INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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CDEEP IIT BOMBAY

COMMUTATIVE ALGEBRA:

PROF. DILIP P. PATEL
DEPARTMENT OF MATHEMATICS,
IISc Bangalore

Lecture No. – 05
Associated Prime Ideals of a Module

In this lecture we are going to discuss some basic results about associated prime ideals and support of a module, associated prime ideals and support of a module. These results are basic and most of the time I will not give complete details, however I will give enough hints and enough details so that one can fill up the remaining details, alright.

So first of all our base ring is commutative ring A, this is a commutative ring and V is an A module, we call that A module is abelian group and which has a scalar multiplication by A so that the scalar multiplication and abelian group structure they are compatible with each other, this is equivalent to saying that we have a ring homomorphism theta from the ring A to the endomorphisms of the abelian group (V,+), so that A going to theta A, if theta A is the scalar multiplication on V which maps any a to ax, and this is precisely the A module structure on B.

So for example the kernel of this map, kernel of θ is precisely all those elements $a \in A$ such that this θ_a is a 0 homomorphism that is, so this is same as all those a in A such that a times any x is 0 for all x in V, this precisely what we call it annihilator of the module V, this is the definition of the annihilator of V,
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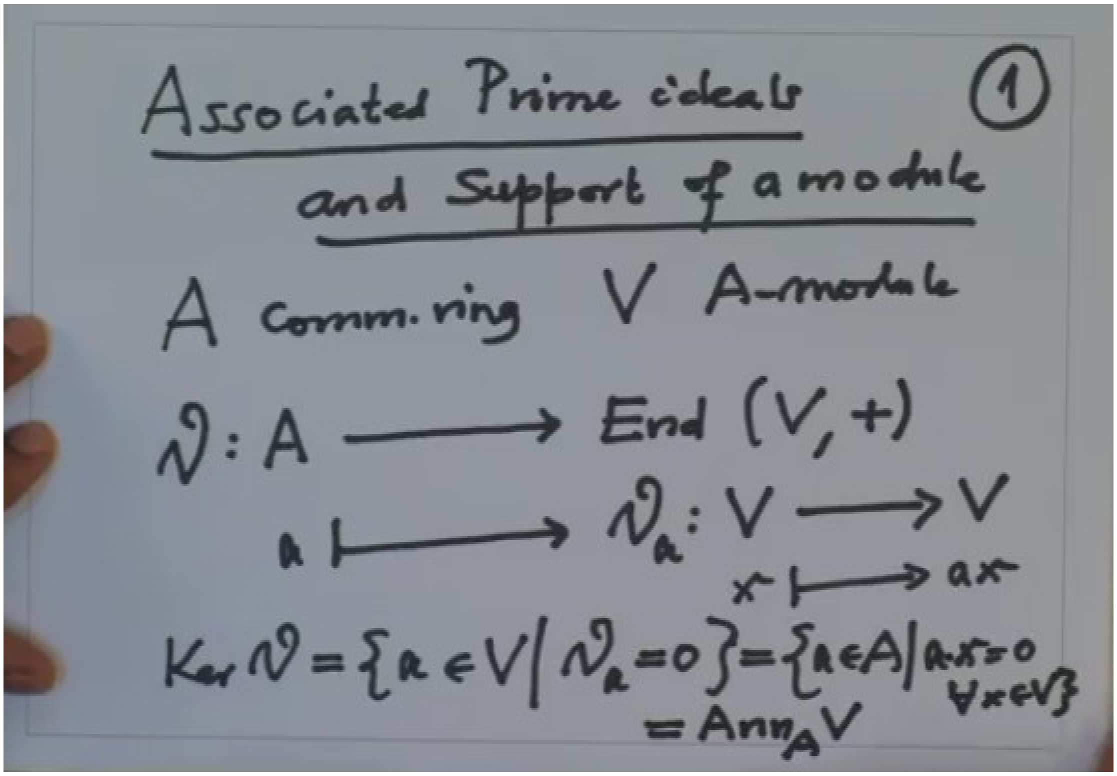

so first of all this is clearly so annihilator of V is an ideal in V, ideal in A and also if particularly when you take fixed element X in V then the element which, elements of the ring A which annihilate X means all those A image such that AX is 0, these are also ideal, so these are ideals in A.

And among them now we are looking for the prime ideals, so this is the definition, so definition associated prime ideals of V, associated prime ideals of V as A module these are precisely prime ideals in A, prime ideals in A that is E is in spec A such that P is annihilator of some element, for some X in V, for some X in V then you call that prime ideal to be the associated and set of all these prime ideals we call associated prime ideals of V, and now this set is very important, so this set,

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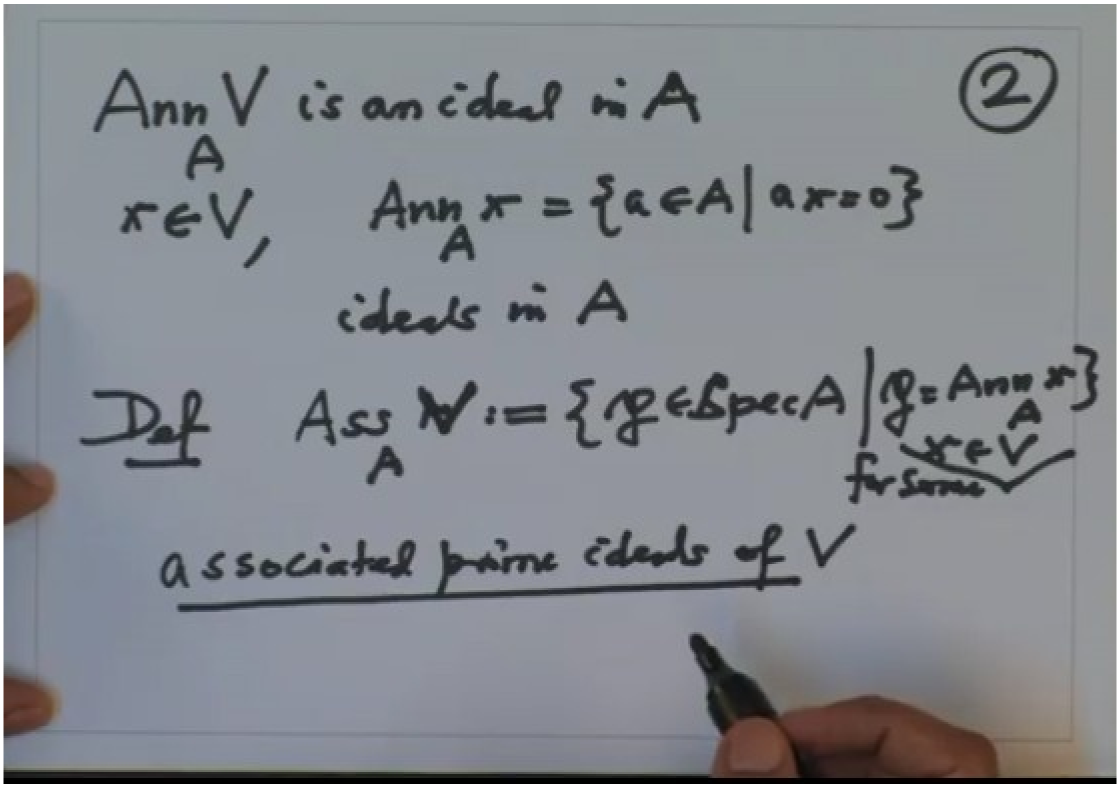

for example it is finite or not and how to find elements etcetera, this is very important for, because in the one of the lecture we have studied the Zariski topology on the spectrum and this particularly for V = A will give some information more finer information about this topological space, spec A, for example it will tell what are the irreducible components of this topological space, so for that this set of associated prime ideals is very important to study.

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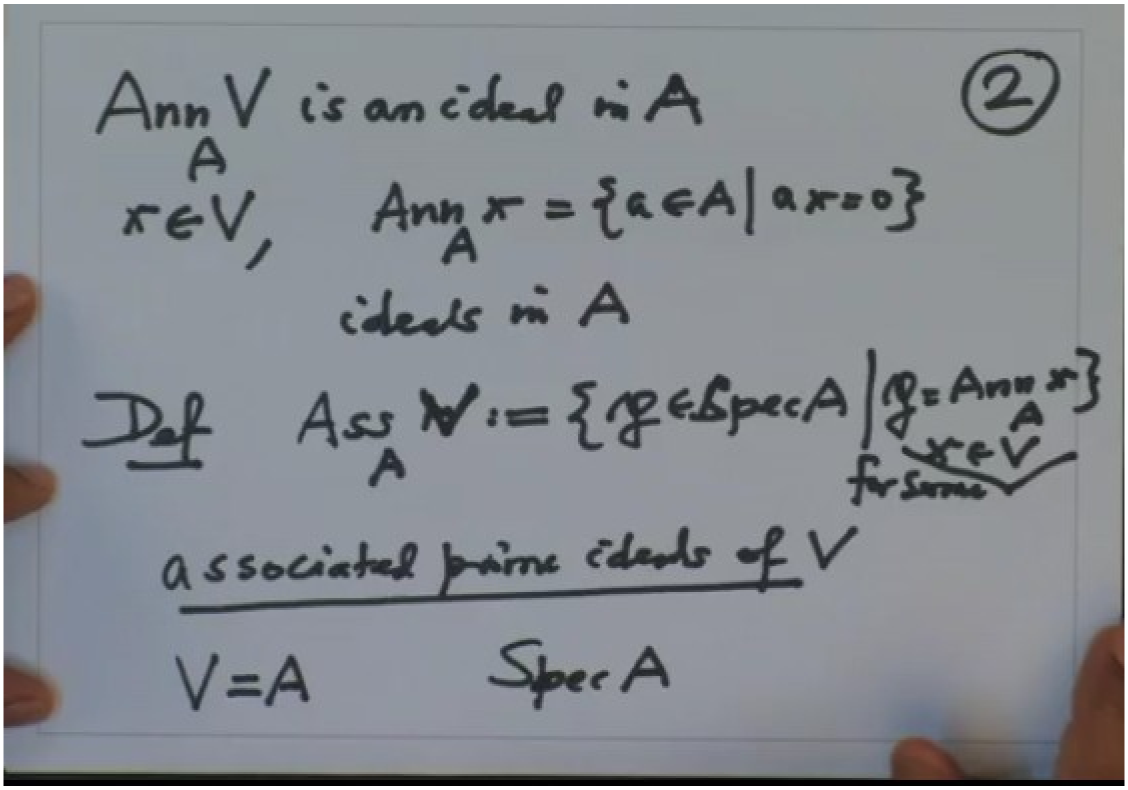

So first of all note that, so note that V is 0 if it only if the associated primes of V this is an empty set, so there is no prime ideal in that because V is 0 therefore all the annihilators will be the whole ring and so on, so there is no prime ideal in that case, alright.

So another important result I want to write it as, so one of the important consequence of this is, so let me write it as a corollary, if A is Noetherian is very important, assume that A is Noetherian that means every ideal in A is finitely generated and V is any A module, then the set of zero divisors, the set of zero divisors for V which is we denote by Z(V), this is by definition all those elements $a \in A$ such that their exists an element $x \in V$ such that with $ax = 0$, and x is nonzero, so these are called zero divisors of the module.

Assertion is the zero divisors, this set is precisely equal to union P, where P varies in the associated prime ideals of V. This is very easy to prove.

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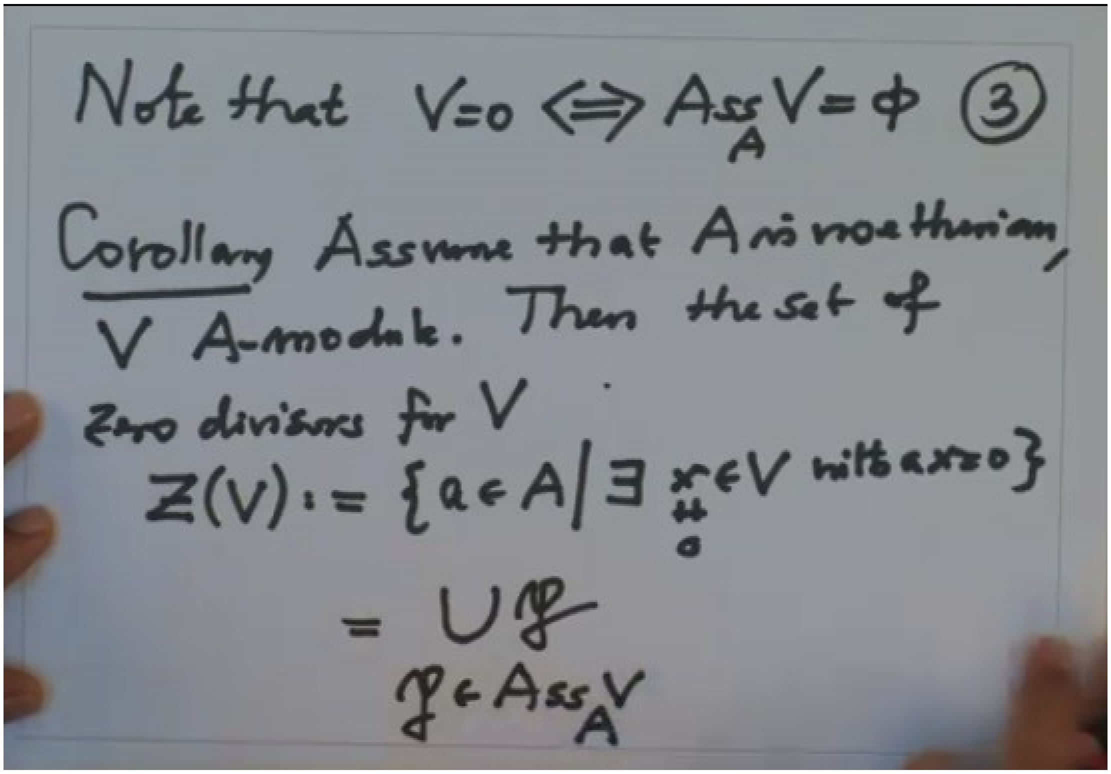

For example, so proof suppose if P is an associated prime of V, then by definition P is annihilator of x for some $x \in V$, some $x \in A$, so because P is a prime ideal, P is by definition, P is not equal to A therefore this x has to be nonzero, and therefore all elements of P, so it is clear that clearly P is then contained in zero divisors of it, because every element of P which is, when you multiply by this nonzero x it is zero therefore it is, they are zero devisors, so therefore P is contained in the zero divisors.

Conversely, if some element a belongs to the zero divisors of V then a times x is 0 for some $x \in V$, x nonzero therefore annihilator of x, this is nonzero, this is an ideal in A and this ideal is not equal to the whole ring, so that is a non-unit ideal, so if I look at the family of annihilators, now if I look at all annihilators $Ann_A y$ such that y is in V, y nonzero, and this annihilator contains the annihilator x,

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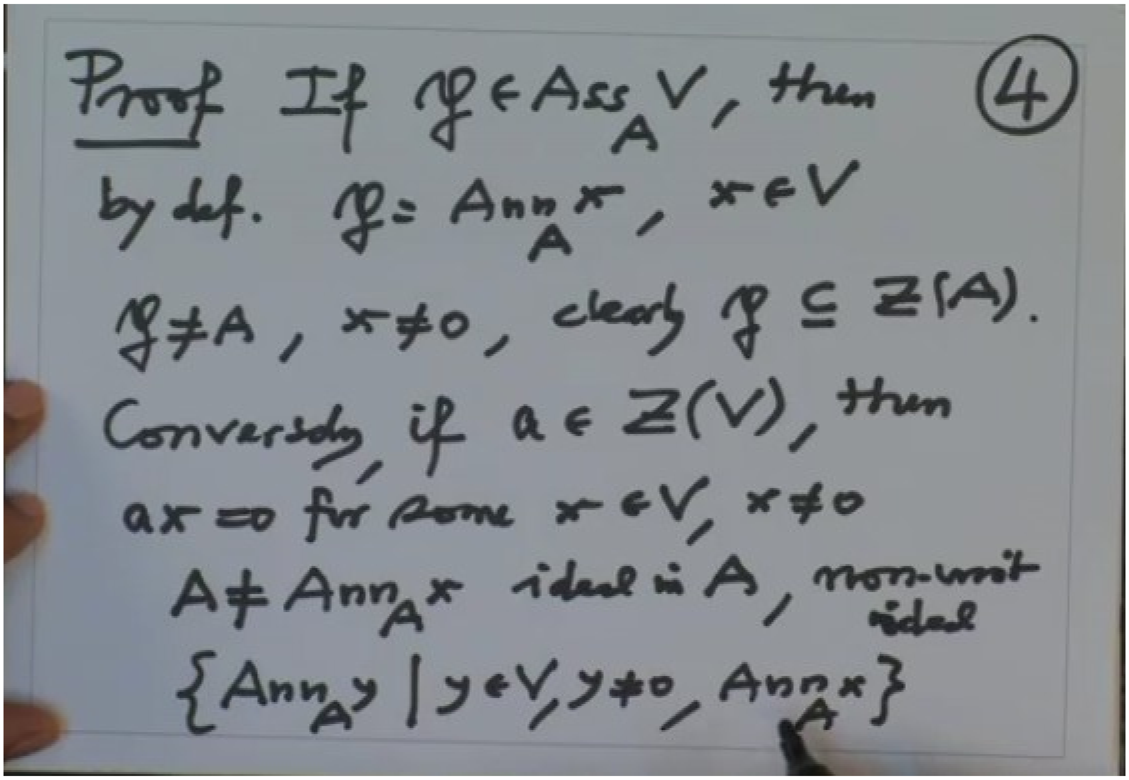

this family of ideals will have a maximal element and that maximal element is a prime ideal and that prime ideal will be therefore associated prime so, then that implies there exists a prime ideal P of the form annihilator of y, where y is in V, y nonzero and this by definition over choice this contains annihilator of x, so therefore this P is therefore by definition an associated prime of V in A and the given A belong here, so A belongs to this, so altogether we have proved that the set of zero divisors of the module V is contained in the union of P in associated primes of V, alright.

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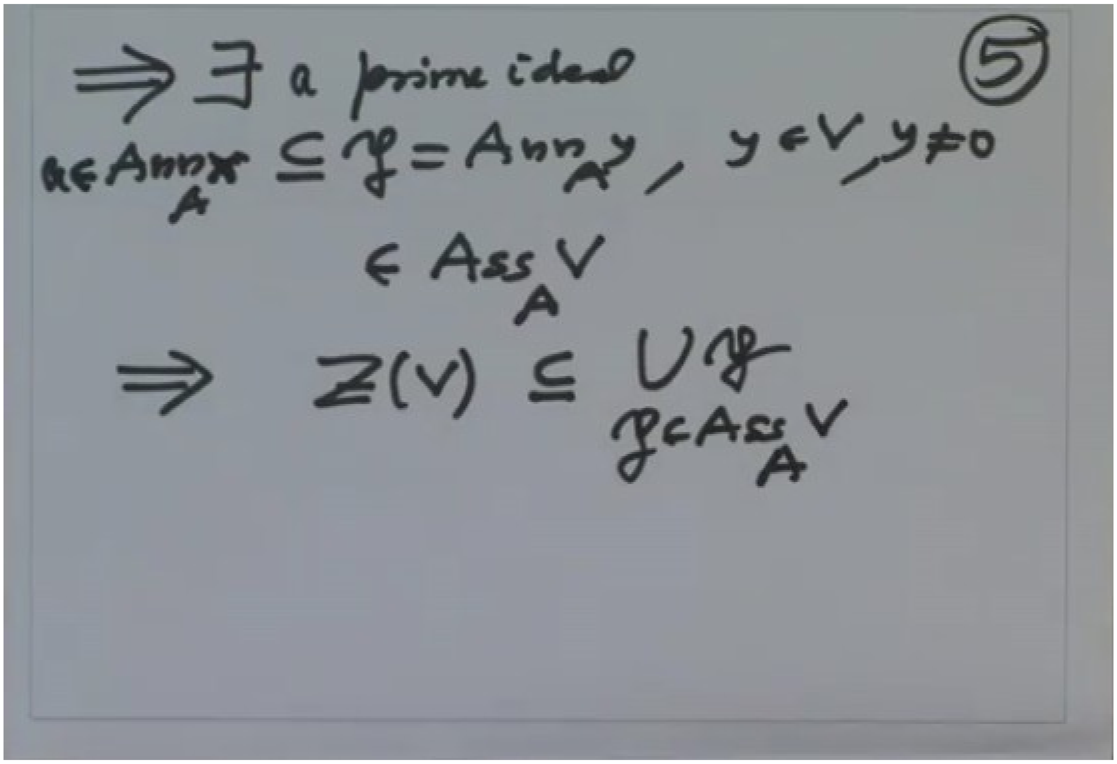

So now the other most important result I want to prove, this is, remember we have used, we have used here the ring is Noetherian because we have used that some family of ideals has a maximal element and that maximal element is a prime ideal that we know from the, that is very easy to check that such a maximal element has to be a prime ideal in that ring, so that was it.

Now another important corollary, all this results which I'm doing in today's lectures are used without any explicit reference in the other lectures, so therefore I'm only giving you a sketch of this results mostly lot of proofs I'll leave it, so let A be a Noetherian ring, and V be a finite A module, finite A module means it is finitely generated as a module that means there is a finite system of generators for V as A module, then there exists a chain 0 which is equal to V_0 contained in V_1 etcetera, etcetera, contained in V_{n-1} contained in V_n which is V, there exists a chain of sub modules of A sub-modules of V such that the successive quotients such that $\frac{V_i}{V_{i-1}}$ this is isomorphic to $\frac{A}{P_i}$ where P_i 's are prime ideals in A.

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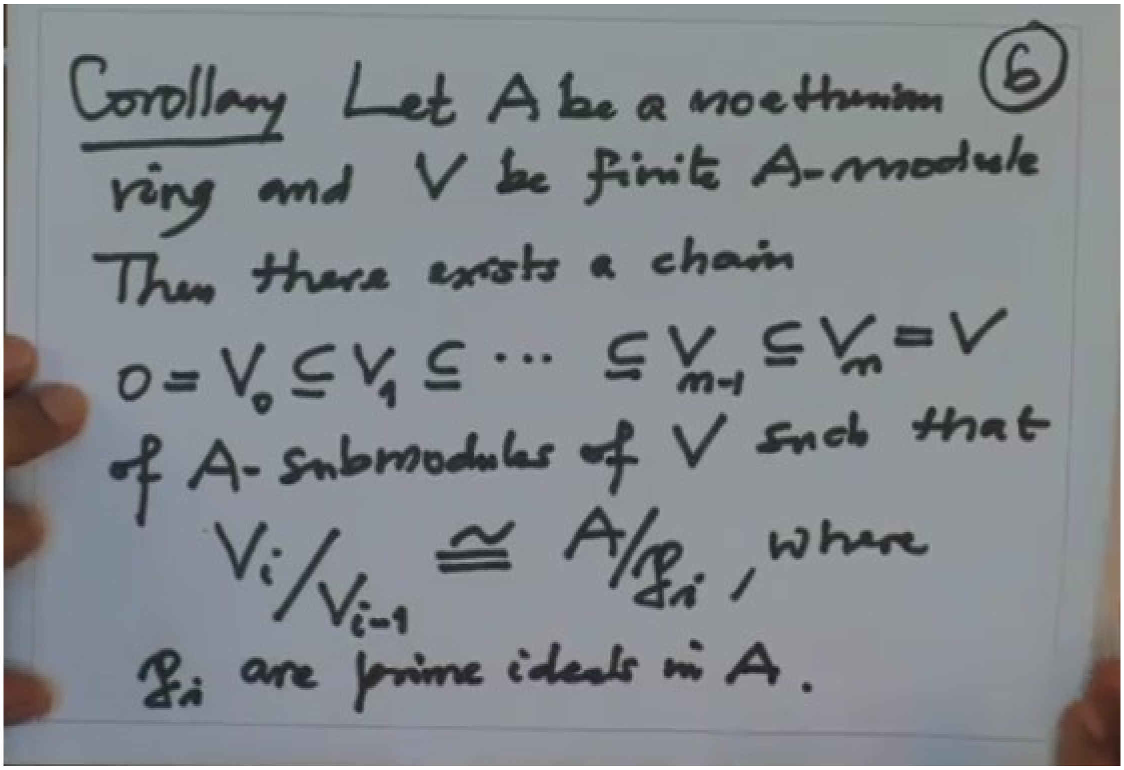

So proof, alright so we may assume V is nonzero otherwise there is nothing to proof, V is nonzero so we know because V is nonzero by just we have remark that there is definitely an associated prime ideal of V, this is a nonempty set so choose, so let P_1 be an element in associated prime ideals of V, V_1 is a prime ideal and this P_1 by definition P_1 is annihilator of some element, some element x where x is an element in V and because prime ideals are proper ideals, this x cannot be 0 element so x is nonzero and this is so, now therefore put V_1 equal to the sub module generated by this x, Aa and this sum module is obviously isomorphic to $\frac{A}{P_1}$, because you can take a map, you have A here, you can take a map here, this map is any A going to Ax and kernel of this map is precisely P, because P is annihilator of this x, so this map will factor through this, and this obviously want to, and because I've mod to kernel this map is injective therefore this is indeed an isomorphism, so we have an isomorphism like that, so we have V_0 here and we found V_1 , and $\frac{V_1}{V_0}$ is Ax which is isomorphic to $\frac{A}{P_n}$ so our chain started.

Now we have V here, there all contained in V, if $V_1 = V$ we stopped there,

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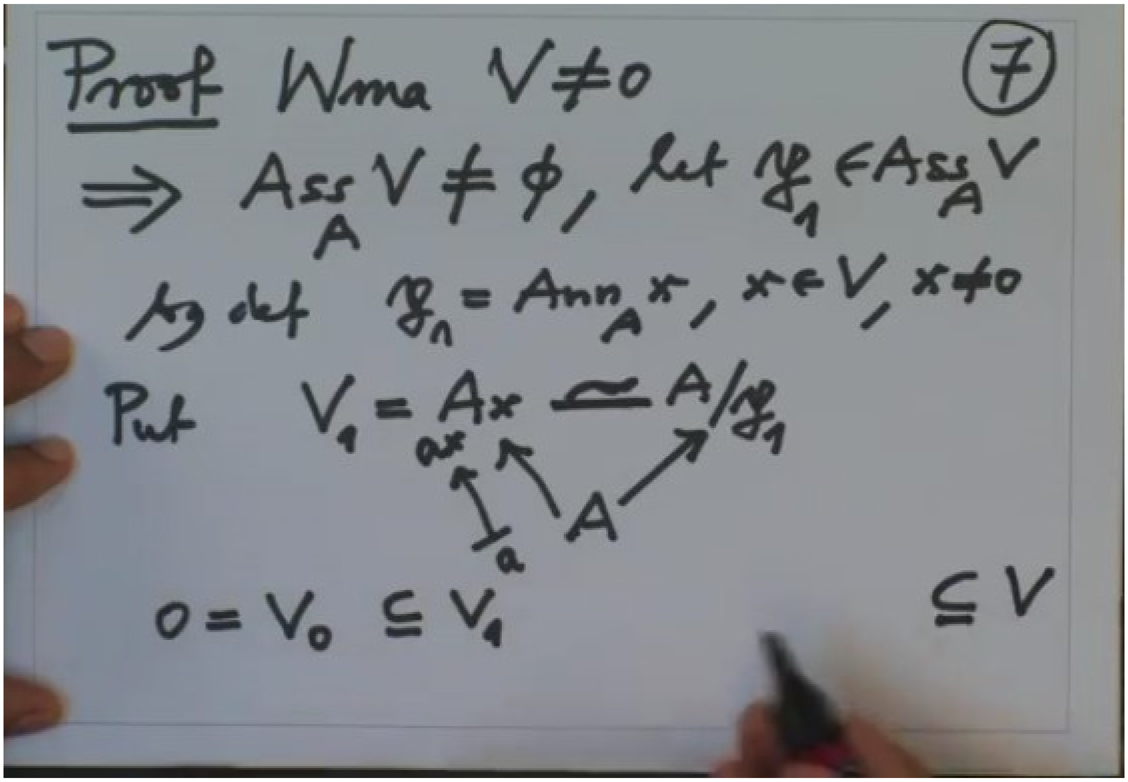

and so if V_1 is not equal to V then we should be able to continue in the same way, so now look at suppose V_1 is not equal to V then we look at the quotient module $\frac{V}{V_1}$ this is nonzero and therefore we can choose associated prime ideals and therefore we can find a sub module here by the first, early step so we have $0=V_0$ contained in V_1 and then we will find a sub module here which is of the form $\frac{V_2}{V_1}$ so that this quotient is, quotient of this, this is prime ideal, associated prime ideal so this is V_2 and continue like that, so ultimately we will reach a stage where $V_n=V$.

So this quotient we found $\frac{V_2}{V_1}$ in inside this, so that these quotient is isomorphic to $\frac{A}{P_2}$, so it is quotient in $\frac{A}{P_1}$, this quotient is $\frac{A}{P_2}$ and we can go on this and this chain has to stop at V because V is finite module, finite A module and A is Noetherian therefore V is Noetherian module, and Noetherian module means every ascending chain of sub modules should become stationary, so therefore every ascending chain of sub modules is stationary, so therefore we will finish at V and that proves this corollary, alright.

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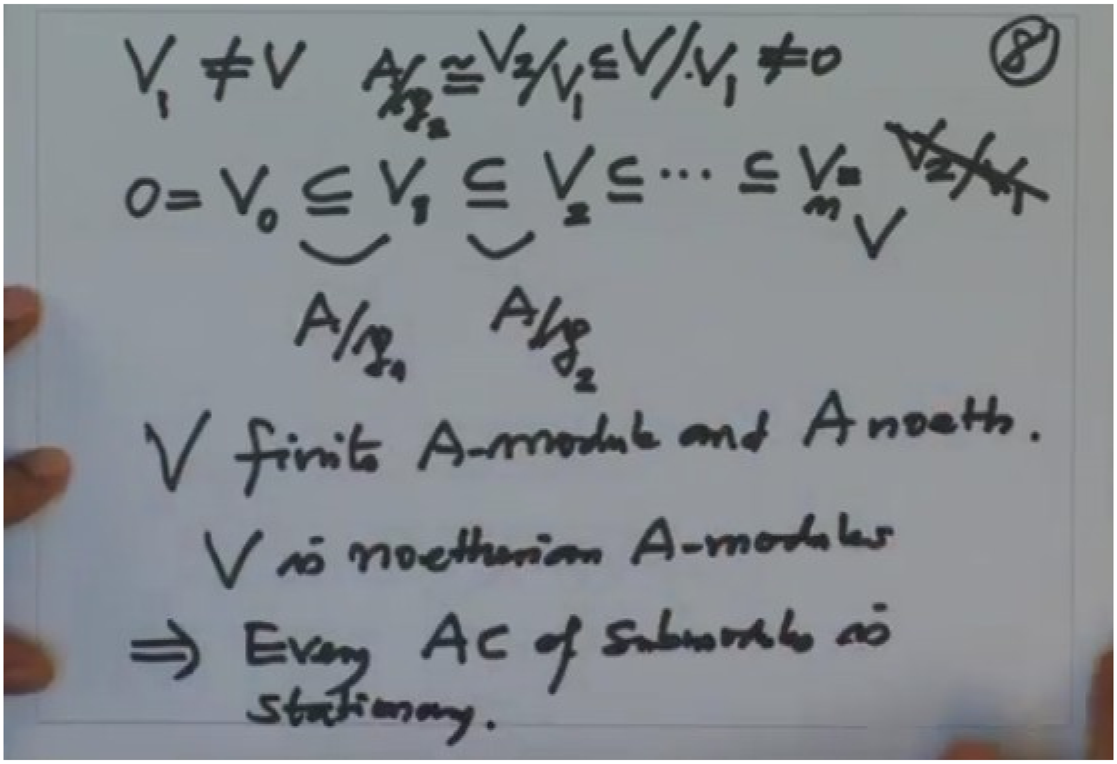

So let us see one example to illustrate this, so example so I want to take the ring to be $\mathbb{Z}$, this is ring of integers and the module is also $\mathbb{Z}$ and so now what is the series so that, so successive quotients are prime ideals and they are coming from the associated prime, so for example we have this chain 0 contained in $6\mathbb{Z}$, contained in $3\mathbb{Z}$, contained in $\mathbb{Z}$, this quotient is, this is also series, this is also chain of sub modules, but here note that what are the successive quotients? Here it is $\mathbb{Z}_3$, here it is $\mathbb{Z}_2$, and here it is $\mathbb{Z}_6$, so it's not, the 6 is not prime ideal generated by this, so therefore we cannot find this, this chain is not the required chain but however we can consider 0, this is V_0 and contained in V which is $\mathbb{Z}$, now this quotient is $\mathbb{Z}$ and that serves our purpose, so that the only associated prime ideal of $\mathbb{Z}$ is 0.

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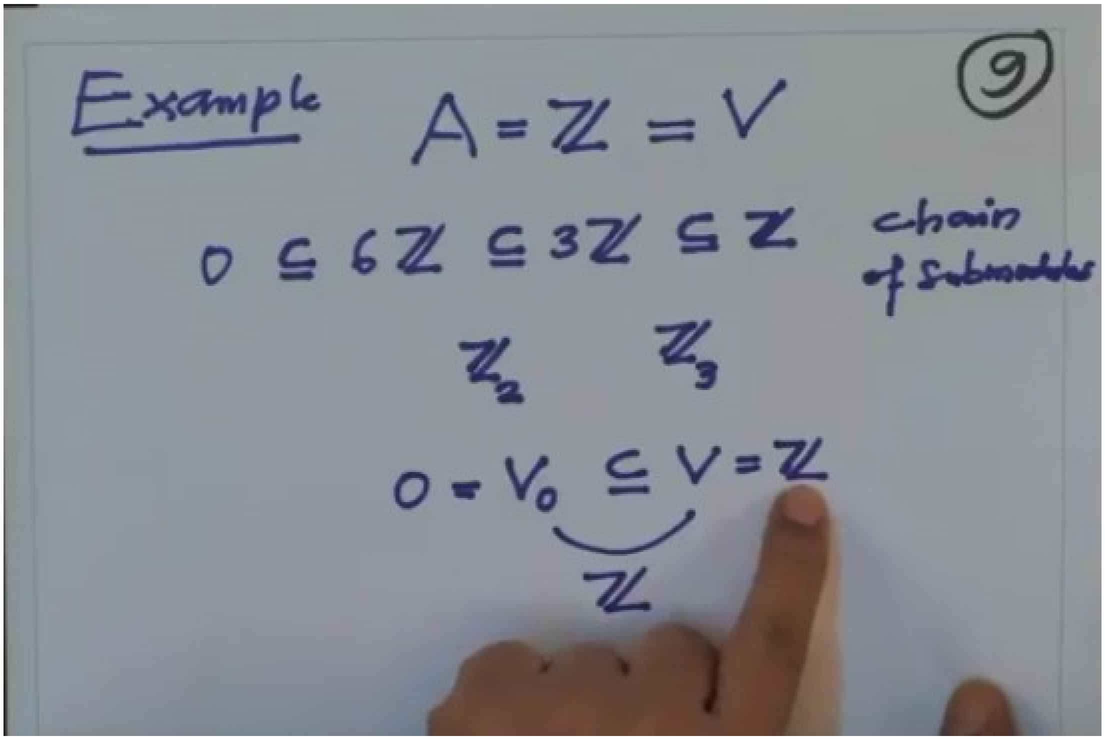

So here note that associated prime ideals of ℤ as if ℤ module, this is only one element namely the zero prime ideal, this is zero divisors in ℤ, there is no zero divisor in ℤ other than 0, so 0 this singleton 0 which is obviously everything, so this is not the required chain, but this is a required chain, alright.

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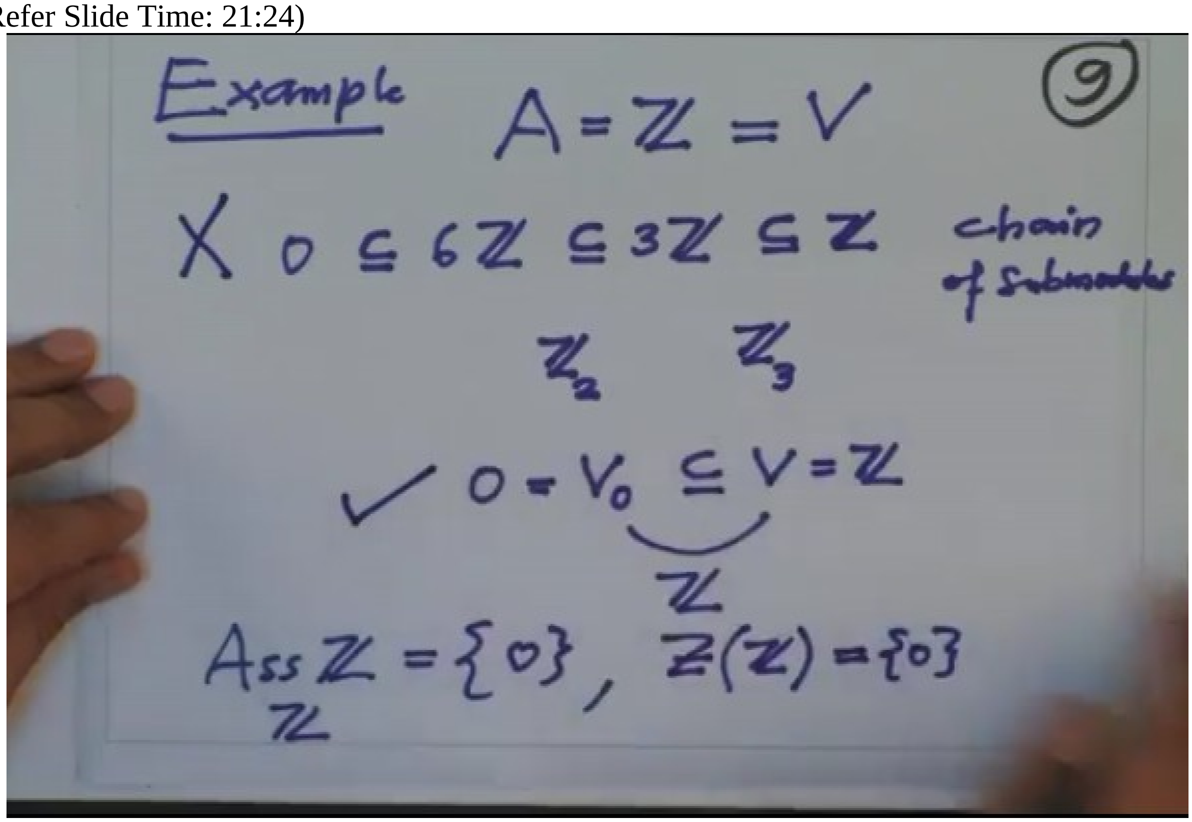
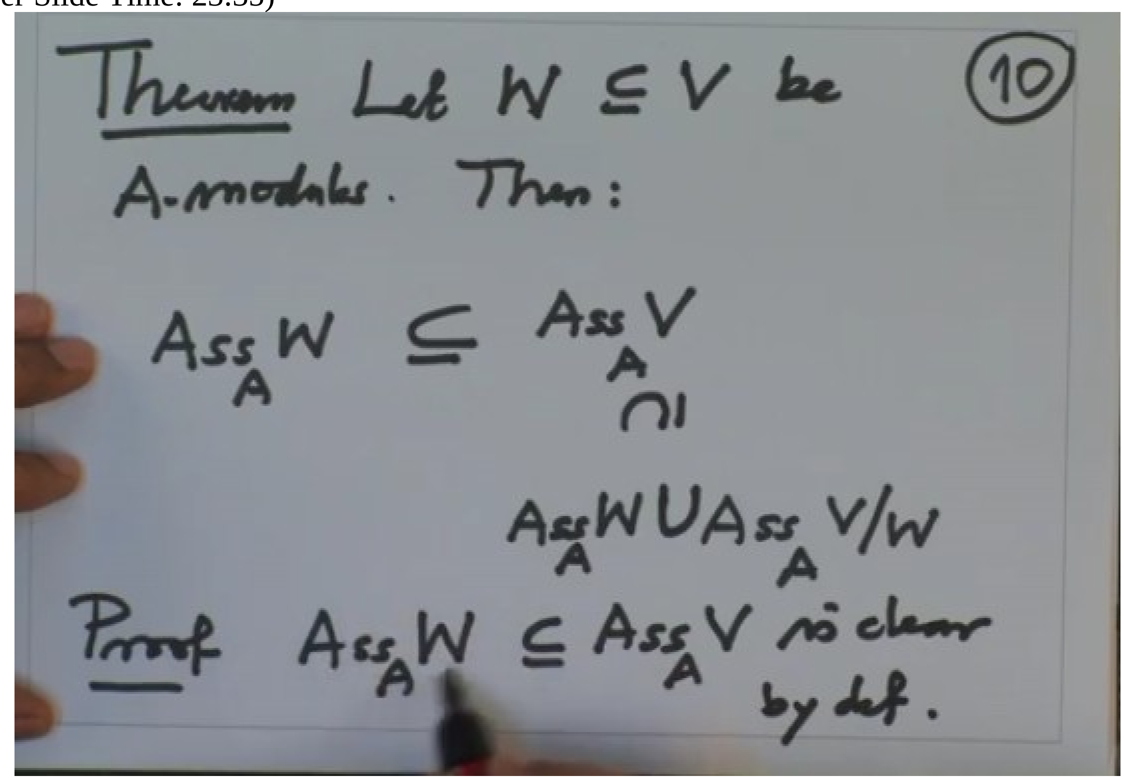


So now for induction, so we want to prove what is called primarity composition, so for that I will need couple of other observations which I will list now, so for example this is very useful observation which I want to call it as a theorem which gives a relation between the associated prime ideals of the module, sub module and the quotient module, so let W contained in V be A modules, so W is the sub module of the A module V, then we want to prove that what is the relation between the associated prime ideals of W? Associated prime ideals of V and associated prime ideals of $\frac{V}{W}$, these three things what is the relation? This is some set of prime ideals which are the annihilators of the elements from W, these are prime ideals which annihilators of the elements from V, and these are prime ideals which are annihilators from the elements of the residue class module.

So the assertion is this is contained in this, and this one is contained in the union of this with this, union, so we want to prove these two inclusions, so proof, by definitions this inclusion is very clear, it's clear by definitions, by definitions of, because this means what? This means the prime ideal if you take an element here it's a prime ideal which is annihilator of some element in W, (Refer Slide Time: 23:35)



but the same element in W is also an element in V so therefore it is annihilator of some element in V, so this one is very clear.

So the next one let P belonging to associated prime ideals of V, that means P is annihilator of some element $x \in V$ and x is nonzero element, that's a definition of the associated prime ideals, alright. So now we have a map A to Ax and this one is contained in V, this is a sub mod we generated by x, and this map is A going to Ax and we saw this kernel of this map, kernel is precisely equal to the annihilator of x which is P, therefore $\frac{A}{P}$ is isomorphic to Ax, and this is a sub modulo of V, so therefore I take a sub module of, so W, no E let us call it E, E is a sub module of V which is isomorphic to $\frac{A}{P}$, that simply means I'm taking the image of $\frac{A}{P}$ inside V, so look at E.

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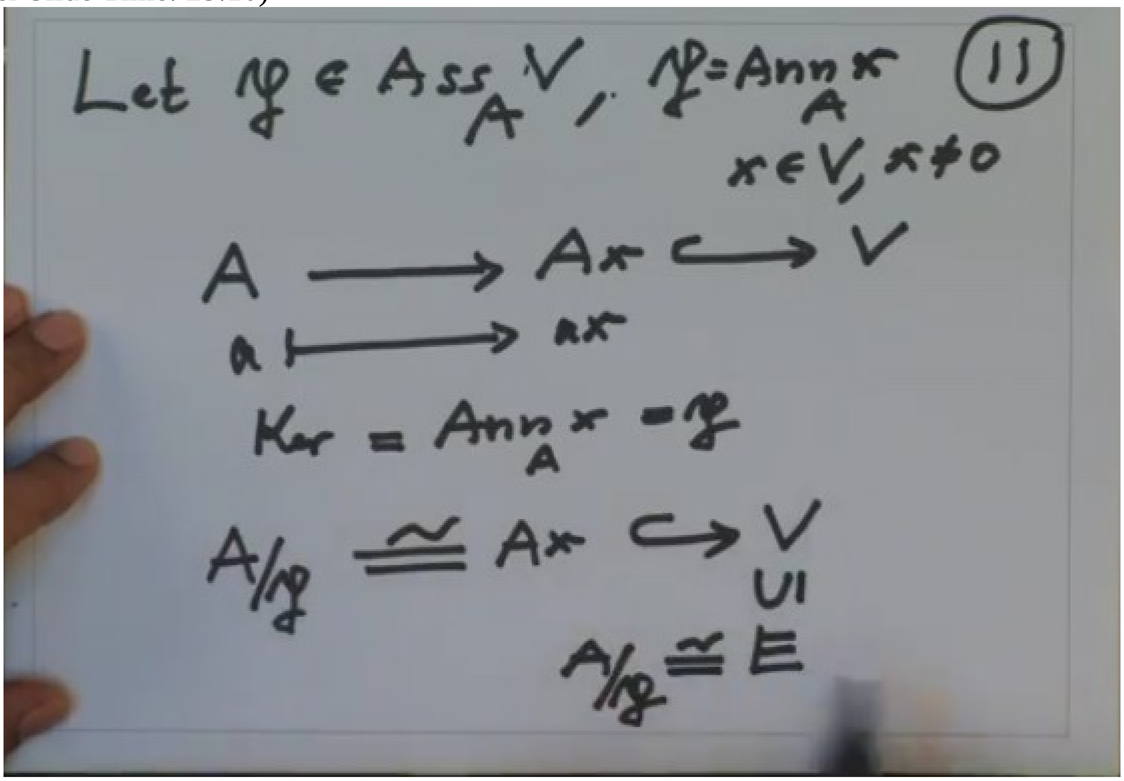

So now E intersection W, look at E intersection W, if E intersection W is nonzero, if this, this can either be nonzero or zero, so if this is nonzero then we know that P, this is the only associated prime of E intersection W, this is because just now we have proved that this is a module whose associated prime ideal is only singleton P, and this is contained in because this is a sub module of V it is contained in the associated prime ideals of W, with this inclusion we already have proved it, so this one is contained in P, alright.

So in this case we proved that P is contained in associated prime ideals of W, other case if E intersection W is 0, then E is isomorphic to sub module E + W modulo W and this is a sub module of V mod W, this is one of the isomorphism theorem, this module is isomorphic to E modulo the intersection, but this intersection is 0, see this is the isomorphism in general, but this is 0 we are assuming so this is isomorphic to E, so therefore and P is then belong to the associated prime ideals of $\frac{V}{W}$, alright.

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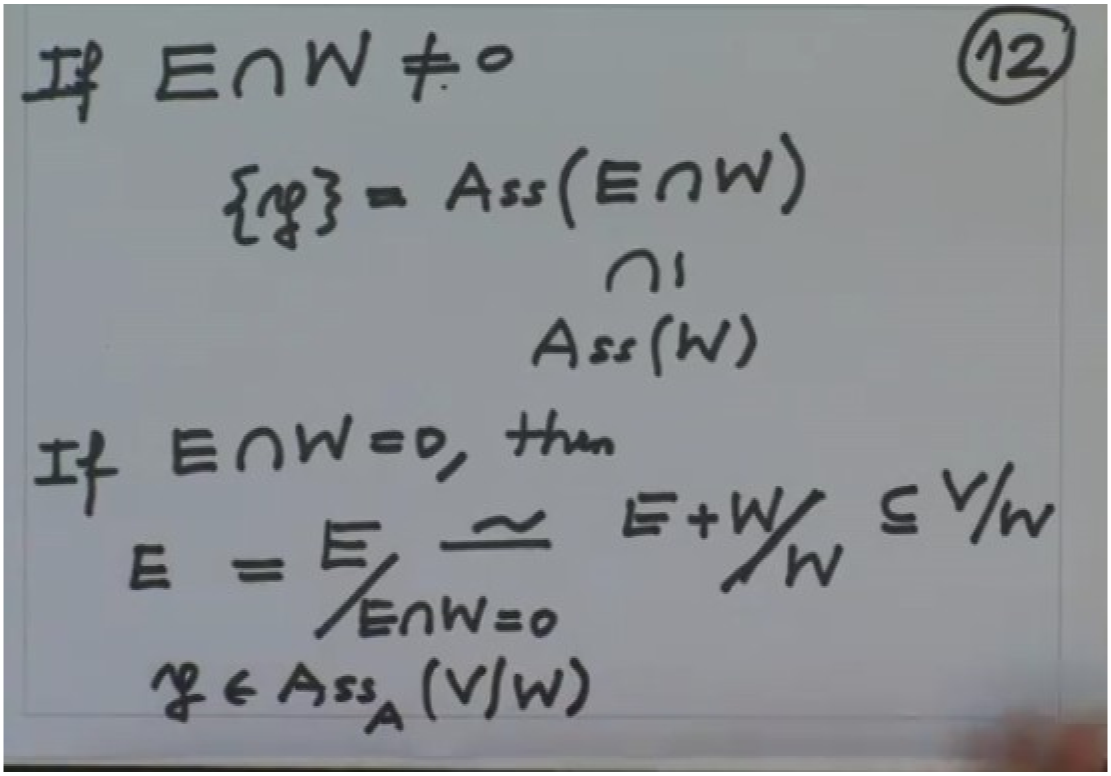

So we have proved this theorem and one corollary I will mention, so let V be A module, and suppose I have a chain $0=V_0$ contained in V_1 contained in so on, contained in $V_n=V$ such that the successive quotients are $\frac{V_i}{V_{i-1}}$ this is isomorphic to $\frac{A}{P_i}$ with where P_i 's are the prime ideals, then the associated prime ideals of V this is precisely contained in the set P_1 to P_n .

So proof, this is a successive application of the theorem, so proof we have this inclusion associated prime ideals of V, this is contained in the associated prime ideals of V_{n-1} union associated prime ideals of $\frac{V}{V_{n-1}}$, this is the second inclusion in the theorem and keep doing this.

Now this one, so this one is, so this is contained in, this is equal to this cell, this is also shielded prime ideals of $\frac{V}{V_{n-1}}$ union, this one $\frac{V}{V_n}$ that you further write it as so, I should have written here this V_n , V is V_n also and this one is $\frac{A}{P_n}$ therefore this associated, this only one associated prime ideal namely P_n ,
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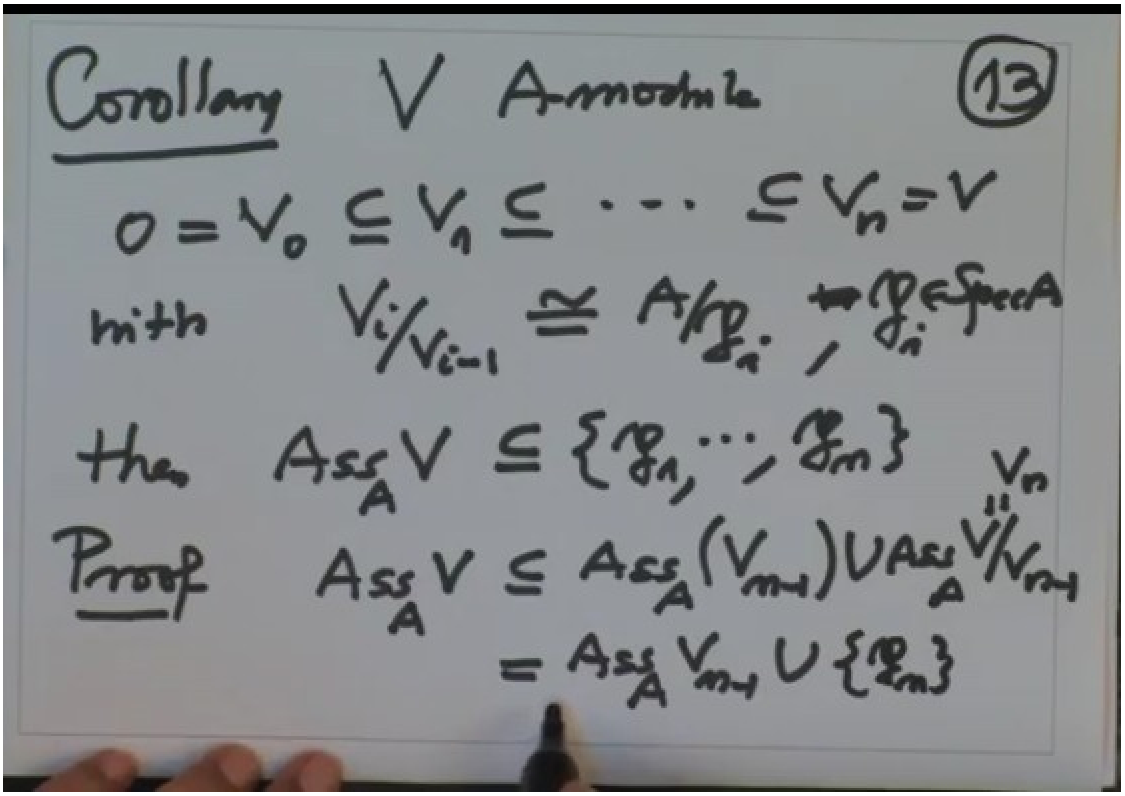

and now keep doing this, write this, this is now this is the bigger module so this is contained in associated prime ideals of V_{n-2} union associated prime ideals of $\frac{V_{n-2}}{V_{n-1}}$, modulo V_{n-2} this is again applying the theorem and this P_n I kept as it is,

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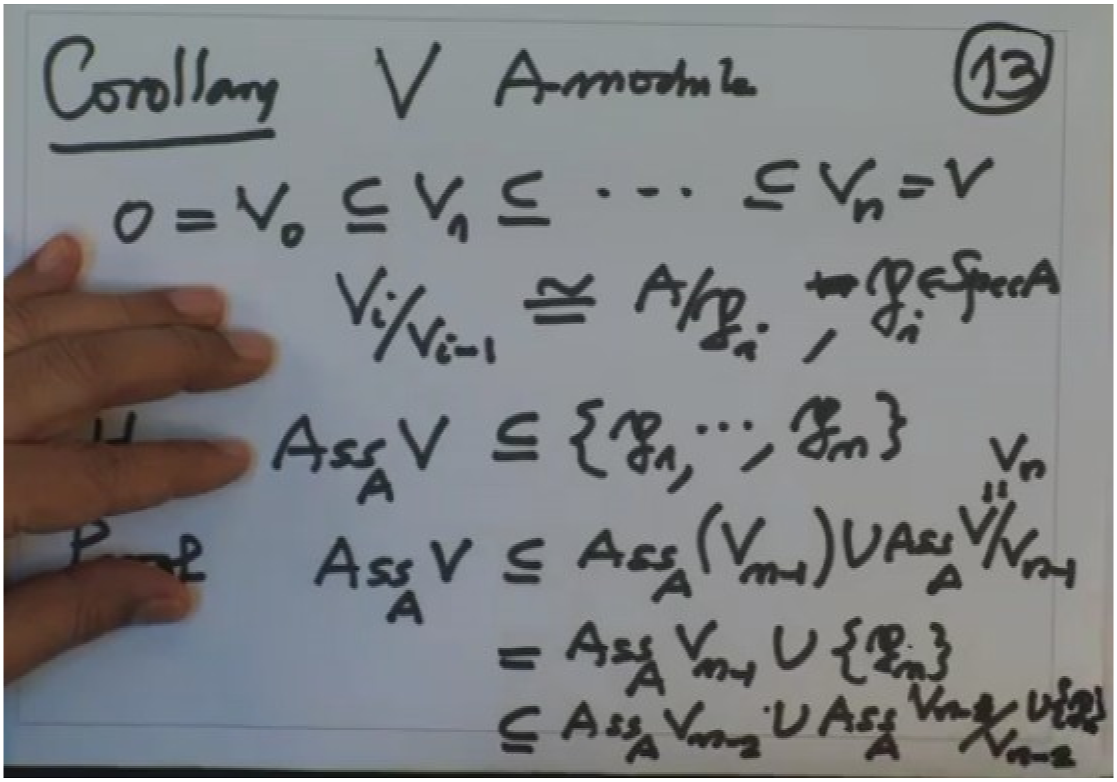

and this one now I'll write on the next page, so the associated prime ideals so this is equal to associated prime ideals of V_{n-2} union associated, so that associated prime ideal is precisely P_{n-1} and union the singleton P_n as it is, so and keep doing this, so ultimately you will reach P_1 to P_n , so therefore one important observation is, so the next corollary is, so V finite A module and A is Noetherian, then associated prime ideals of V, this is a finite set, this is a finite set,

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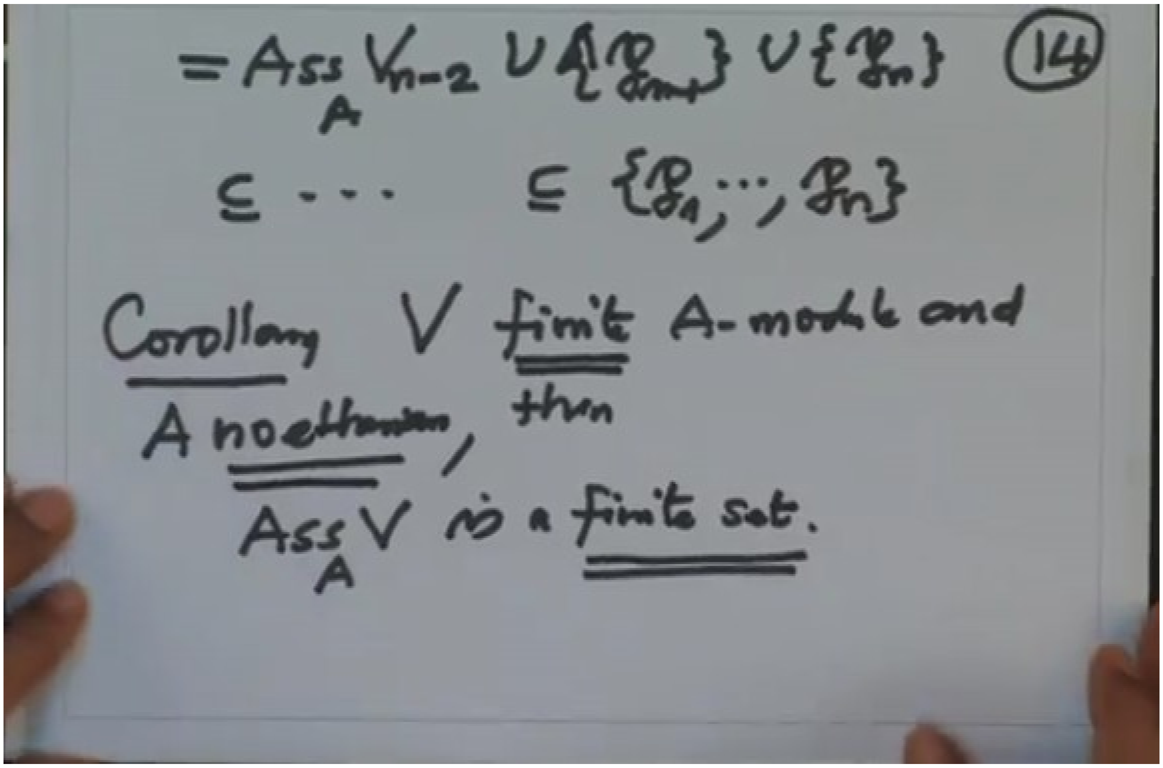

this is very very important, we have used this and remember the assumption V is finite and the ring is Noetherian, so we will continue after the break. Thank you.

Prof. Sridhar Iyer

NPTEL Principal Investigator
&
Head CDEEP, IIT Bombay

Tushar R. Deshpande
Sr. Project Technical Assistant

Amin B. Shaikh
Sr. Project Technical Assistant

Vijay A. Kedare
Project Technical Assistant

Ravi. D Paswan
Project Attendant

Teaching Assistants

Dr. Anuradha Garge

Dr. Palash Dey

Sagar Sawant

Vinit Nair

Pranjal Warade

Bharati Sakpal