INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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Lecture No. – 49

Global Dimension

Last time I defined homological dimension of a module over a noetherian ring and we saw some basic properties. So today I'm going to define a global dimension of a ring and related to the Krull dimension of the ring and this will be very helpful for checking the regular ring is locally regular.

Okay so let us recall last time what I did was, I did, so our ring is since A is noetherian ring and the last time we saw homological dimension of a module over the ring, so this was the supremum, this is not the definition but we did with this supremum of n such that there exists a module A-module N with $Ext_A^n(M, N)$ is a nonzero,
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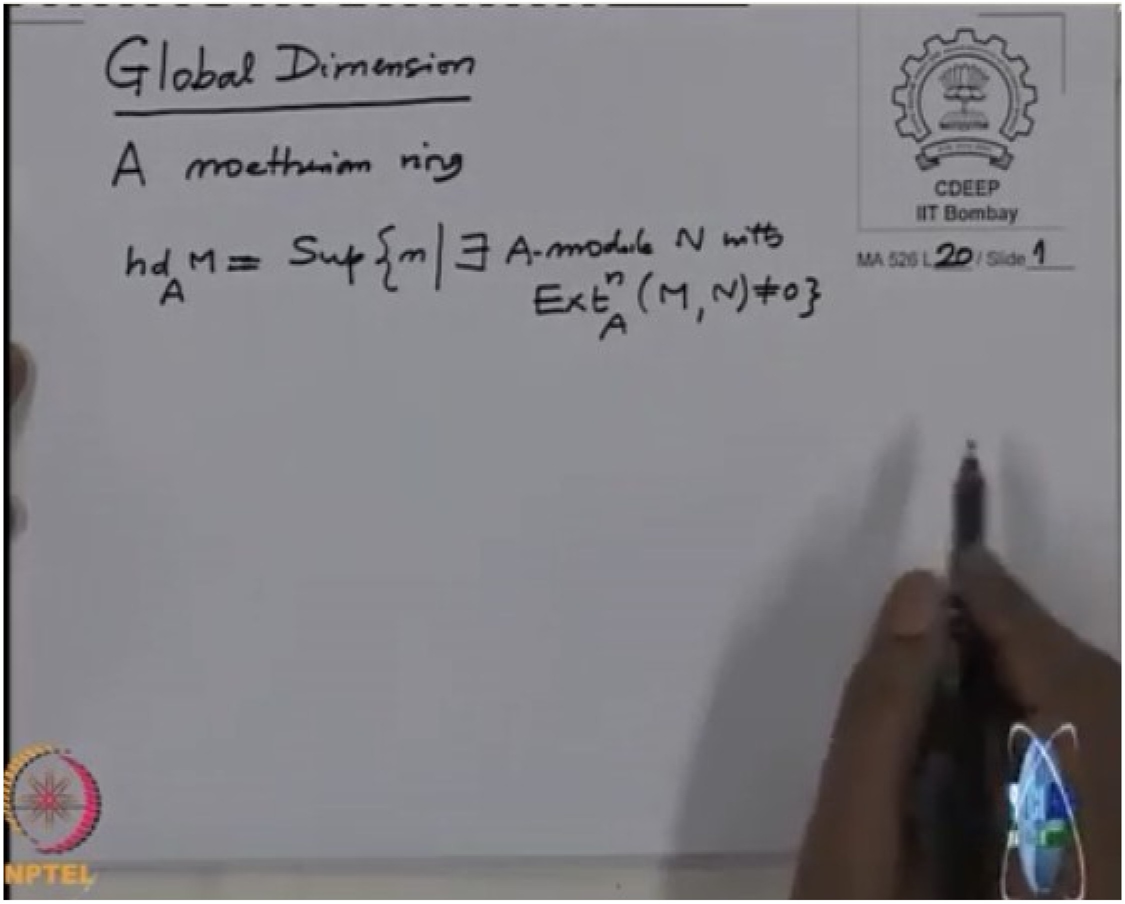

so also this was equivalent to remember checking that if you take a projective resolution of M, then the NH page kernel is projective and after that it is 0, so this was a homological dimension and we saw some properties.

And I will keep losing the Ext, see there are so many things here to check that it doesn't depend on the projective resolution of N, also one can use the injective resolution of N and that tensor with M, no hom with M and then compute the homology or co-homologies the results are the same. So here I am concentrating on, you take the projective resolution of M and hom it with N, okay.

So what is a global dimension? So global dimension usually denoted by GL dim A, this is by definition you take supremum over M, M is an A-module and take the homological dimensions, this is the global dimension definition, you take this as a definition.
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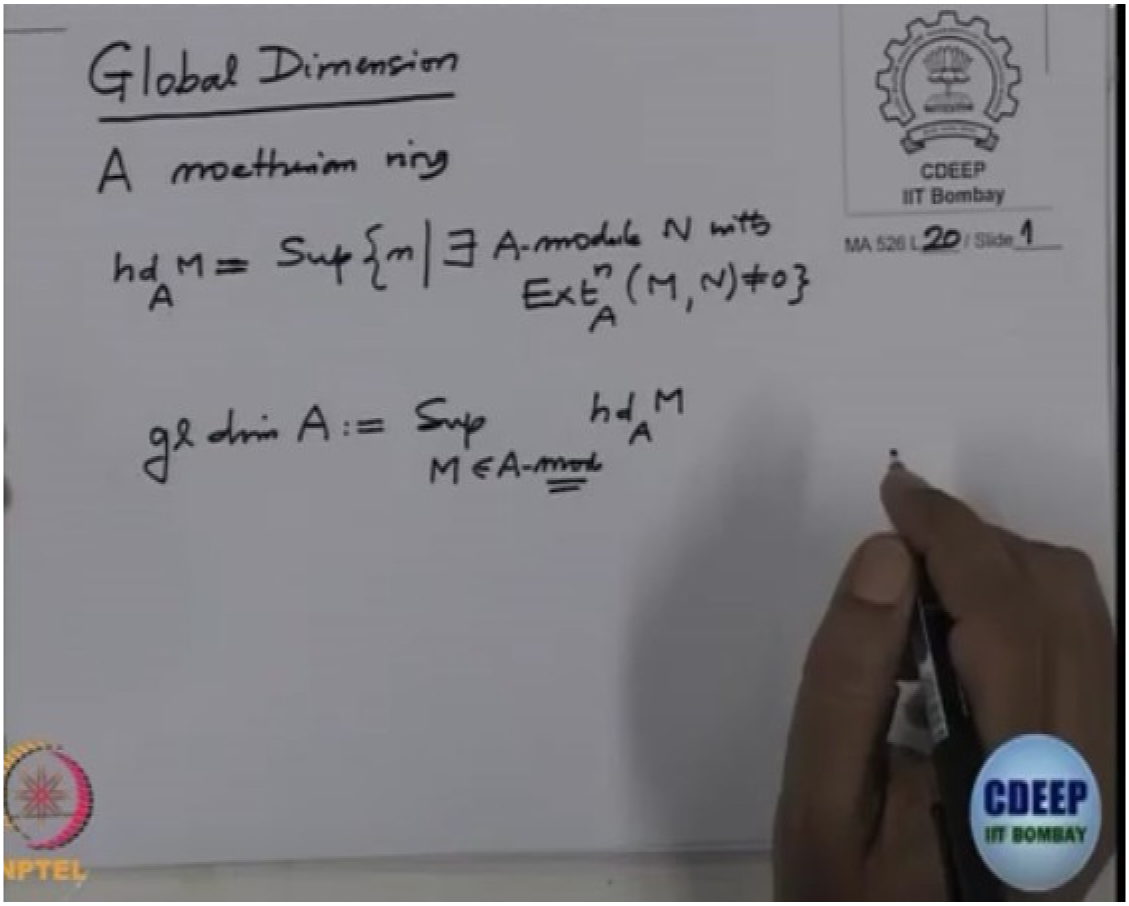

And now we are going to concentrate on the local ring, and we will prove that this, when this is finite that is equivalent to saying the ring is regular local, and in that case global dimension will be equal to the Krull dimension, so that is what the plan will be, it will take 2, 3 couple of lectures, so we are going to assume now A is local, A, M noetherian local ring and we are going to relate these two integers GL dim A and Krull dimension of A,

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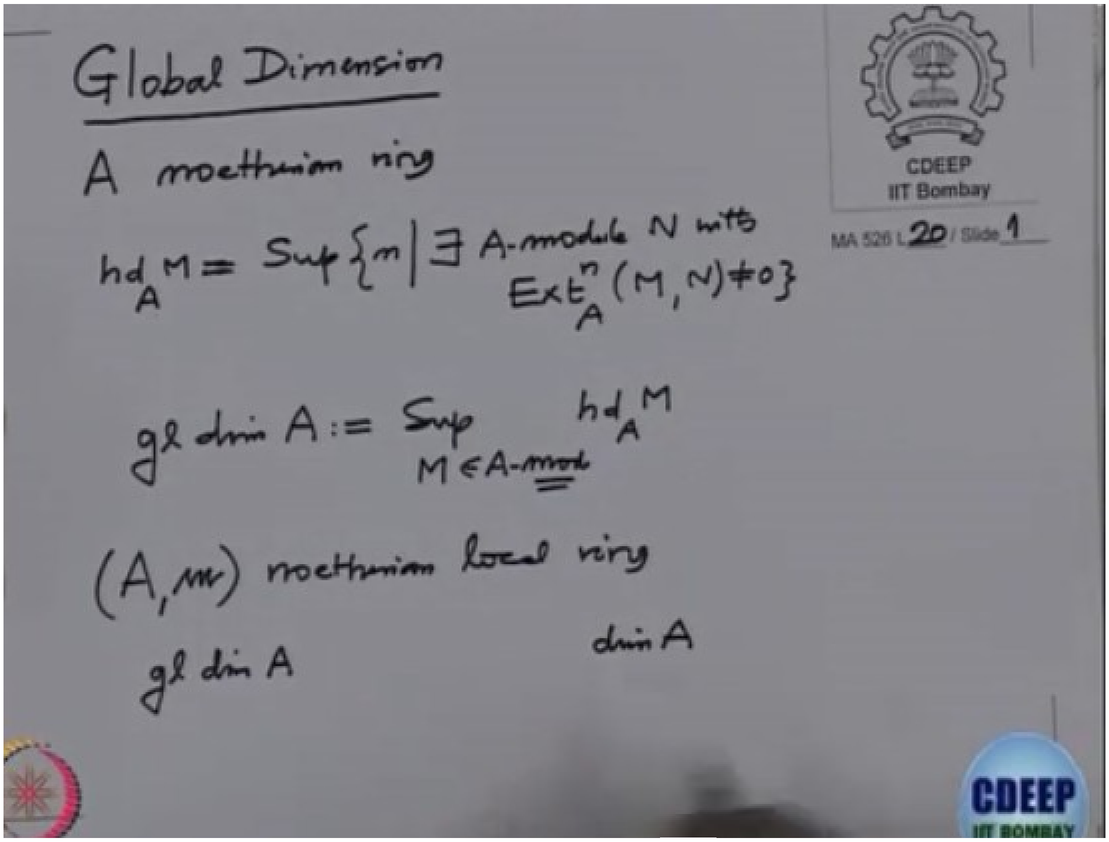

and because it's a local ring we have a residue field, this $\frac{A}{M}$ and we will first show that this supremum we can only concentrate on this residue field as A module, so we'll have much less calculation, so for that I'm preparing now, so what we will prove is global dimension is finite if and only if the ring is regular, that is what the theorem will be approve, alright.

So now we are concentrating on the local ring, so let us recall quickly how do we translate modules being projective or free, how do we test? In general we have to test that the hom P dash this one that is exact, so that one can be little bit improved, so the proposition I want to note first time, we have now (A, M) local, I will always assume the rings are noetherian, local contains as a part noetherian rings, and M is finitely generated A module then the following are equivalent. 1 M is free, 2 M is projective, 3 $Tor_j(M, N)$ is 0 for all A-modules N, and all j big or equal to 1, and 4 $Tor_1(M, \frac{A}{M})$ is 0,

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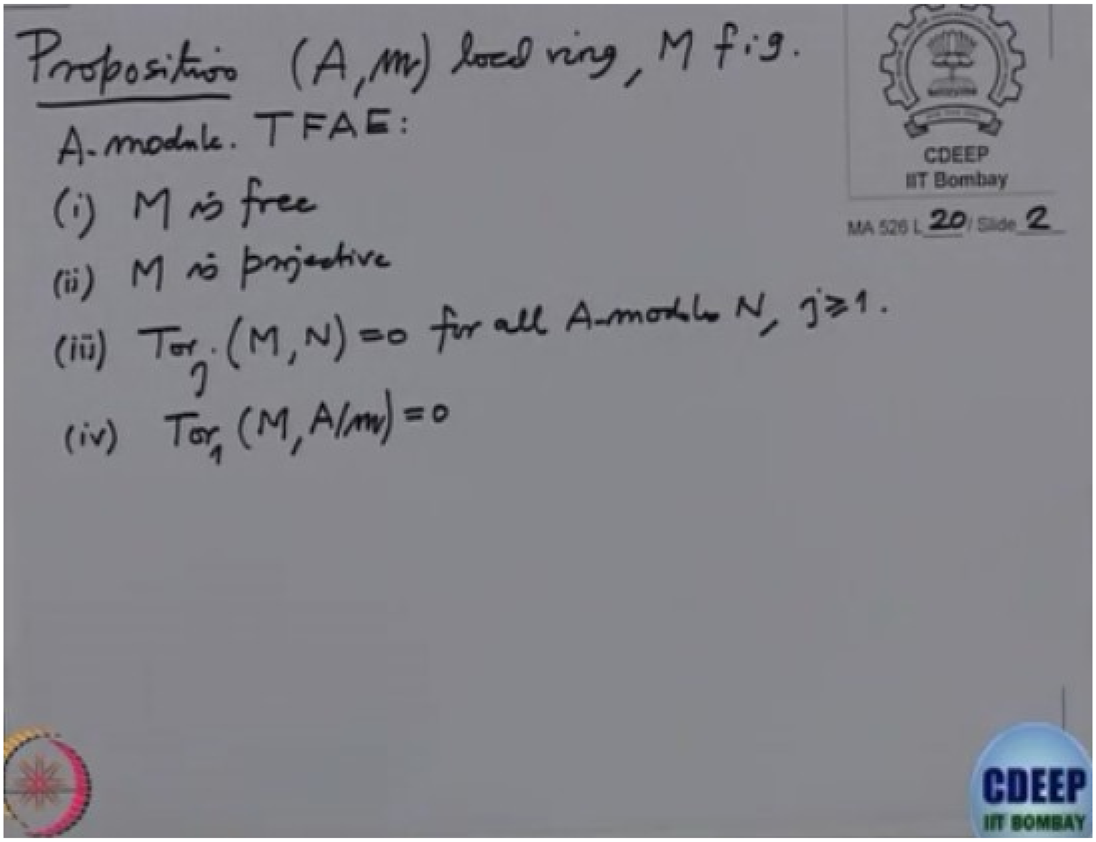

so note that advantage of proving such a proposition is, fourth condition involves only the module M and the residue field, and only the tor 1, so we have very economical way of checking it here.

Another thing is here also one if and only two the projective modules are free over local ring, see in general that was a big question whether projective modules are free or not, okay. So let us quickly prove this, so proof so first of all 1 implies, 2 is trivial, every free module is projective, 2 implies 1 let us, 2 implies 1, so we have given the module is projective and we want to produce a basis, so what you do is take a module M, and now we are over a local ring so therefore minima number of generators for the module $\mu(M)$, this is nothing but the vector space dimension of $\frac{M}{mM}$, this is precisely this follows from Nakayama's lemma or moreover if you take any set of elements in M,

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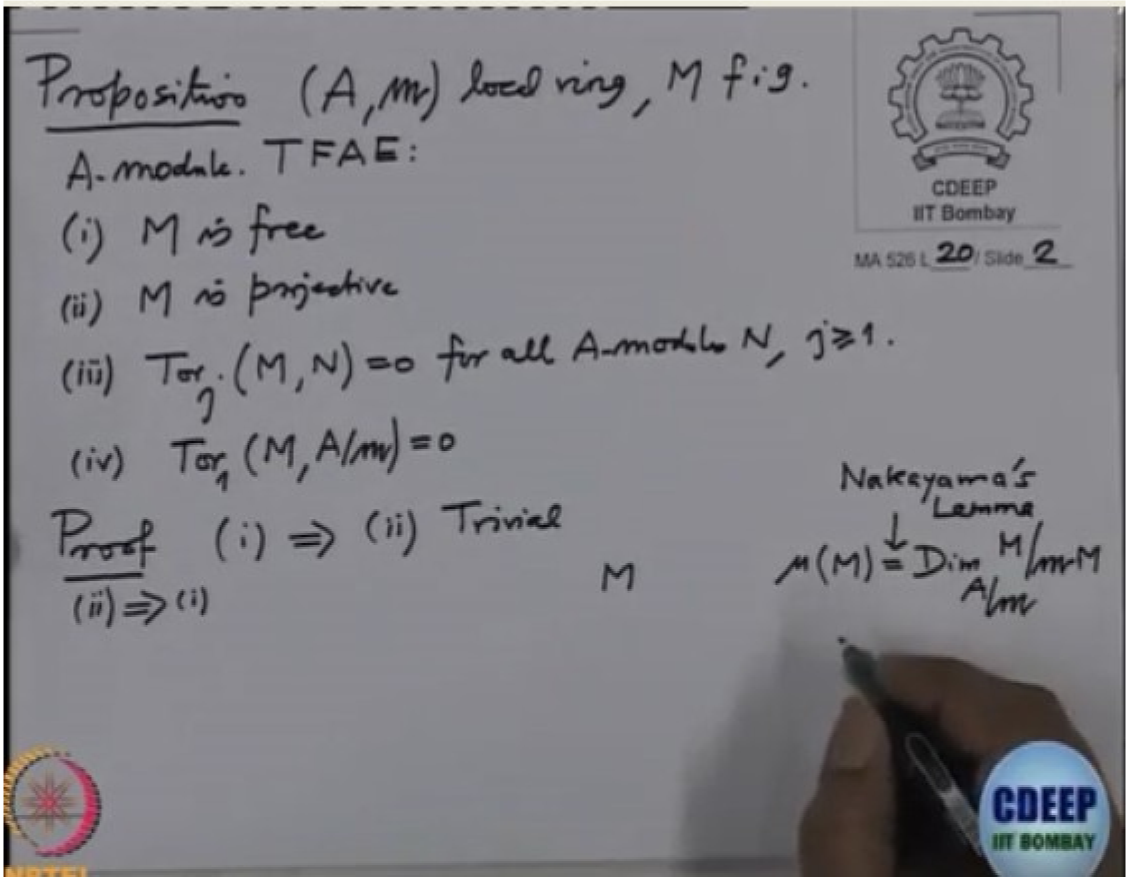

if mod M it’s a basis which is already generate and it’s a minimal set of generators, so that’s what I’m going to do, I choose a minimal set of generators $x_1, \ldots, x_n$ is a minimal set of generators here for M, and then write it as a quotient of free module, so now the free module is what do I denote? F, this, this is free of rank n and e_i is going to x_i, obviously this map is subjective and we are going to prove the kernel is 0, if you prove kernel is 0 then this map will be an isomorphism in particular M will be free, that’s what one we do.

Okay so let us look at the kernel,
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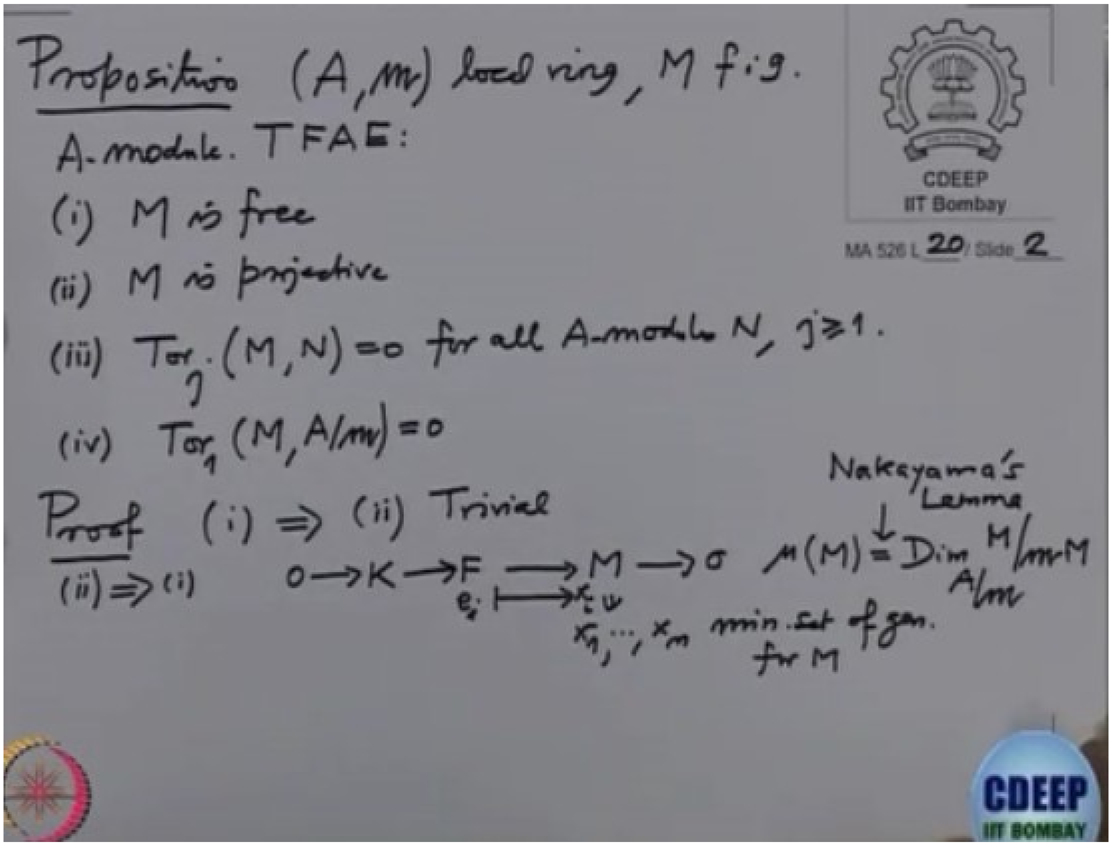

so we have such an exact sequence and I want to prove this kernel is 0, so to prove kernel is 0 I will use Nakayama lemma, I have to use, so to show K is 0, it's enough to show, to show K = MK, first of all it is finitely generated because our ring is noetherian, F is a free module finitely generated therefore noetherian, therefore sub-module are finitely generated, so K is finitely generated.

And if I prove $K = mK$, then by Nakayama lemma K will be 0 and once K is 0 this will be an isomorphism and M will be free, okay. So to show $K = mK$ I'm going to tensor this short exact sequence with the residue field, okay first of all note that this module M is projective therefore this short exact sequence splits, once it splits that means it is splitting here, splitting is the map they say so that it is composite identity on M, so once it splits then if I tensor it will remain split, in particularly it will remain exact, so therefore I will get a sequence 0, $\frac{K}{mK}$ to $\frac{F}{mF}$ to $\frac{M}{mM}$ to 0, this is we have obtained this sequence by tensoring over A the residue field, and it remain exact because the original sequence was split, but now this is a vector space of dimension N, this is also vector space of dimension n, and this is a subjective map and so finite dimension vector space is subjective, is it injective, is it bijective? So this is isomorphism, therefore the kernel is 0 here that means $\frac{K}{mK}$ is 0, so that is $K = mK$, and therefore K will be 0 by Nakayama lemma and therefore we will approve the fact that M is free.

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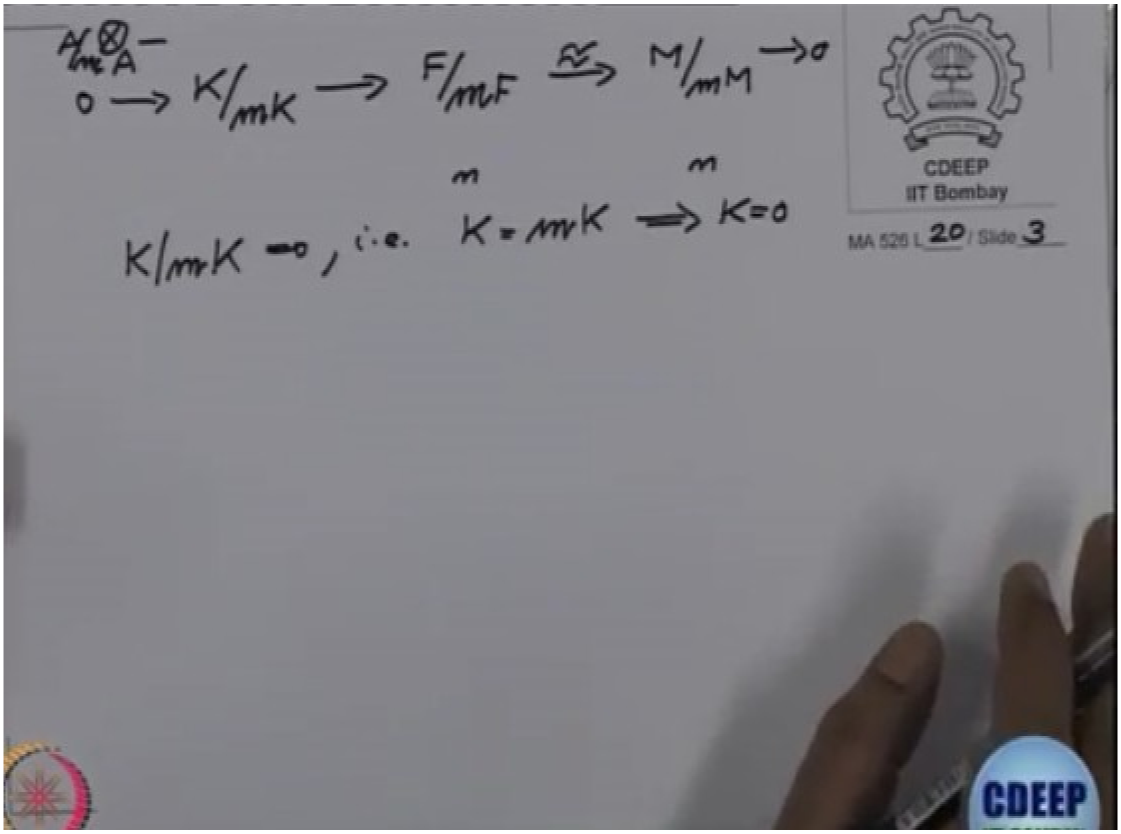

Okay so that was to imply, so we have finished the proof of 1, if and only if 2, now 2 implies 3, so remember we have finished one if and only if 2, then I'm proving 2 implies 3, so this we have done, now this we are doing, okay so 2 what we have given is the modulo M is projective and we want to check that the tor J is are 0, alright, so if M is projective then I will choose a resolution to be that itself, this is my the first component is M, and the remaining are 0, this is my projective resolution and to compute tor I have to take a projective resolution of $M \otimes N$ and take the homologies, so I take this as a projective resolution, and then tensor with N so this is the identity map, then tensor with M and what do I have to do? I have to prove that $Tor_j(M, N)$ is 0 for all N and for all J big or equal to 1, but when I tensor it the only complex, this is the 0^{th} stage and one stage etcetera will come from here but all of them are 0 because everything is 0 after that, so this is done, even 2 is done.

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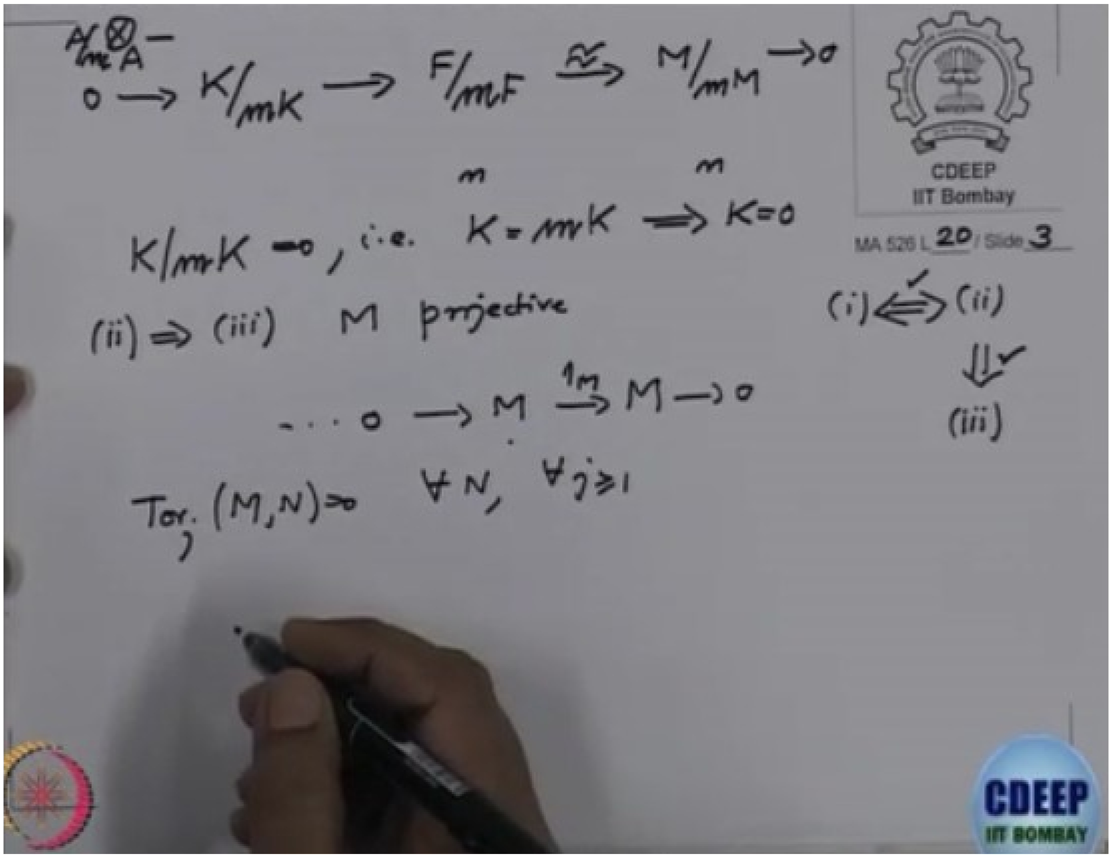

Now 3 implies, now I will prove, 3 implies 4 is trivial, because 4 statement is only tor 1 and the module is $\frac{A}{N}$, so here just to show you this is just tor 1 and the module is the special module the residue field, and 3 implies 4 is therefore trivial, so 3 implies 4 is trivial, so this is trivial, so this is one.

Now we have to prove 4 implies 1 this is what we have to prove, so 4 implies 1 we are proving, 4 says $Tor_1(mK)$ is 0 and I want to prove the module is free, alright, so this proof is similar to that what we proved 2 implies 1, you remember we assuming M is projective, we proved it is free, so it is similar to that, okay. So again start with, I've given 4 that means I have given $Tor_1(mK)$ is 0, so you take a resolution of M, so it is F here, and you take K here this is free, and we are taking resolution of this K is the module, K is the kernel of this, this is projective and therefore free because we already prove 1 if and only if 2,

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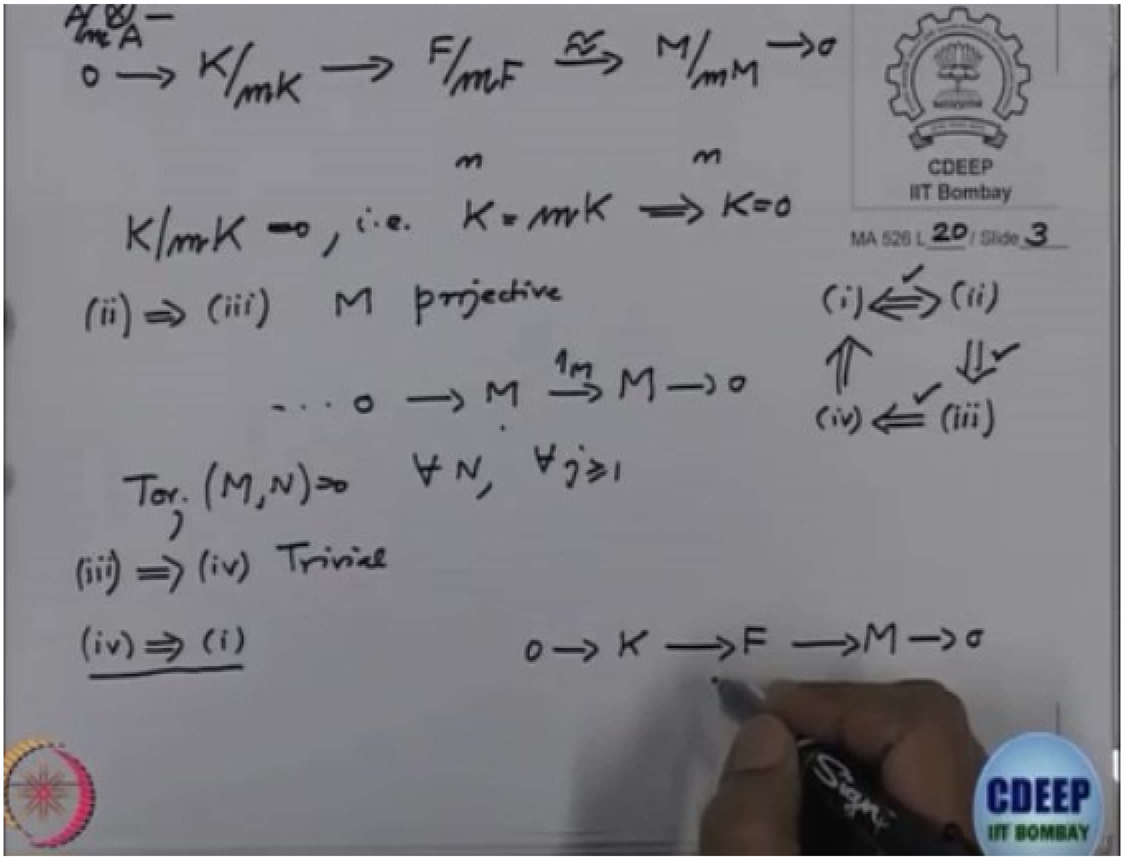

and now we want to do it tor 1, so now when I tensor this sequence with the residue field the long exact, what do I'll get? If I tensor this with $\frac{A}{M}$ I'll get $\frac{M}{mM}$, 0, $\frac{F}{mF}$ and $\frac{K}{mK}$ and here the term will come $Tor_1(\frac{M}{mM})$ the residue field, this is the part of the first, part of that long exact sequence, but and then so on, but this is 0 that is our hypothesis 4, when this is 0 this sequence is exact,

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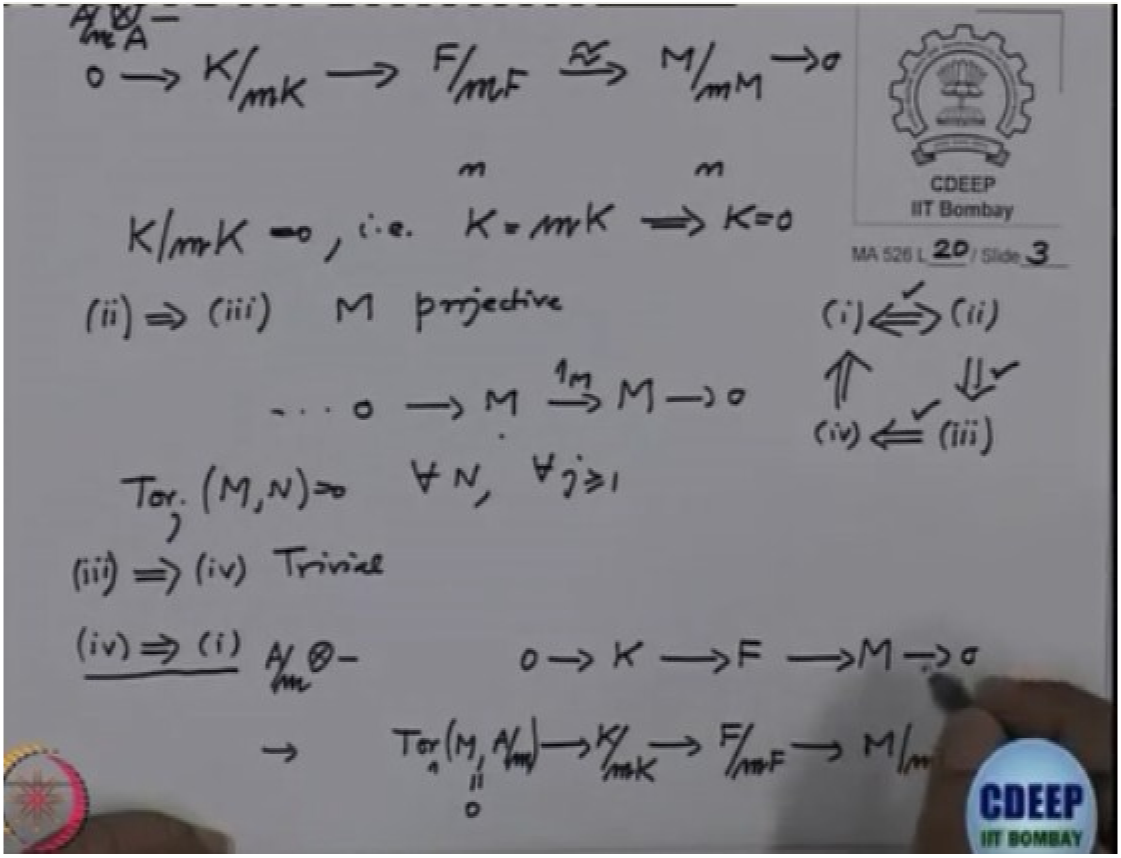

this sequence is exact and while choosing this projective resolution I would have chosen the minimal number of generators of M here and this F is of rank n, so therefore this two will be isomorphic, and again the same proof as in the proof of 2 implies 1, $\frac{K}{mK}$ is 0 and therefore K is 0 by Nakayama lemma.

So same as, by then K is 0 by the same argument as in the proof of 2 implies 1, okay, so we have, to check some module is projective we just have to check $Tor_1(mK)$ is 0 or not, okay, so that will also help us for the homological dimension now, so this proposition is computing homological dimension of the module over a local ring, so A local noetherian always our assumption and M is finitely generated A module, then the following are equivalent. One, homological dimension of M is less equal to N. Two, tor n or $Tor_j(M,N)$ is 0 for all n, and for all j big or equal to n+1. And third, $Tor_1(M,K)$ is 0, n or n+1 I'm not sure it will come out in the proof, I think n+1, so proof, okay.

So 1 implies 2, this is similar to the earlier proposition what we have proved, so we have given homological dimension is less equal to n,
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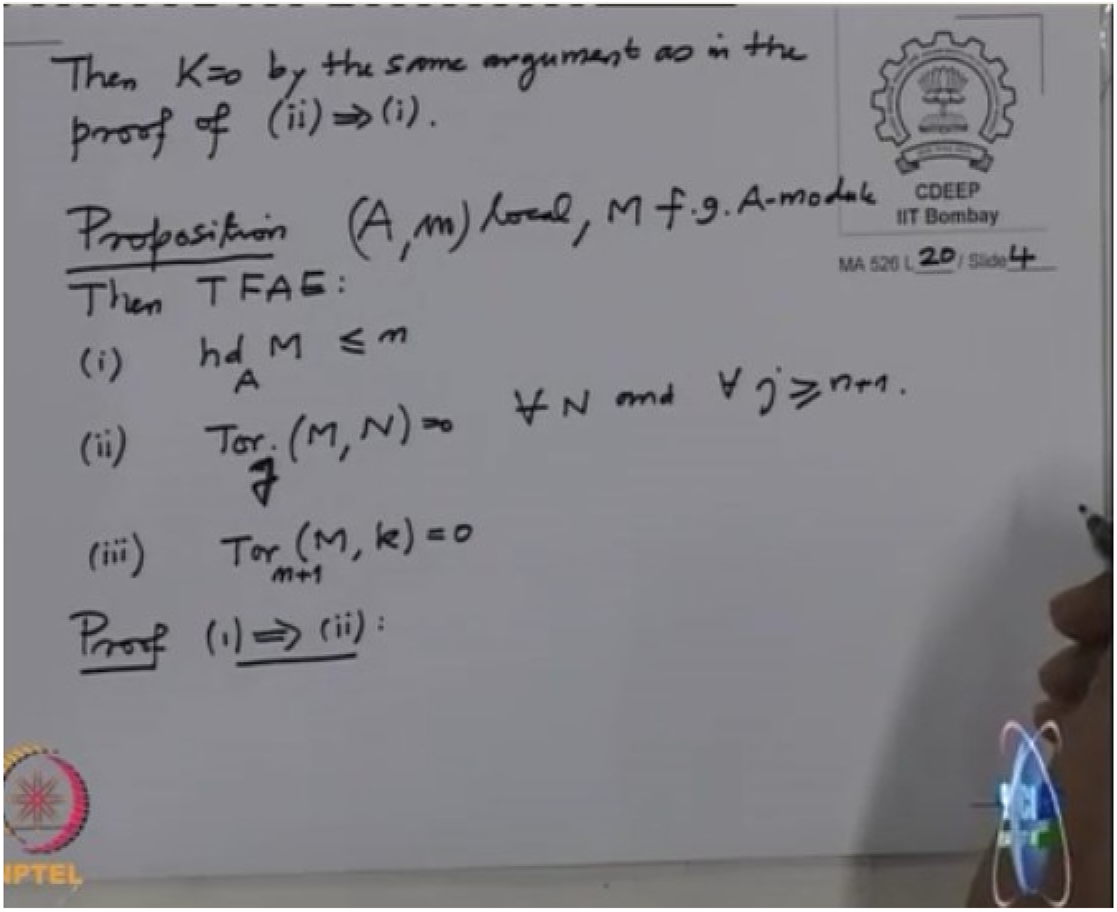

that means we have a projective resolution whose length is less equal to n, so that means projective resolution looks like this M_0, then there is a P_0 and so on, and there is a P_n, and then 0 after that, this length is n, right.

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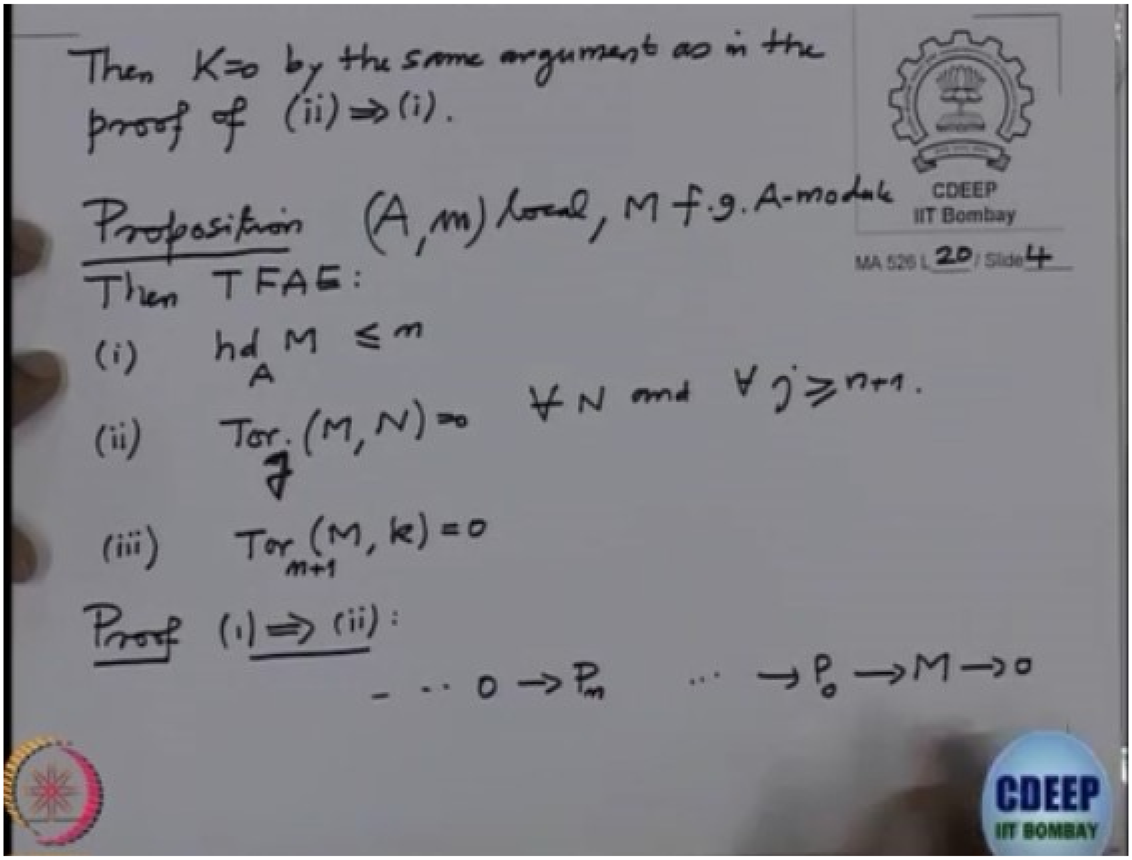

Now when you tensor this with n, tensor this with n, then obviously only up to here the homologies will be if at all nonzero there, this stage onwards they will all be 0, because 0 tensor and etcetera, so that proves 1 implies 2.

So 2 implies 3, this is trivial because I can take N = K, and this is the J = n+1, so N = K, K is a residue field, and j = n+1, the real implication one need to prove is 3 implies 1, so 3 implies 1, this we are going to prove it by induction on N, proof by induction on n, n = 0, what we have given is tor 1, 3, that means we have given $Tor_1(M, K)$ is 0, but by the earlier proposition, so $Tor_1(M, K)$ is 0, this is given, this is given and from here we want to conclude that homological dimension of M is less equal to n, that means homological dimension is 0, that means M is projective, but then by earlier proposition M is projective by earlier proposition, that precisely means that is homological dimension of M is 0.

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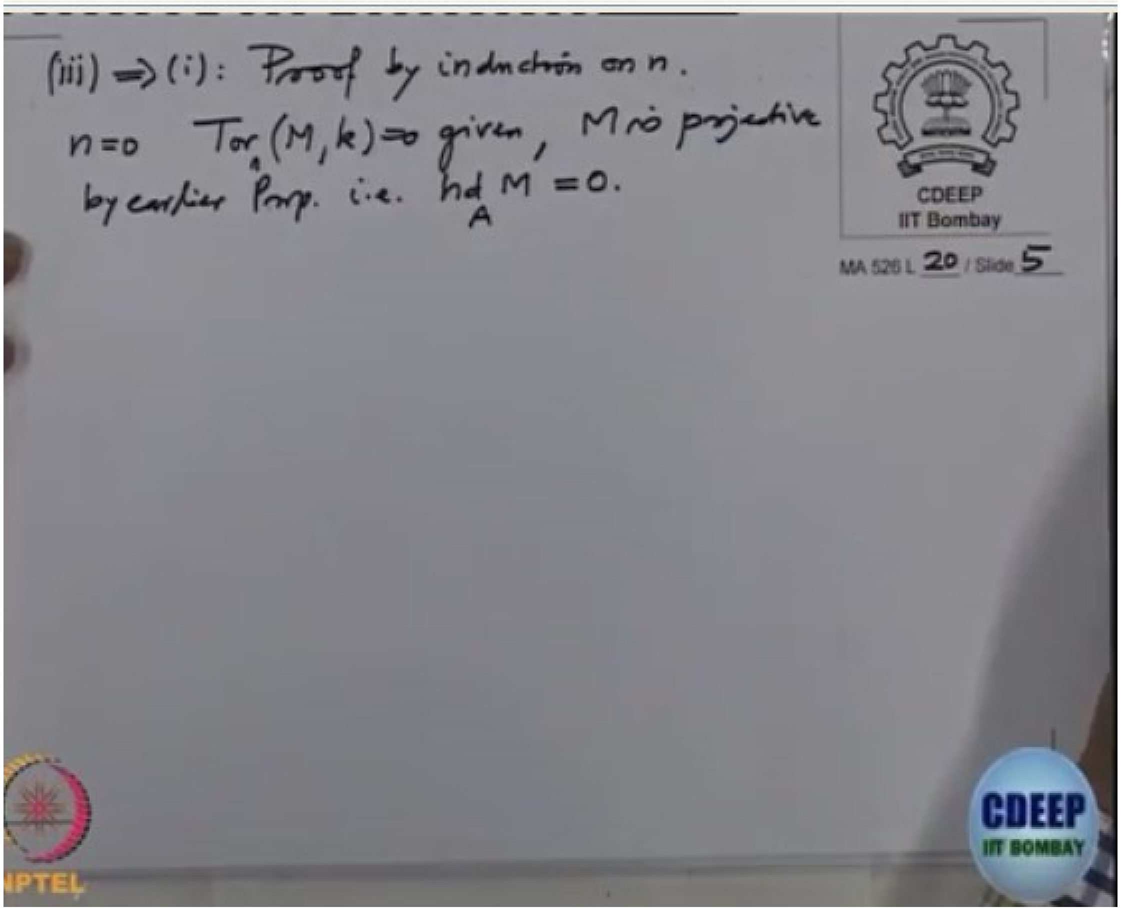

So induction starts, still it is free, if a projective and free are equivalent on local ring, therefore we are assuming now n is at least 1, and we resolve M, take M, take a projective module which subjects on M and look at the kernel that I call it M', 0. And remember that P is projective, this is projective.

Now when I apply tor, when I tensor this with K and take the long exact sequence, what will I get? I will get, there is a connective homomorphism here from $Tor_{n+1}(P,K)$ this is connective homomorphism here, $Tor_n(M',K)$ to $Tor_n(P,K)$, I'm only writing that part only then here it will goes on $Tor_n(M,K)$ and so on, this is a connective homomorphism, this is the delta, connective homomorphism, this is exact we know, but then what is our assumption?

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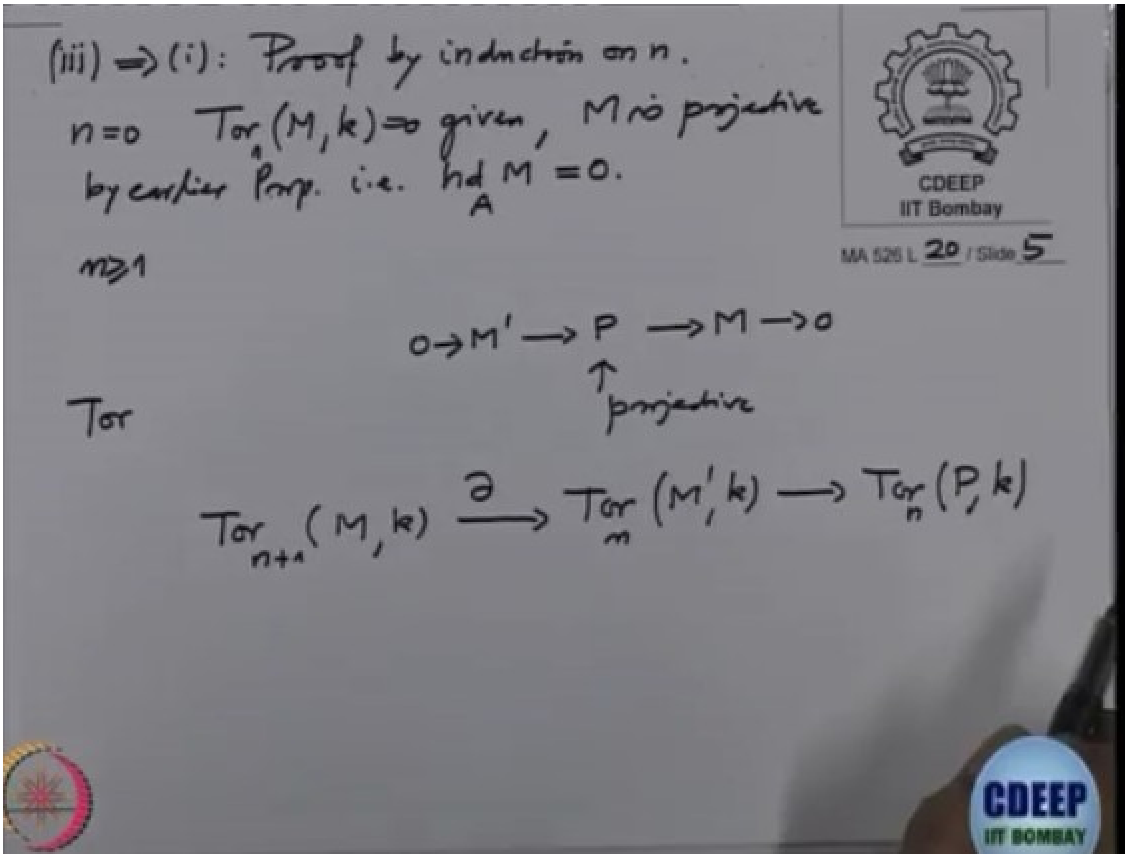

By assumption this is 0, by assumption 3, we are assuming 3, so this is 0 by that therefore and this is 0 because P is projective, this is the same earlier proposition it's projective therefore tor n, n is at least 1, after 1, tor 1 and before that everybody, after that everybody is 0 so this is 0, so therefore this will be 0, so this is 0.

Now apply induction hypothesis to conclude that, so that will imply by induction hypothesis homological dimension of M' is less equal to n-1, and then apply one of the theorem we have proved, one of the lemma we have proved that in such a situation homological dimension of M will be less or equal to n,

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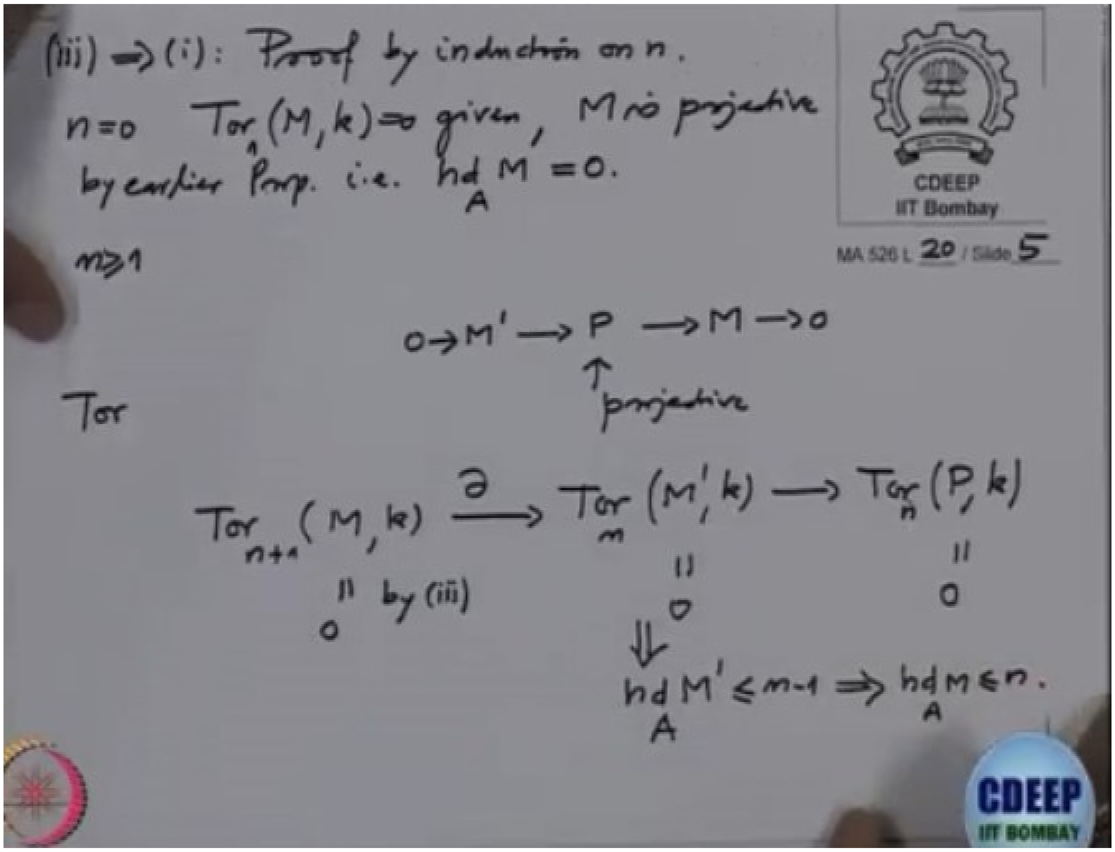

so that implies homological dimension of M is less equal to n, so that proves 3 implies 1 and it finishes the proof of the proposition.

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