INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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Lecture No. – 48

Homological Dimension and Projective module

Okay so we have this one, now let us go back to the, what we want to define, so now here you have lot of things to be checked, but as I said you can take these TFR complete details all the definitions and this. Okay so now I want to define two things, so for a ring I want to define what is called a global dimension of A.

And then after we have done this then we will have its, what is its relation with the Krull dimension, so that will come later, but how to further definition of global dimension all modules are inverse, so this global dimension is usually denoted by GL dim A, this is by definition supremum of homological dimensions of M over A, so this supremum is running over all modules M is A module, so you see it's very big to understand this even to say it is finite contains lot of information about the ring,
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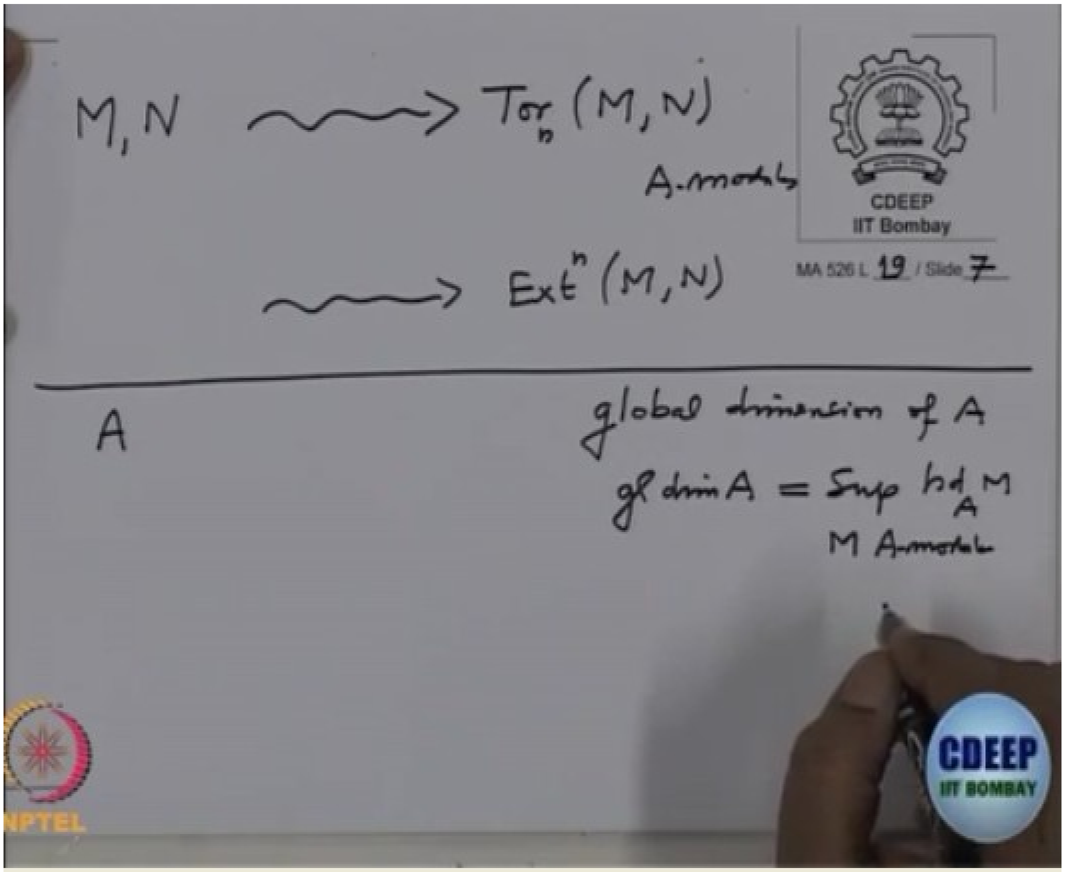

so what we will prove the theorem, what we want to prove is if the global dimension is finite that is equivalent to saying the ring is regular and in that case global dimension equal to Krull dimension, okay so this is what will come in the full lecture, true lectures ahead.

So we have to define what is a homological dimension? So homological dimension, so first of all suppose given a module M and we have this projective resolution of M, projective resolution, now we will say this projective resolution has length n if after nth term all the terms are 0,
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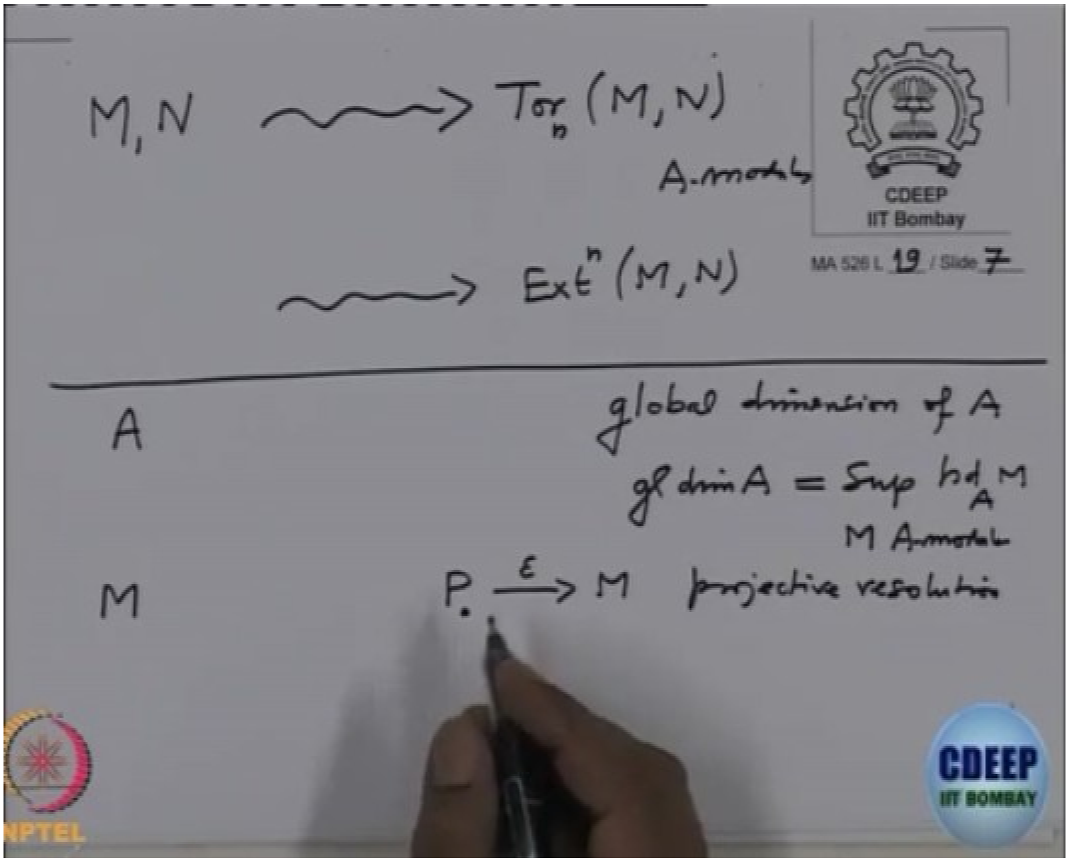

so that means so P has length n means P_n is nonzero, and later ones are 0 and P_i 's are 0 for all I bigger than n, so that simply means there is a projective resolution of this type $P_0,\ldots,P_n$ and then 0, and this definitely nonzero,

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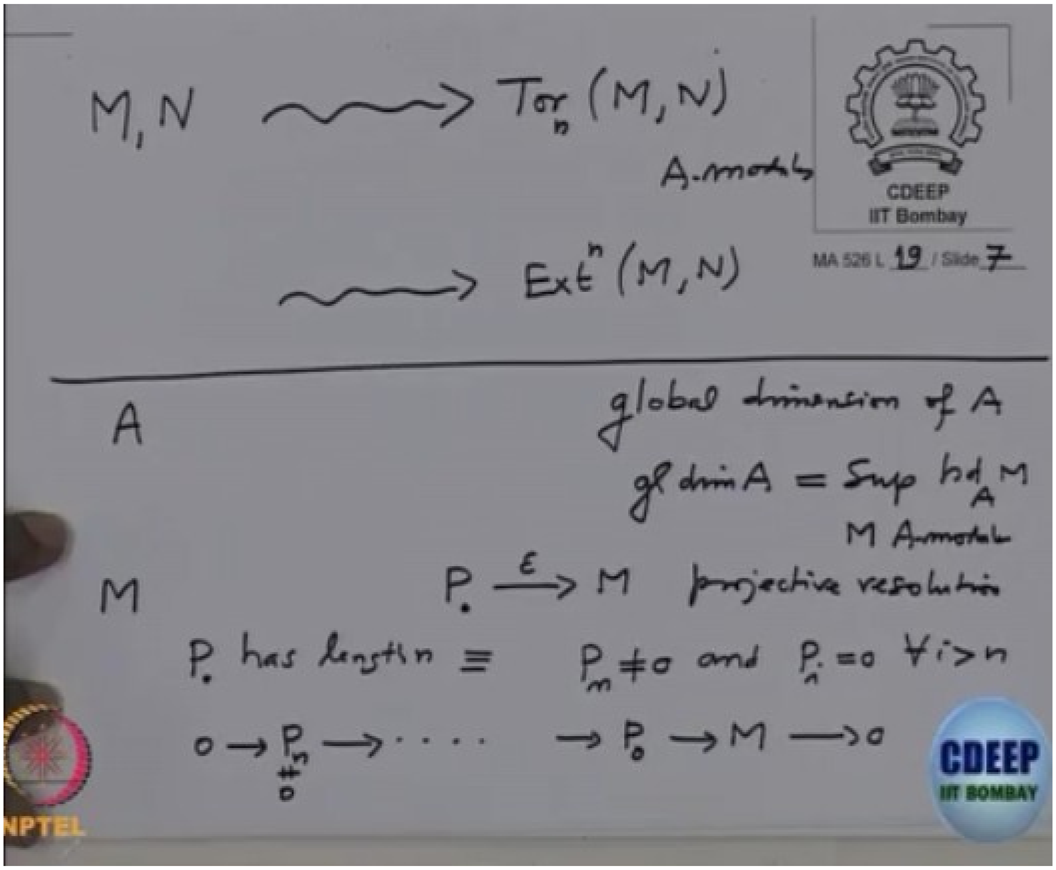

if it is 0 then we will not write that, so this means it's a length, this n is called a length of the projective resolution.

So now homological dimension of the module is by definition the least n such that there exists a projective resolution P. M of length n, if it doesn't exist, if there is no such integer then you denote it to infinite, so otherwise infinity, that means all, in that case all projective resolution will have, none of them will be finite, so such a thing is called the homology, so this integer n is called the homological dimension, homological dimension of a module can be infinite or finite, but if the module is 0, then we put homological dimension of the module to be -1, so for example when is the homological dimension is 0? Homological dimension is 0 if and only if, what is our definition say? That means this least integer is 0, that means there is only one term, there is the projective resolution is only one term,

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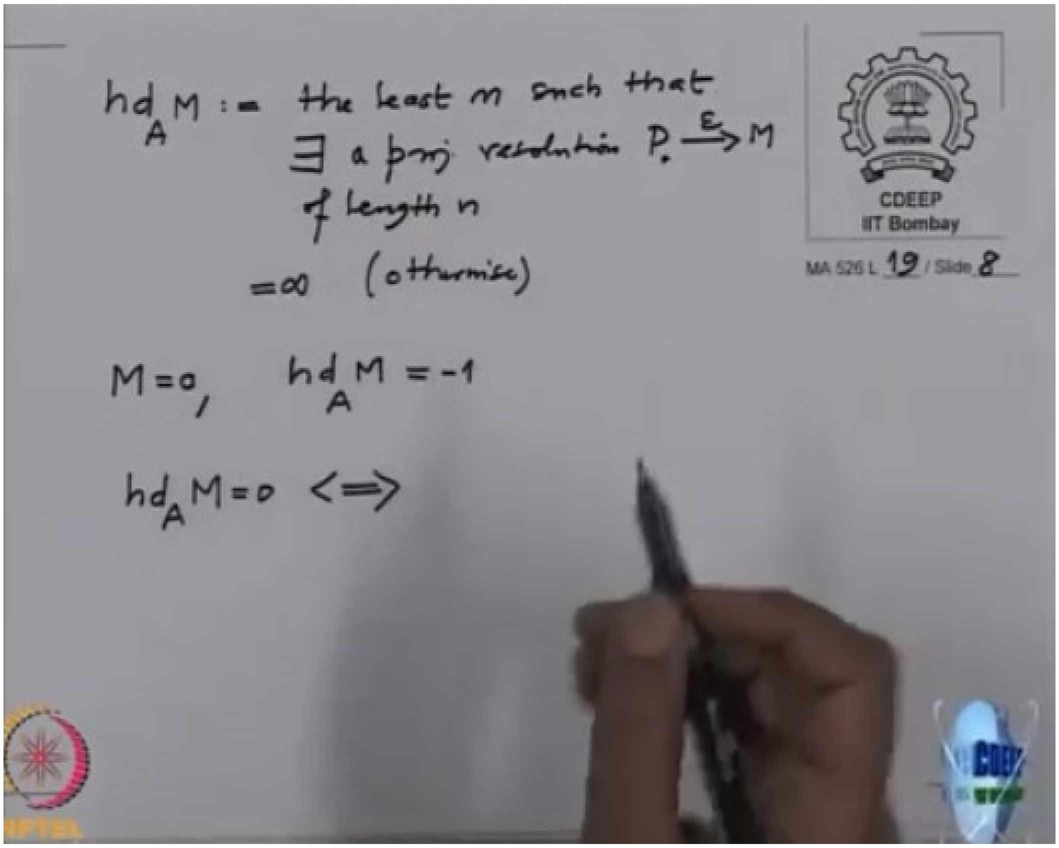

so that means this if and only if P_0 to M, this is a projective resolution this term is 0, but that means this is an isomorphism, so that means P, that means this M is projective.

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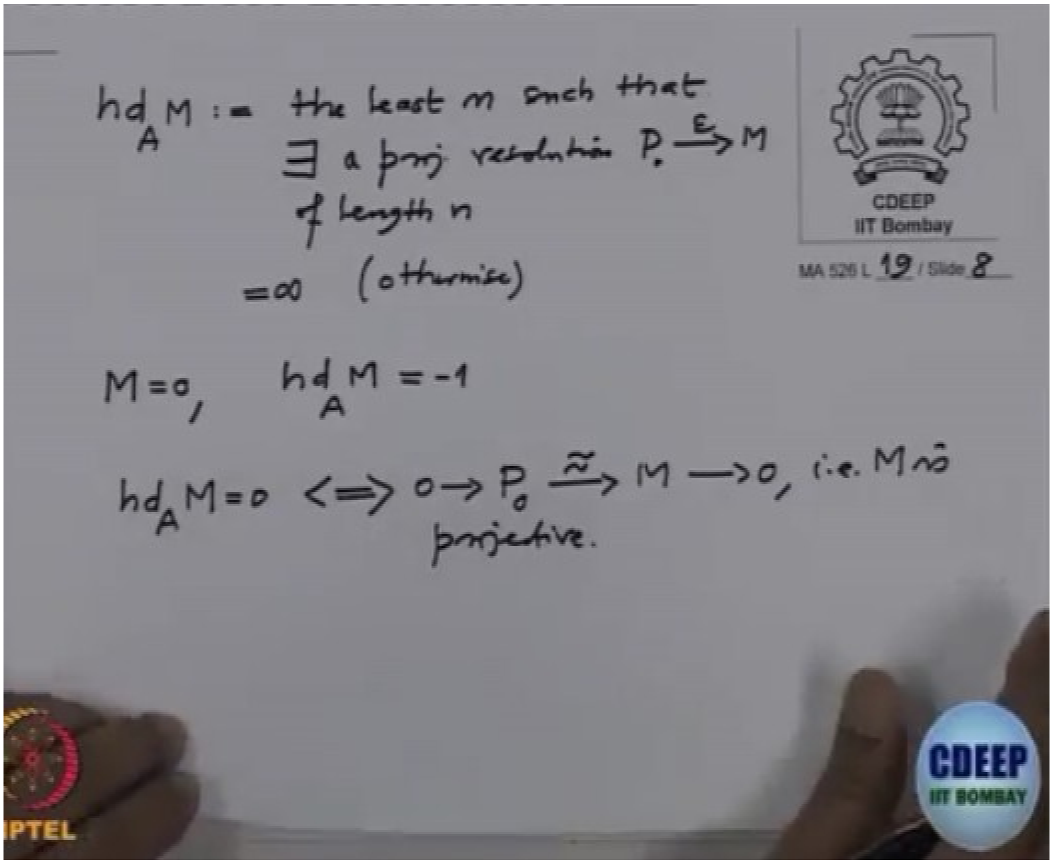

So in some sense this homological dimension majors, it's a major, how far is the given module from a projective module, so if the homological dimension is 0 that means the module is projected. So in particular homological dimension of a free module is 0, okay.

Now let us make some remarks or some basic properties about the homological dimension, how does one calculate and what is its, for 0 homological dimension it's a nice description, but otherwise we'll have to work with the module X, so for example this proposition, proposition so M is A module, so many things here may work out without assuming the ring is Noetherian or modules are finitely generated but I suggest when one learns first time one assumes, easily assume the ring is nice, Noetherian and modules are finitely generated, because in the first time itself the difficulty is, so one will not know where is, what thing is important and where is, why the conditions are put like that, so the following are equivalent, for a module the following are equivalent.

So one I'm just recording this, the earlier homological dimension 0 case and then we will generate, M is projective. 2, see there is only one term so therefore if I take this projective resolution definitely and if I take the, if I apply hom and compute the homologies and all other terms are 0 therefore there won't be any contributions from kernel and image, so that simply mean that $Ext^j(M,N)=0$ for A modules N and all indices j bigger equal to 1.
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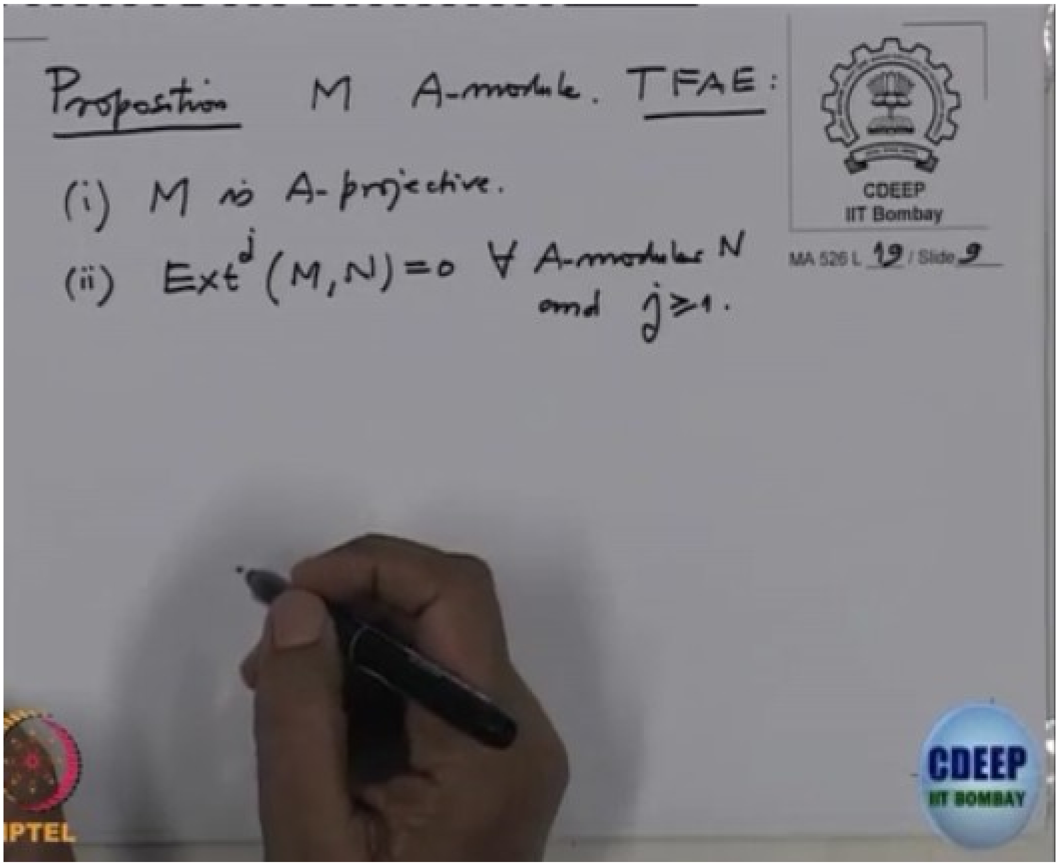

3, $Ext^1(M,N)$ is 0 for all A modules N, so proof, see condition 3 is better because you only have to check only one condition Ext^1, so first of all 1 implies 2 is, because if M is projective then I could simply take this as a projective resolution, this is my P naught, and this is my dim at, so this will be projective resolution and to compute this X I will apply hom functor, hom - M to N to this exact sequence and compute the X, and whatever comes after that they will be 0 that is where this condition too.

And 2 implies 3 is obvious, now the real implication one needs to please 3 implies 1, so if $Ext^1(M,N)$ is 0 for all modules N then I want to prove M is projective, that is a real, so simplification is the real implication, so let us prove this, so I want to prove the module M is projective, that means what? That means a definition, was what? If I have N and N'' 2 modules and a subjective map and given a map from M to this, ϕ and f, and I am looking for lifting like this, $\bar{f}$ so that this diagram is bounded that is what, that is how one rules the module is projective, okay.
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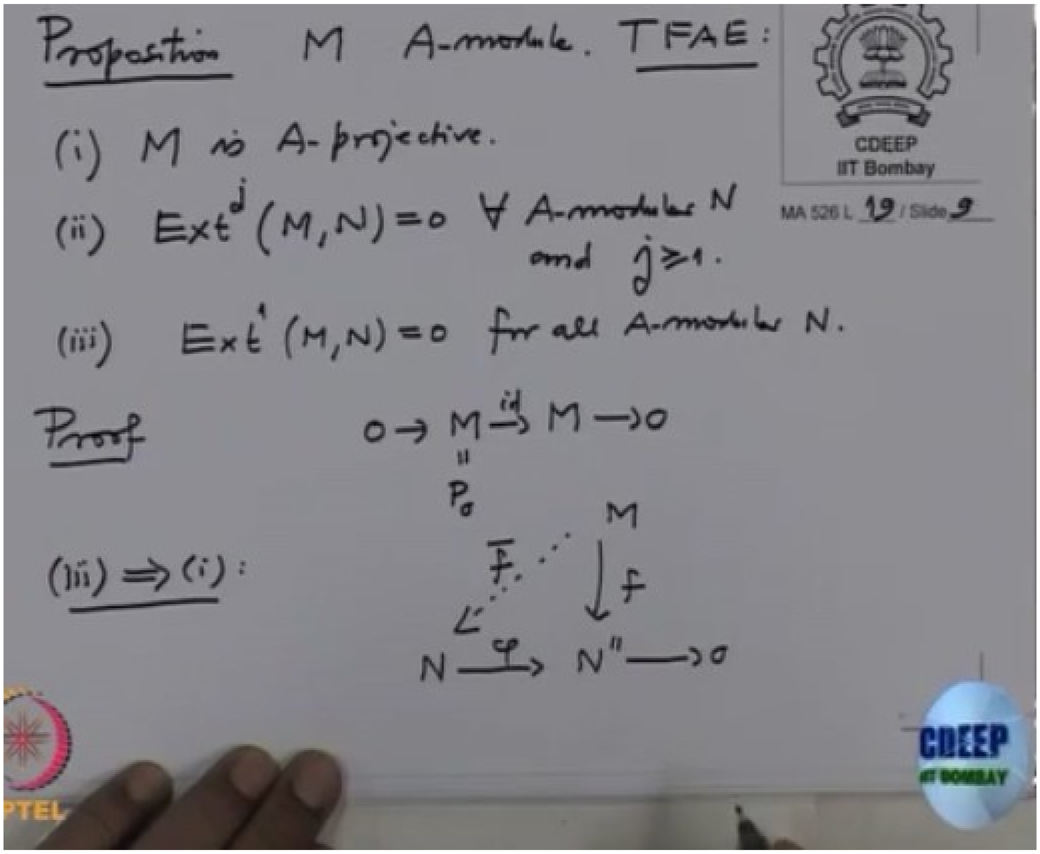

But once we are given subjective map like that then I can write a kernel here, and then we get an exact sequence, once I have exact sequence like that, short exact sequence like that and when I apply a functor $Hom_A(M,-)$, if I apply this, so let's call this as N',

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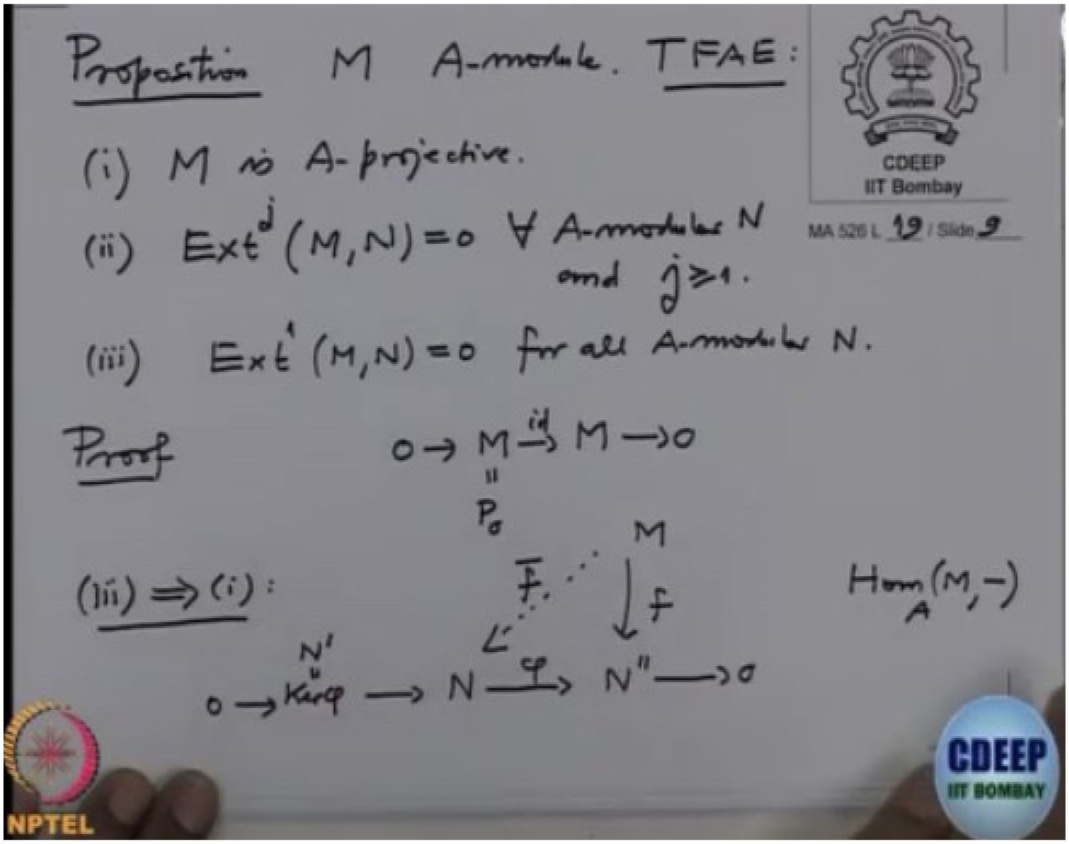

then what do I get? Hom is left exact, so I will get, so hom A, oh it's a contravariant, no it's a co-head, so this $Hom(M,N')$ hom A N, that I have change, no correct M, N and then $Hom_A(M,N'')$ and 0 is, this is the hom is left exact.

And what will be the next term? So they will be connecting homomorphism here and the next term will be $Ext^1(M,N)$ and so on, and what is that we want to prove? We want to prove that this is, we have given this is 0, this is 0 is that is what we have given in 3 because $Ext^1(M,N)$ is 0 for all modules in particular for, so this is 0, but this 0 means precisely this map is subjective, but that precisely means the functor hom M is exact and that was one of the equivalent condition for a module M to be projective so that proves 1, okay, so write it, so that is how one usually deals with this,

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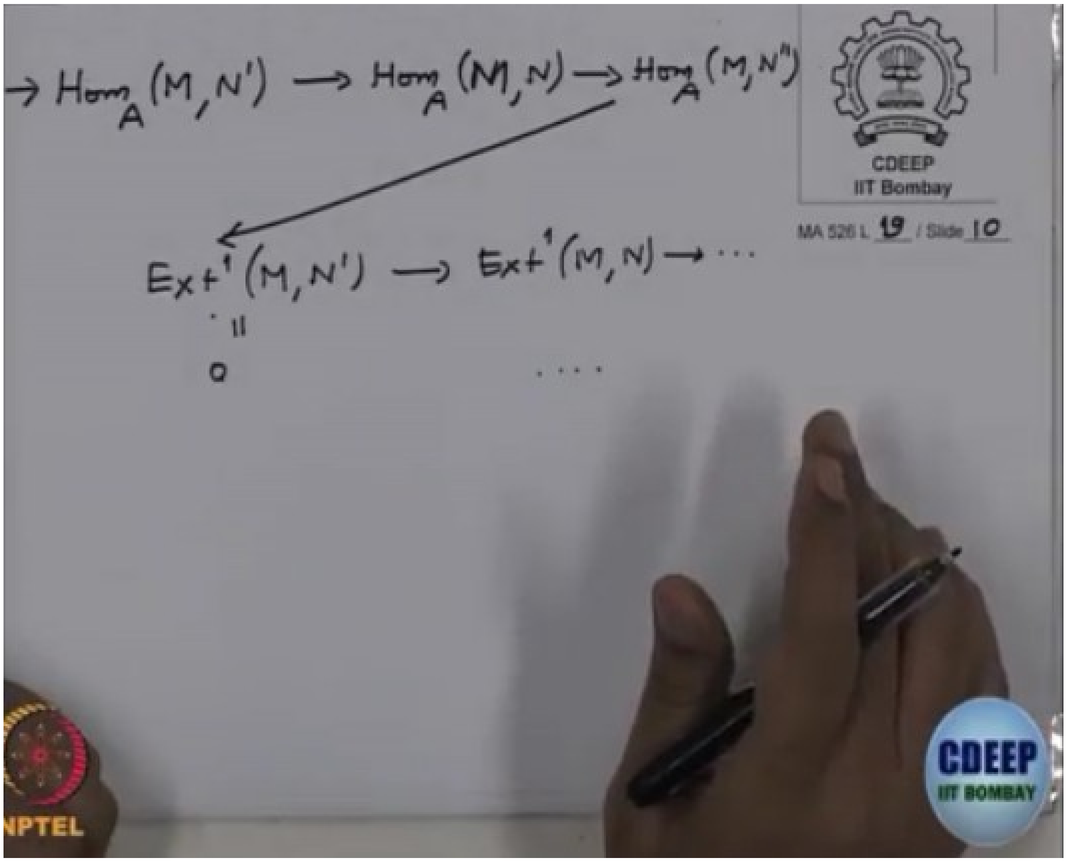

that usual calculation if the X modules will be done by a diagram chasing and writing a big long exact sequence of Ext module which are connected to each other by connecting homomorphism.

Okay so we can now, so this is like homological dimension 0, this means homological dimension of M is 0, now we will state an analog of this proposition there homological dimension is bigger, so that one is proposition, so M is an A module then the following are equivalent, so the first condition will be homological dimension of M is less equal to N, so this means, what does this mean? This means there is a projective resolution of M whose length is utmost 10, because the last nth term maybe 0 or may not be 0, if it is 0 it will be still smaller and so on.

So this second $Ext^j(M, N)$ is 0 for all A modules N and indices j big or equal to n+1. Third, $Ext^{n+1}(M, N)$ is 0 for all A modules N, so this n and this so, we have to compute Ext^{n+1} and test whether it is 0 or not 0, if it is 0 homological dimension is less equal to n, okay.

Fourth condition, if this is a restatement, so but it is useful to state it so given M, so there if, so if M to $P_0, \ldots, P_{n-1}$ to K^n to 0, if this exact sequence of A modules with $P_0, \ldots, P_{n-1}$ are projective modules and K^n is the kernel, now so when you start dissolving M you start writing a quotient of a projective module and then take the kernel and do that, repeat the process and you stop it here, but this is only the kernel here, now the next step will be resolve this also, that means write K^n as a quotient of a projective module and club it here, but if you stop there than this, then K^n is already projective that is the condition 4, so when you try to write the projective resolution, 4 says when you try to write the projective resolution for M keep doing this, at nth stage you already get a projective module, and that is equivalent to homological dimension is less equal to m.

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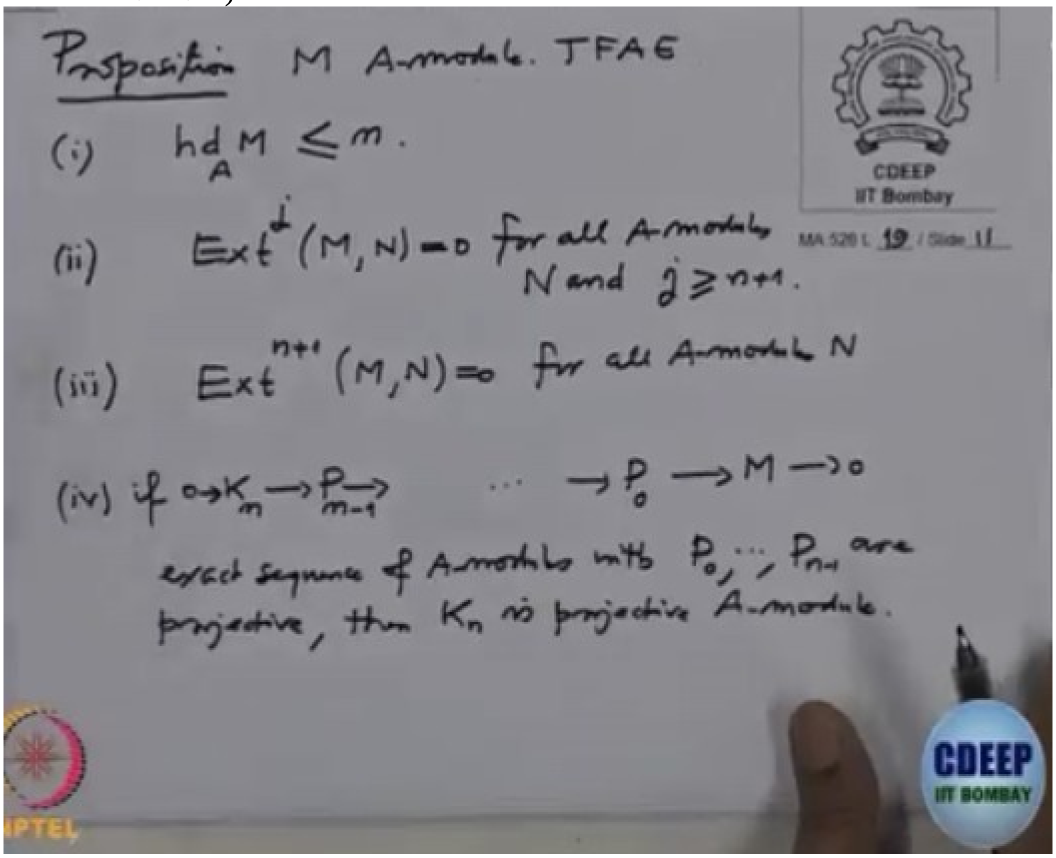

And the proof of this is same like earlier, now you have to take this definition, apply hom M, N and then take the long exact sequence and do that thing so, and I will not prove this, so proof see TIFR pamphlet, okay so as a immediate corollary you can write it homological dimension of the module M is supremum over N where there exists and A module N with $Ext^n_A(M, N)$ is nonzero,

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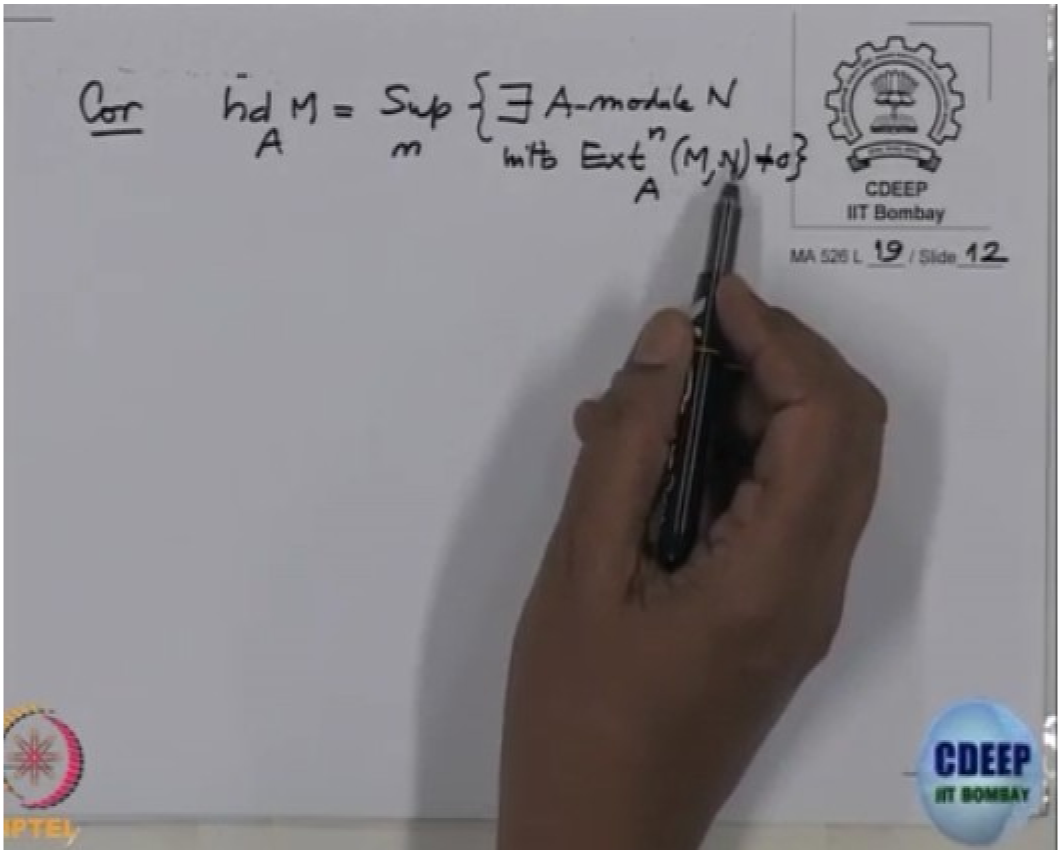

the largest n so that for some module N this Ext_A^n is nonzero that is the homological dimension of M there, because you see here this proposition says homological dimension less equal to n is equivalent to saying that Ext^{n+1} is 0, so if it is equality here this is 0 but earlier should not be 0, so that is $Ext^n(M, N)$ should not be 0, alright.

Another corollary I want to write it for future use, another corollary is suppose you have a module M, M is A-module and N' is a sub-module, A sub-module, if M' is a direct summand of M, then homological dimension of M' is less equal to homological dimension of M.

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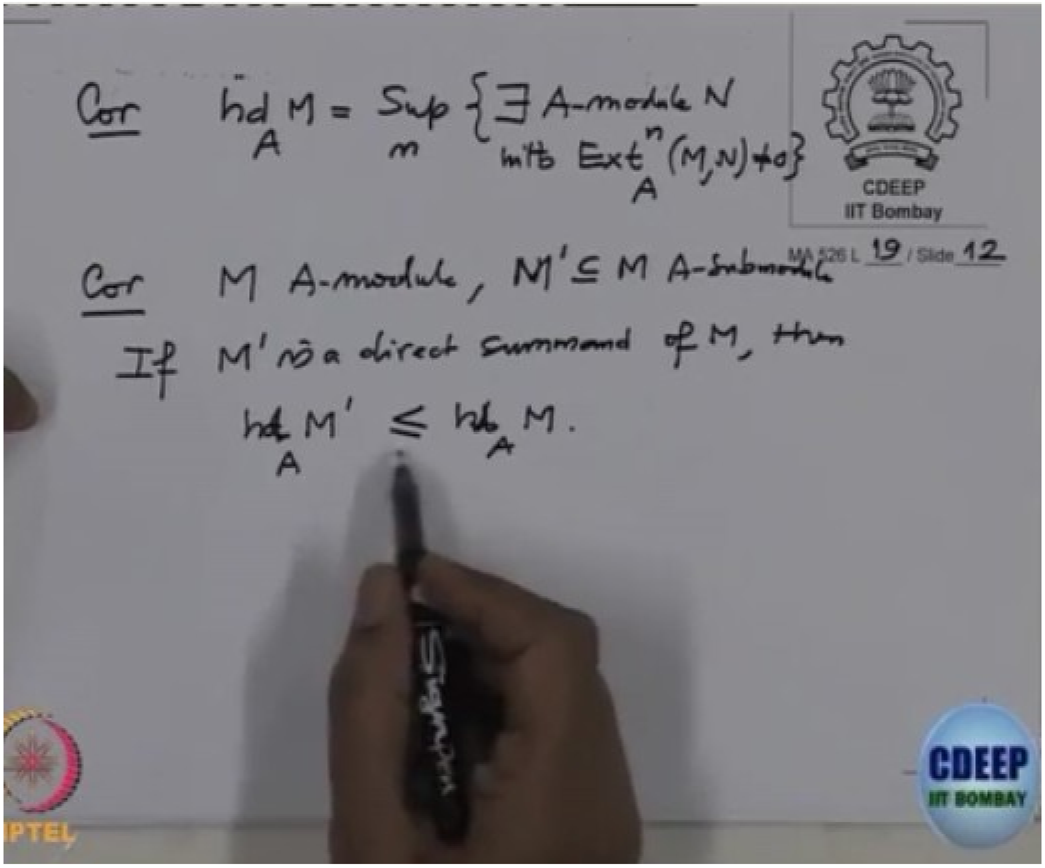

Homological dimension will not be more than the original module if it is a direct summand, so proof so I'll prove, I'll use earlier corollary, because I want to prove this homological dimension is small or equal to this, so first of all let us call this homological dimension to be n, then what do we know? We know that Ext^n there is a module N so that this $Ext^n(M,N)$ this is nonzero for some A module N, this is by earlier corollary.

Well okay, now when this is a direct summand that means what? That means we have $M^{'}$ here, M here, and this means that this M is $M^{'}$ direct sum $M^{''}$, so you can take the projection here and then this, that means this short exact sequence is split, so this is a split short exact sequence.

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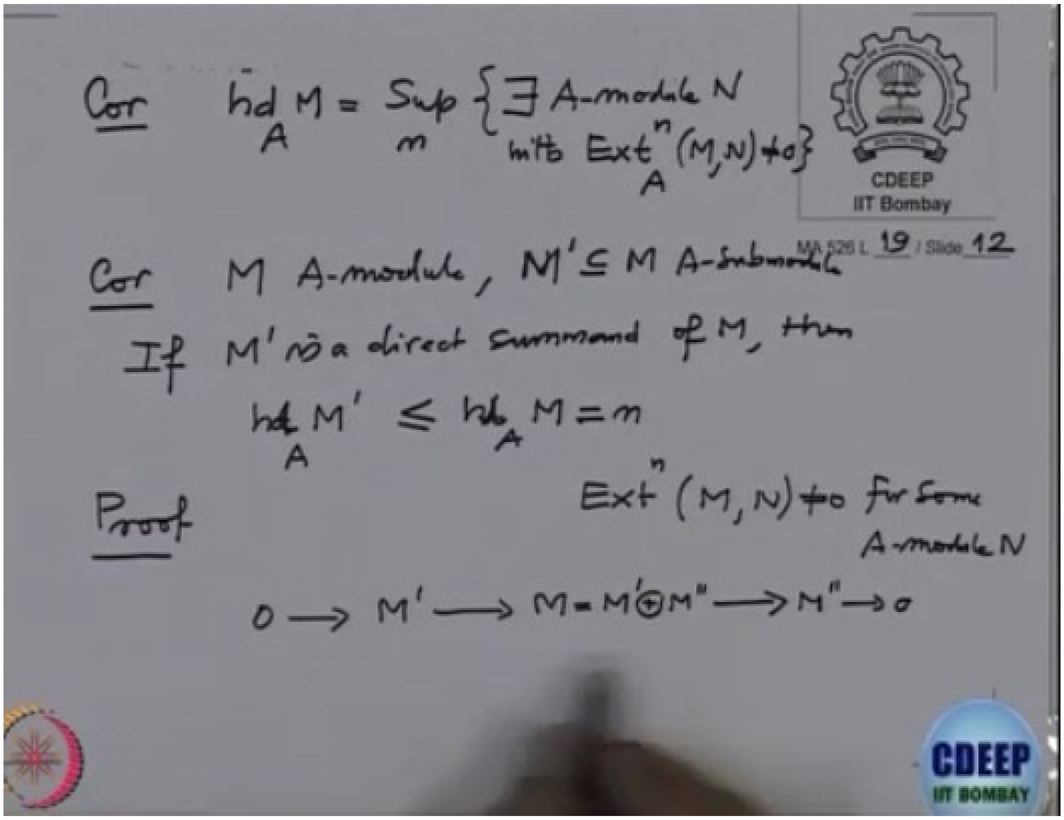

And you know when you have a short exact sequence which is split then it will hom, when you apply hom it will remain split so it will remain split, so it will remain exact so when you, therefore as a result this if I compute by using this and the long exact sequence then $Ext^n(M', N')$ is a direct summand of $Ext^n(M, N)$ this is nonzero,

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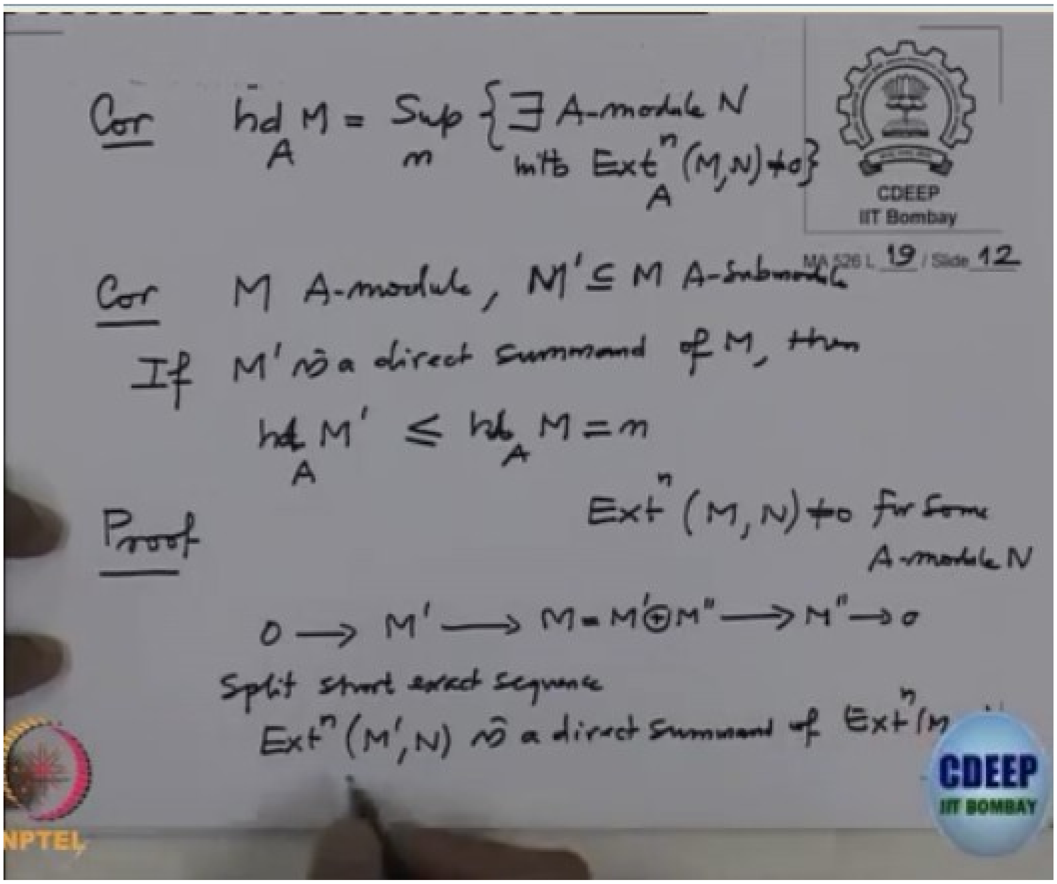

so this direct, it's a direct summand here so it may be zero, it may not be zero, and this is true for, so therefore a small or equal to. So the sub modules of a module will not have more homological dimension, okay.

So now, so another observation I will need is let us write it as a proposition, this is when we want to compute the homological dimension, so the earlier corollaries homological dimension of a sub module will be finite and it will not be more than over general module if the homological dimension of the original module is finite, so this is similar, so proposition so suppose we have short exact sequence like this M' to P to M'' to 0, and where with P projective, we will arrive such a short exact sequence typically when you try to resolve some module and break the big resolution into short pieces, because what will you do then, you will take the image and kernel, so this will be the image and that will be the kernel, so this will be the typical situation there.

So then when is this projective? And when is this projective, that is what our concern is, and what are the homological dimensions, right. So then one says if M'' is projective, then this is also projective, then M' is also projective. 2, if homological dimension of, so this is like a, this one is very special case, so you should read like this M'' projective means homological dimension is 0, this homological dimension is also 0 because of its projective, so then this is projective then this homological dimension is also 0.

Now I'm with the second statement I'm writing is if this homological dimension is something then what is the relation between the homological dimension of this, this and this? So if homological dimension of M'' is big or equal to 1, then homological dimension of M'' then what it is, this one is homological dimension of M' + 1. And it may happen that both these are infinite size,

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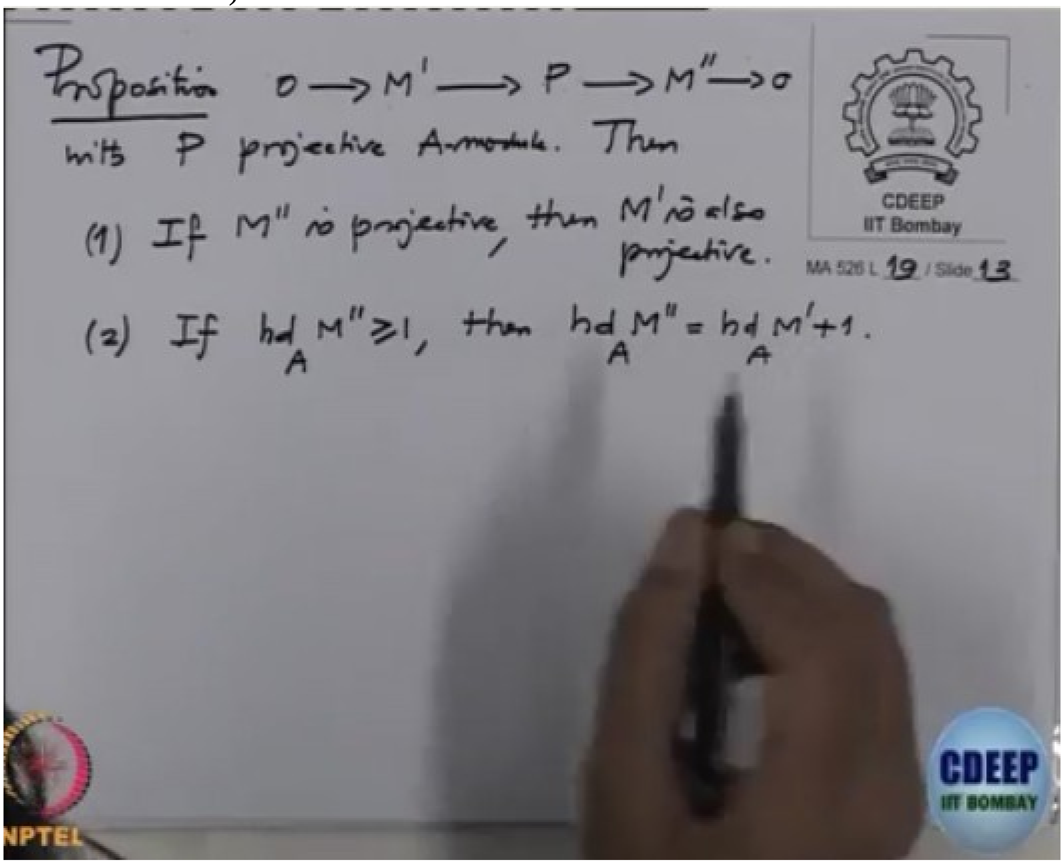

both these are it may happen both sides are infinite, you know in that case this is nothing to prove, so okay, so let us prove it that will reveal, okay so first of all the one is clear because we have this exact sequence and we are assuming this is projective, if this is projective then it will be split exact sequence, that means this M'' is a direct summand of P, but direct summand of a projective module is also projective, and direct summand but what will be the other complement will be this M', therefore one is clear, so one is clear since direct summand of a projective is projective, okay.

2, okay so when I try to compute now the Ext from this sequence is what we have to do is we have to compute the Ext to decide what is a homological dimension, so when I apply, so you take the projective resolutions of this each one of them, apply hom to hom dash end to that and then compute the homology, and then we will have, we will get exact sequences of complexes and then we will have to use that connective homomorphism, but then see in the middle module is projective so I will take this special projective resolution for that, namely only one term projective resolution, and all this so if, so in that case for, I'll write the result for an A module N and any integer n, what will we get? We get $Ext^n(M', N)$ and then $Ext\, n+1(M'', N)$, see this will be connective homomorphism,

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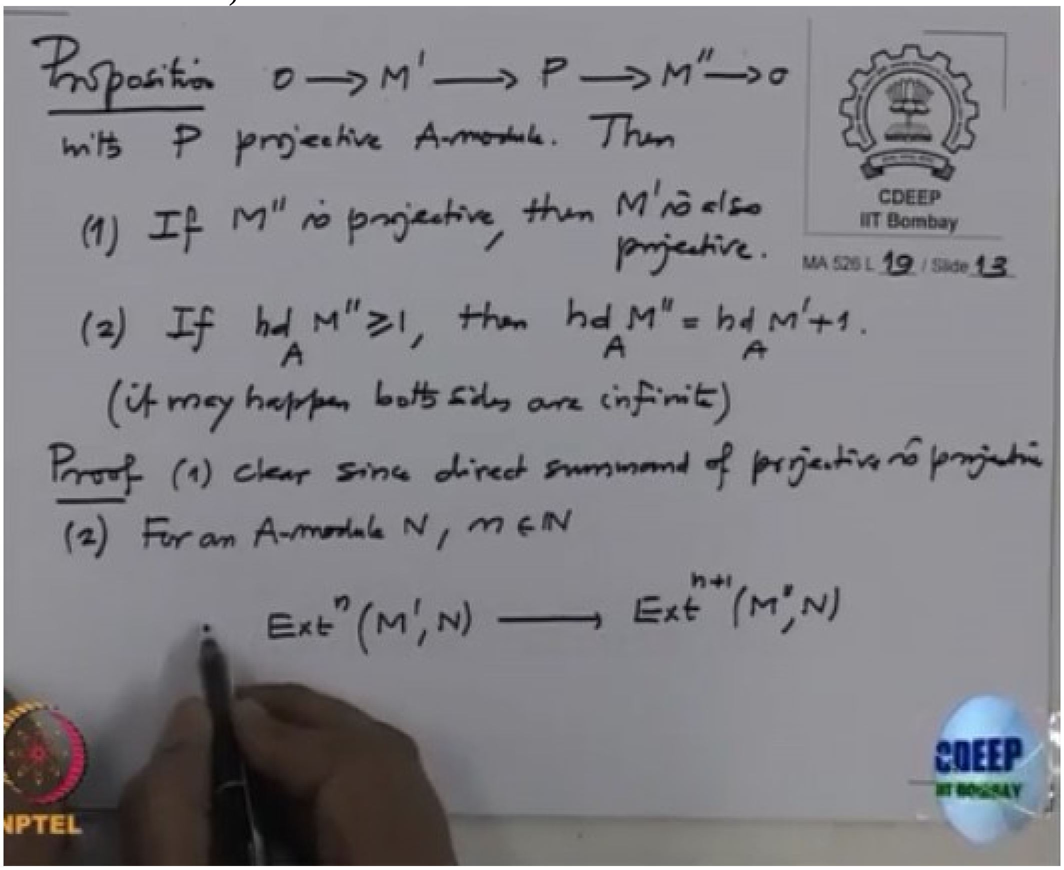

see before that this thing will come, but that will be 0, and after that also it will be 0 because then I'll put 0 here and put 0 here, so putting 0 here means this modules are isomorphic, but once this modules are isomorphic, so since here we have used the fact that, since P is projective,

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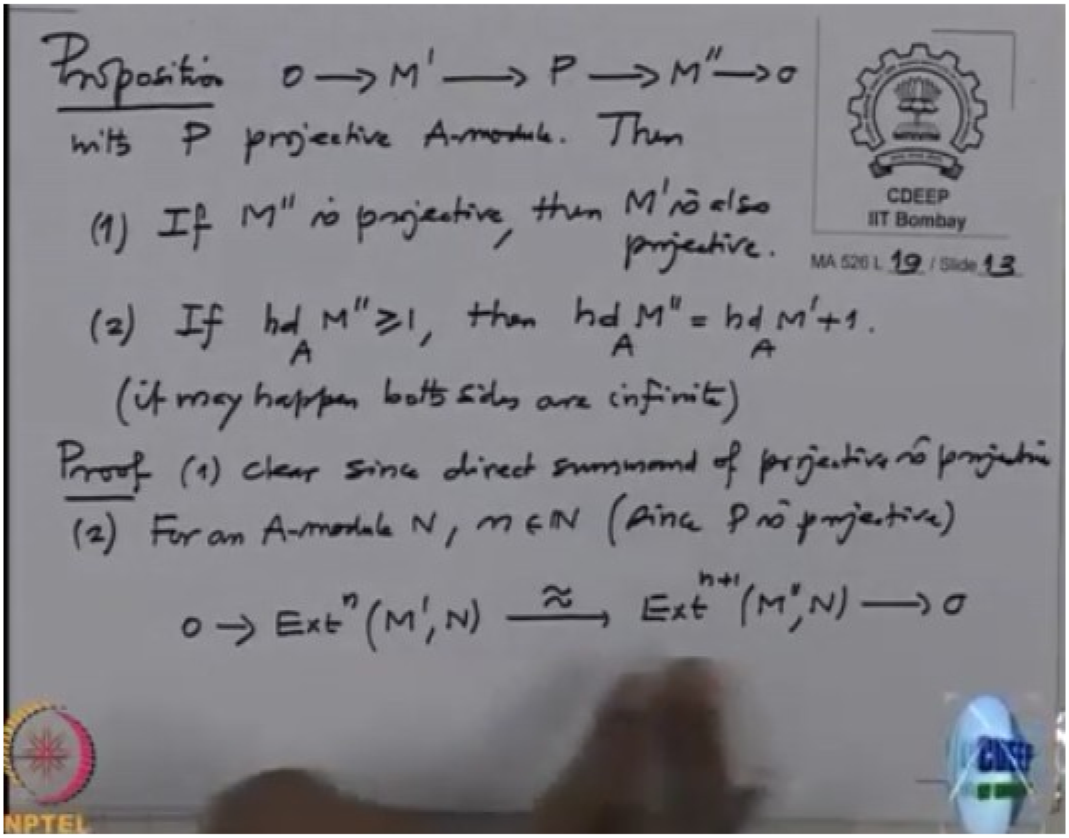

so if this is 0 some shares in that will be 0, but that will mean this formula, if this is 0 then this is 0 so that means the homological dimension of M'' will be exactly one more than that, so that proves this one.

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