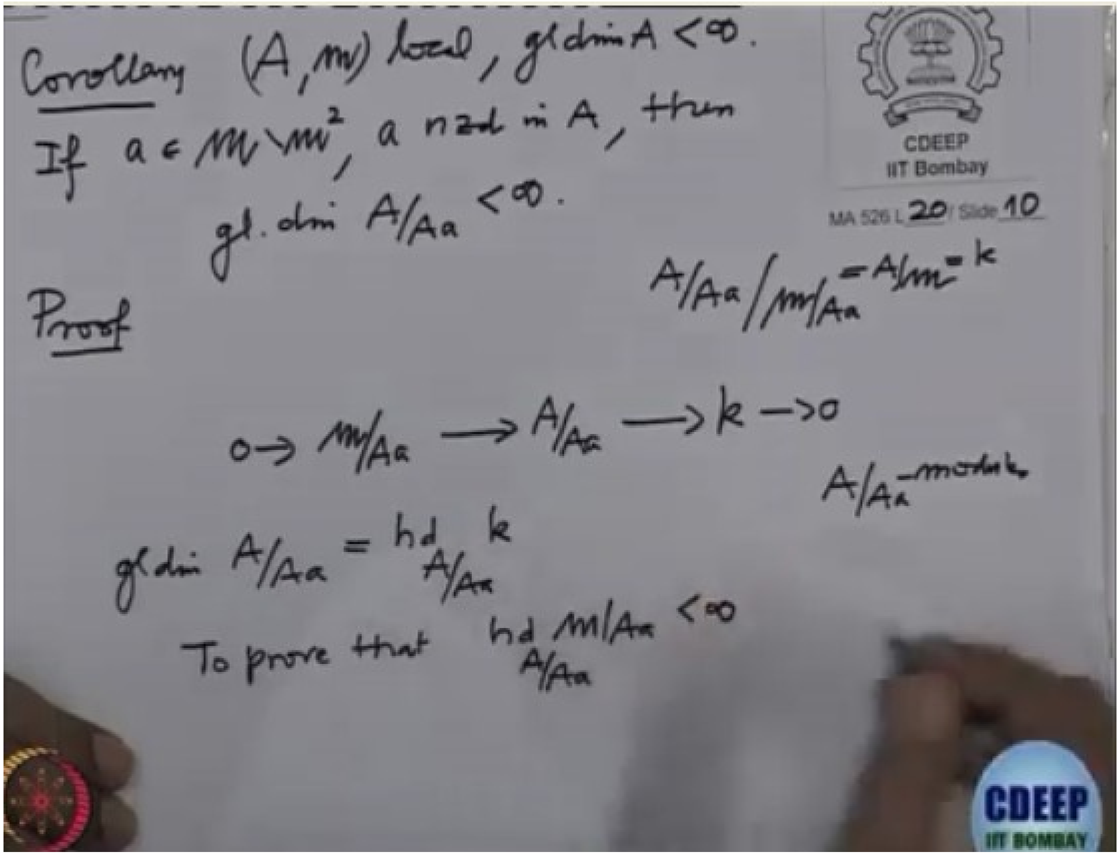

but we know now, our assumption is global dimension of K is finite, so that means homological dimension of, as A-module K is finite, this is precisely the global dimension of A, and if this is finite that will also mean that homological dimension of M is finite as A module, because again we use the short exact sequence 0, M, A to K to 0, so the homological dimension of this, this is finite, therefore this will also be finite, so this is finite.

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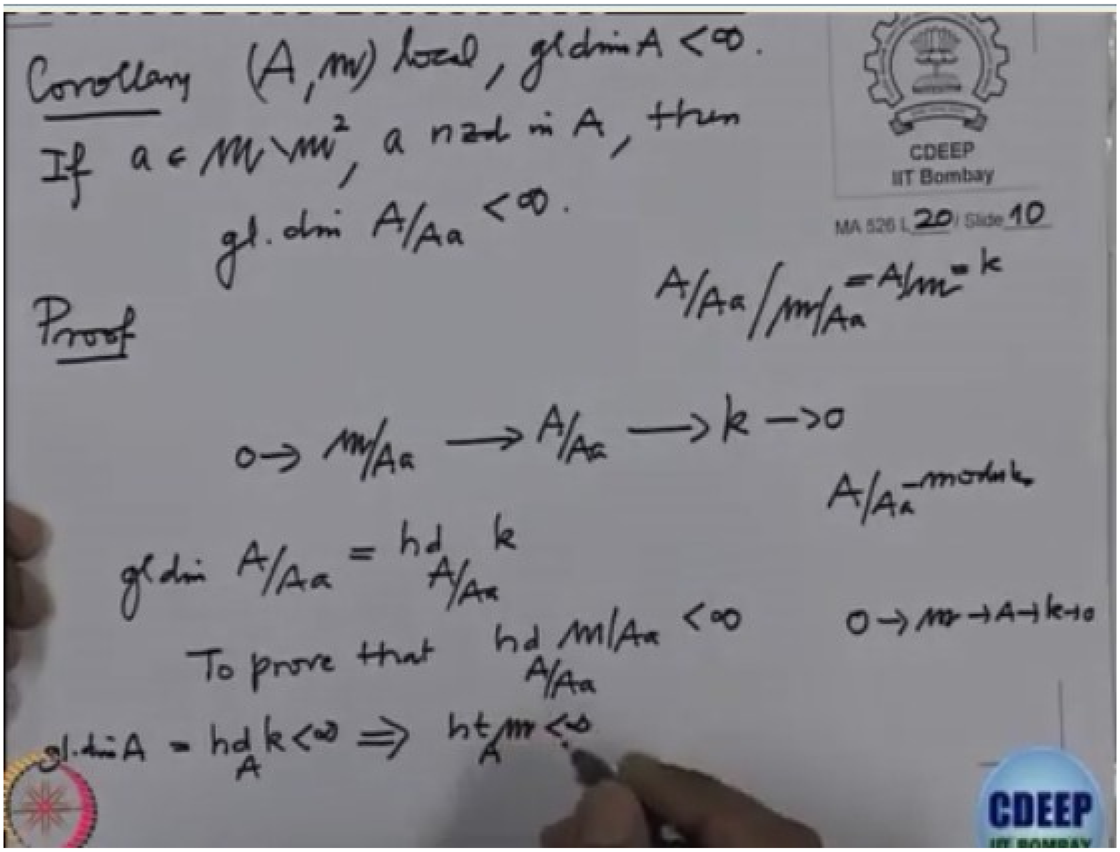

But now remember this A is a nonzero divisor, so I can take a multiplication map on M and then we have a short exact sequence so M, A this is injective, and the quotient is $\frac{m}{ma}$ to 0, so this is exact sequence, so therefore to, now this is A,

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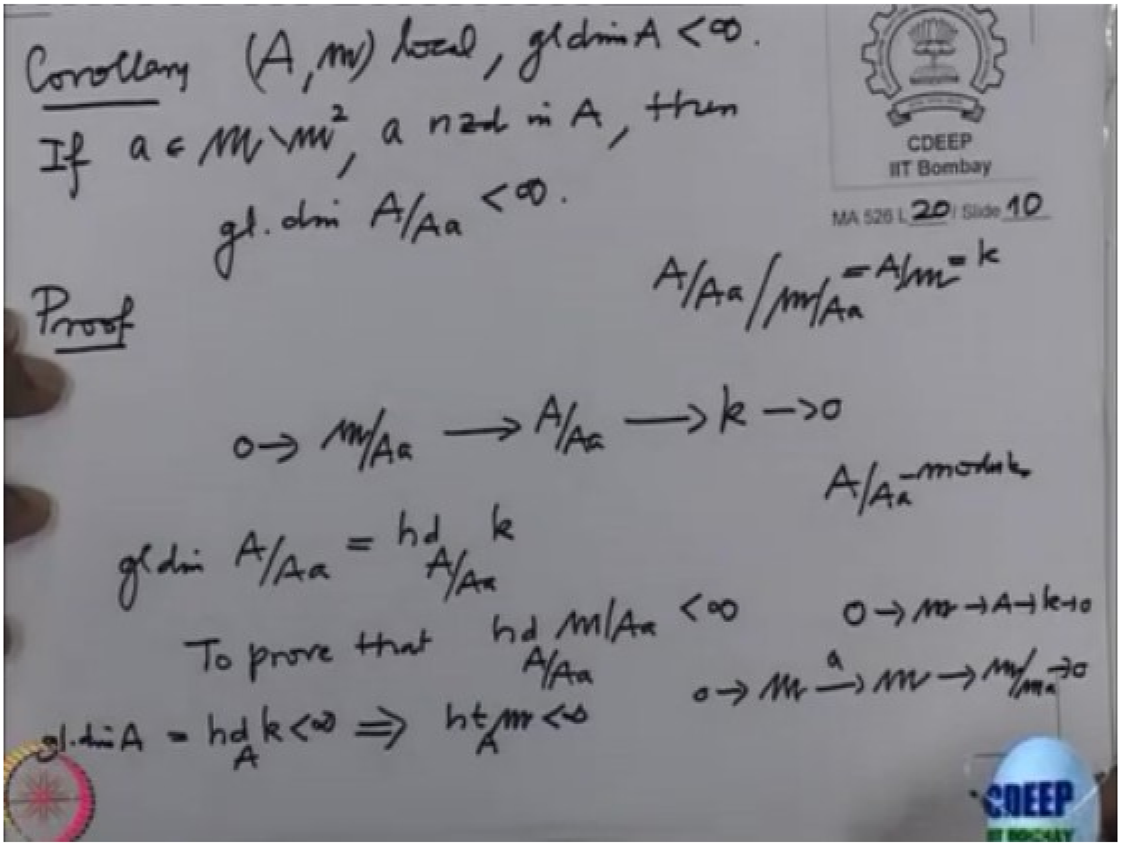

this was ma is correct, image now so this ma is correct.

And above lemma says, above lemma says that this is a direct summand of, the splitting means, splitting means we have proved the above lemma that says that $\frac{m}{aA}$ is a direct summand of $\frac{m}{ma}$, and this one has a homological dimension finite, therefore this one will have homological dimension finite, so direct summand of a, this is homological dimension finite, so direct summand will also finite, so this will mean the direct homological dimension of this, because this is finite by this exact sequence, this exact sequential, this is finite, this is finite, therefore this homological dimension will be finite, but this one,

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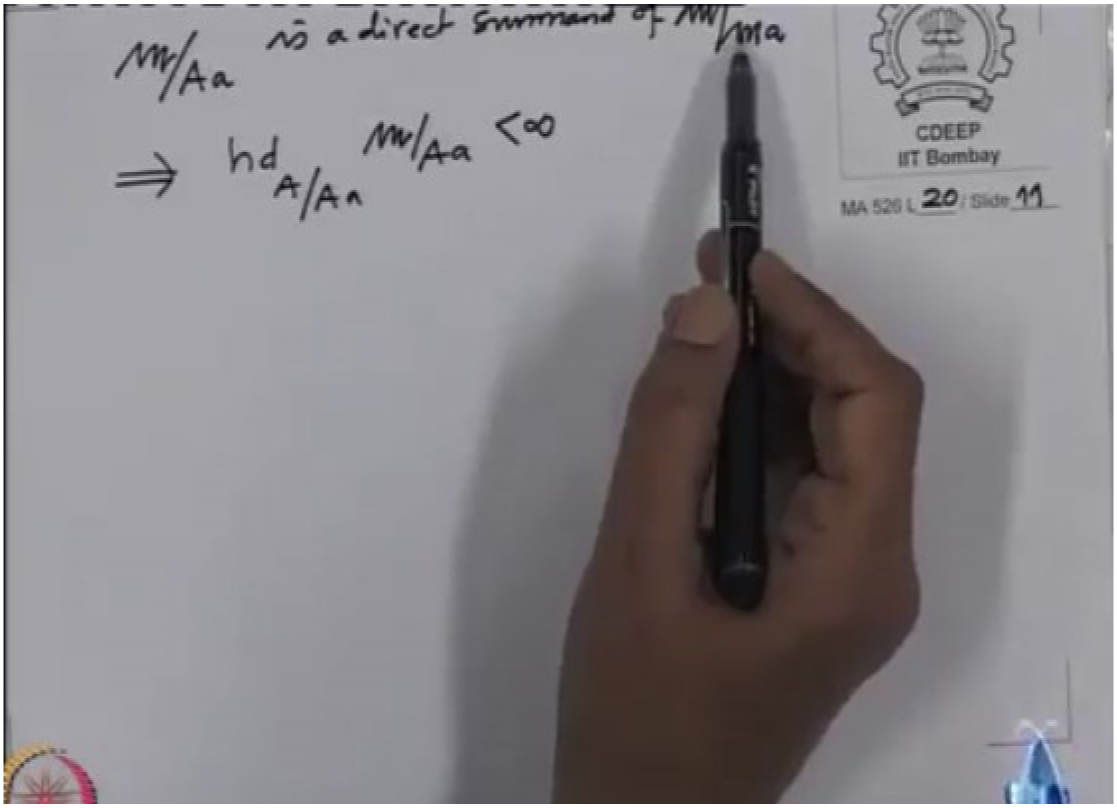

this one is a direct summand of this one by the earlier lemma, therefore this one will also finite homological dimension and that finishes the proof.

So and we will continue this, we have not yet finished the, we just started the homological characterization, so that I think we should be able to finish next time, and then we will spend some time on the fact that the regular local ring the real disk.

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