Lecture – 47

Projective Modules

Gyanam Paramam Dhyeyam: Knowledge is supreme.

Good afternoon. I will continue our homological algebra formula lectures. And in this homological algebra I will not prove almost anything but I want to make itself content by giving definitions and also some examples. So, for example last time we saw, what is a projective module. So, let us recall briefly. So, A is our fixed commutative base ring and we are studying the category of A-modules. And the idea is to get information about the ring from this category. That is the main idea. Okay. So, module P is called projective. If-- we saw definition there was a two equivalent conditions. So $Hom_A(P, -)$ is under, from category of A-modules to category of A- modules is exact. That means it maps exact sequences to exact sequences. And because this Hom is left exact what fails is the right exactness and that this spelled out in the definition. So this is equivalent I'm saying for every subjective map were given M $M^{''}$ to 0, his means subjective. This is p and given any A-module homomorphism from p to $M^{''}$ f, we should be able to lift it here. So that this diagram is commutative. So, given this their exist f bar from P to M, such that with p composite bar is f, then you call that module is projective. This also is equivalent I'm saying that P is a direct summand of a free A-module. And free module is the module which has a basics.

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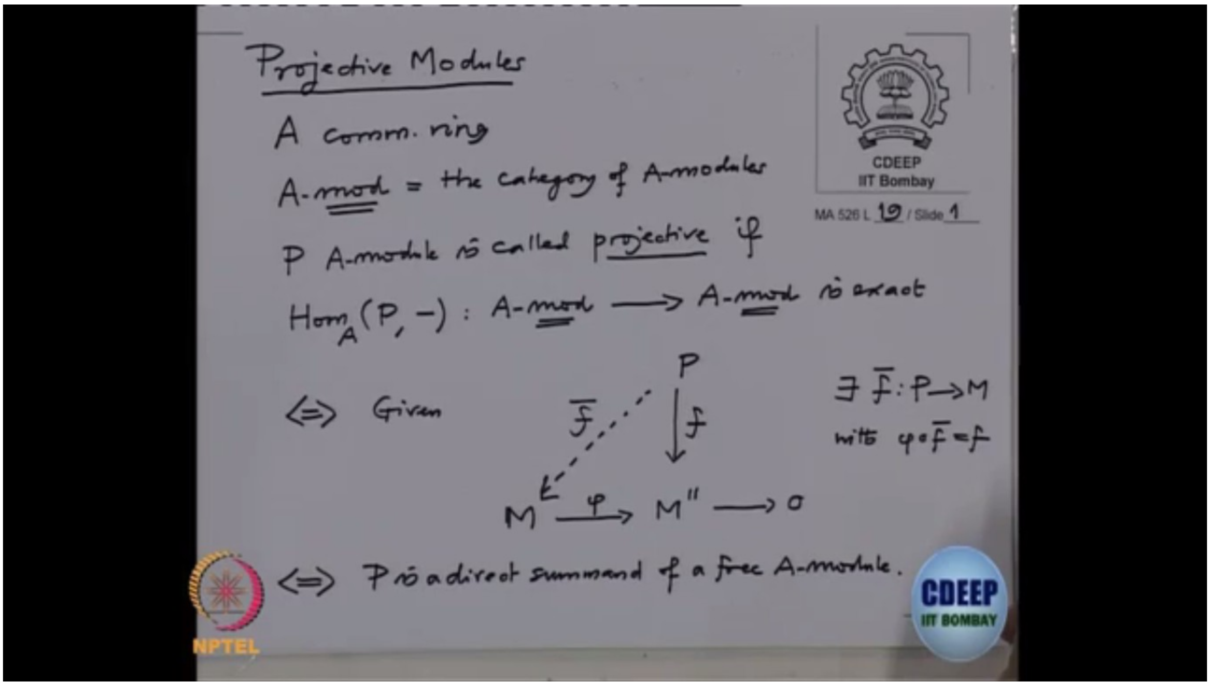

In general every module may not have a basics over arbitrary commutative rings. So if a module has a basic then you call it a free module. And in one proof that the number of elements in a base is constant. I mean, its any two bases of the same cardinality like theorem in linear algebra and for this commutative of the base rings is very, very important because this statement is not true for non-commutative rings. So there are examples of a modules, free modules or a non-commutative ring in which they may have two bases may have different cardinalities. But any way our problem is because we are assuming all the way it is commutative, so we don't have to worry about it. So, actually the third part of the definition, I mean third condition, equivalent condition of the definition tells you which are the examples of the projective modules. For example, So, some examples, free modules are projective. Free A-modules are A-projective, because it's a direct summand of itself. So, just to show that there are examples of a projective module which are not free. So for example, if you take the base ring to the $\mathbb{Z}_6$, then $\mathbb{Z}_6$ is same as a ring as a homomorphism. $\mathbb{Z}_2 \times \mathbb{Z}_3$, so therefore this a free module, A is a free module over itself and this is $\mathbb{Z}_2$ as a module over the $\mathbb{Z}_6$, it is a direct summand but it can't be free because if it is free it will have a base and then the base is, I mean, any free module will have cardinality equal to cardinality of the base ring power to the rank of the module. So it will be, in this case it should have at least rank 6 but it has the cardinality 6, but it has only 2. So, therefore $\mathbb{Z}_2$ is projective $\mathbb{Z}_6$ module but not free. Okay. Another, I will mention this because this topic is very interesting and also right now last many years it is a very front line research topics. For example this is regarding, whenever the projective modules free, So when are projective modules free? So, for example we are interested in a polynomial ring for example. Even for a polynomial ring over a field. So even when you take A is equal to, so, in this ring. Actually this question originated in 1955 by J.P Sears, and he is motivation to read this question over the fallowing. So look at the polynomial ring in n variable over a field, so this is very special ring. So he asked the question is every faintly generated projective module over $K[X_1,\ldots,X_n]$ free? And this question was there for a while and ultimately the proof came only in 1976. Two independent proofs came, one Suslin and the other Quillen. This answer is yes.

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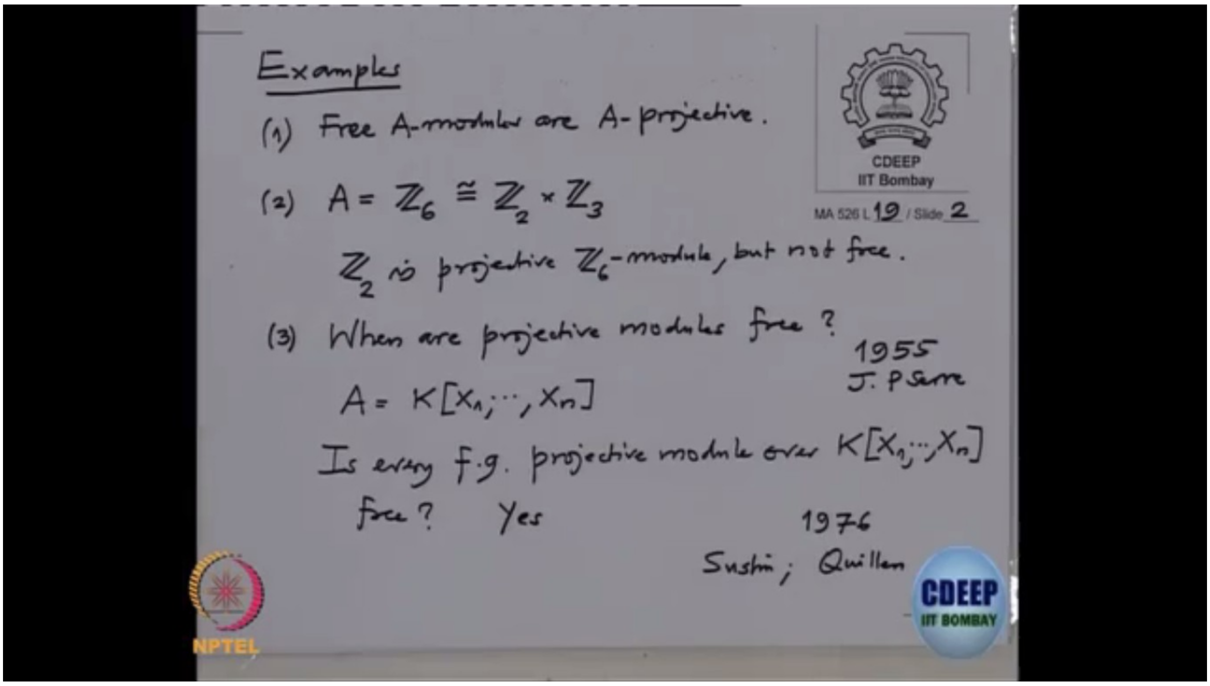

But why did he asked this question, that motivation I want to raise here because it is interesting. So remember these polynomial ring $K[X_1,\ldots,X_n]$ study ideals et cetera in these, you have seen ideals, prime ideals, maximal ideals, study of this, in this ring that was closely associated with algebraic geometry. And naming Spec $K[X_1,\ldots,X_n]$. This topological space, it is I square topology. This is also call some people to find n space over a field. This is also easily denoted by AnK, This is not K^n but this spectrum is also known as a affine scheme, actually it is called affine scheme. So, this is algebraic part, this projective module be free an algebraic part. So what was the geometric part? The question is formulated this algebraic question is formulated from a geometric motivation namely. So you would have shared in algebraic topology, for example Rn this is the Euclidean topology, the usual topology of $\mathbb{R}^n$ or a $\mathbb{C}^n$. This, with usual topology these spaces are so called contractible. If you don't have much knowledge I think you can ignore it but those who have or want to have this is very interesting. So these spaces are contractible. Contractible means identitive map is Homotopically equivalent to the constant map. That is contractible of this space. So, you know, in particular it implies π_1 for fundamental group is 0 trivial. So, now you want to formulate this contractible to this, this is our topological Spec now, and then you can also one would like that analogy here holds so that means this topology, this spectrum, it is a risk topology or it is contractible Spec that is a question. And this question, then he converted into algebraic language it became projective module free or not. So that is why, and beside this affirmative answer to this question will have lot of consequences in algebraic geometry. For example, this also connected to the fallowing fact, So for example, when you take curves, Alzebric curves in three space. Let us see $\mathbb{C}^3$. So typical examples that, for example if you take a cylinder and cut it or cone and cut it. So conic section in this, so curves in these space the question is whether are they defined by two equations. So if you say irreversible curve, they will correspond to the prime ideals p in the polynomial ring in three variables and they will be height too. Height of p is 2 because the dimension of $\mathbb{C}[X, Y, Z]$ mod p should be 1 because there are curves. So height should be 2, and then this, so they lead at least two equations to define but the question is whether are two enough.

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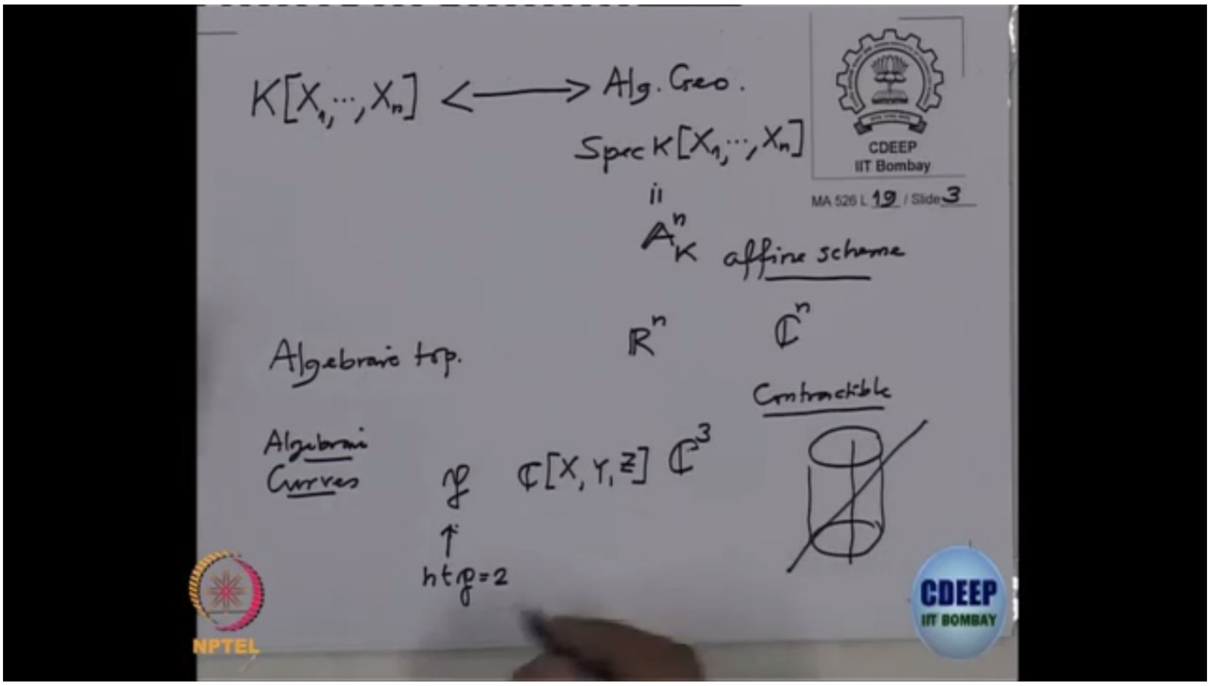

So that means the question is translated into completely algebra by given a prime ideal p of height 2. Height p is 2 is p the radical ideal of 2, ideal generated by the two elements. Because if it is so then in geometric language $V(p)$ will be equal to V of f intersection $V(g)$. So that means these curve is intersection of two surfaces. But this answer is still today not known even for $\mathbb{C}$ and three space. So this not known in general. Yeah, this is called Set-theoretic complete intersection. It's only Set-theoretic not an ideal theoretical. Ideal theoretical, the answer is easier to give examples. So this closely connected to projective modules been free. So more than that I will not go into that because only thing I will say that projective modules over polynomial ring in n variables over a field are free. So therefore, when one looks for example, when can not look over polynomial over a field. Okay. Now, next is, see we are trying to define a homological dimension, so by using homological algebra we want to define what is a homological dimension of a module. So first we note that give any module M or A, you can always write this M as a coefficient of a free module.

If we take any generating set, take a free module with that bases and map the bases element to the generating set. Now you can repeat this process. That means you repeat, look at the cornel and write cornel as a coefficient of a free module and that way you can generate, you can hook up a sequence which is exact sequence of A-modules and all these free, the modules are free. These are all free modules in particular they are projective modules. Such thing is called a projective resolution. So when I do a little bit better by not, what is called minimal resolution.

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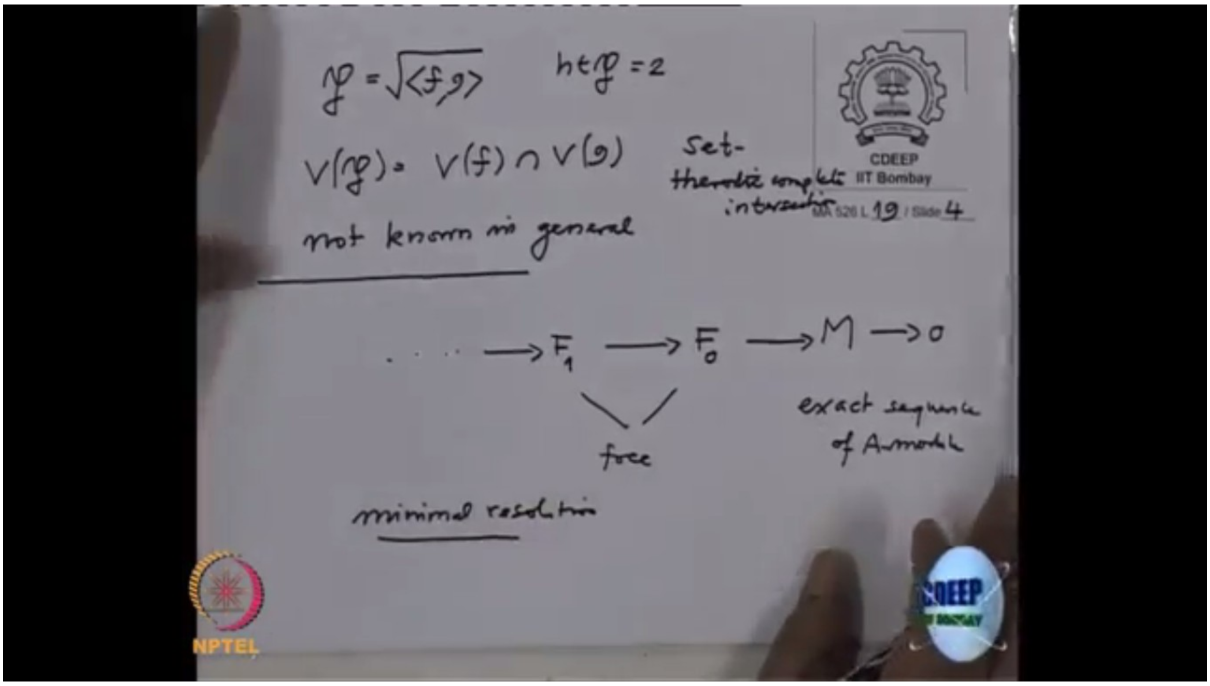

So that is called the projective resolution. So, what we know is every module as a productive resolution. Every module you can write a resolution where each member there in the projective module. Okay, now give this minimal means, minimal resolution makes sense only when the ring is local. So that means the image should not be contain inside the M times, you know, that you choose every time the minimal number, the minimal system of generators. So that means under these maps that M times the next one. So that---

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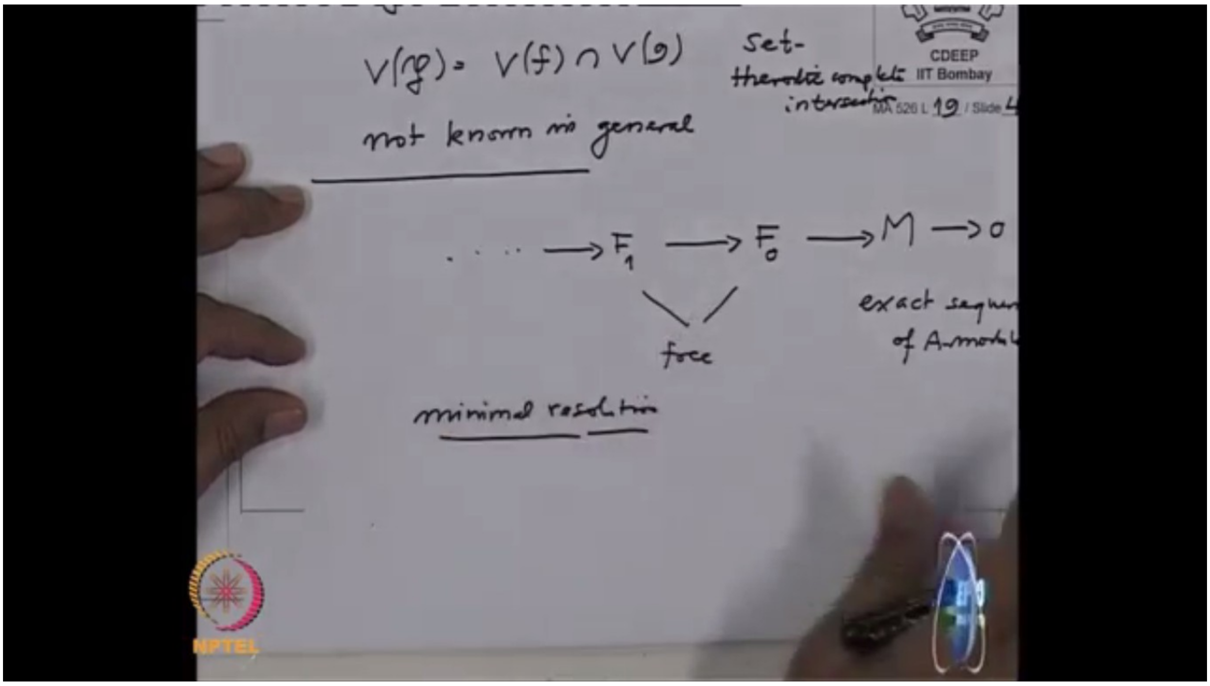

Okay. So, I will come back to it when we calculate something, so every module has a projective resolution and the projective resolution is denoted by $P.$ what do I call it epsilon. This is the projective resolution of M. So I have, when I apply Hom, if I apply now the Hom functor, now I have to be careful because Hom in contra variant so if I apply now, if I apply Hom all these you can, all the proofs you can find in this TFR pamphlet number 5. Okay. There are two things I want to define now. So we have projective resolution and so that is usually denoted by $(P.,\epsilon)$ to M, this is given. Now if I tensor this resolution with some module N, $\otimes N$, if I apply this functor then I will not get exact so I will get only a complex. So that means P dot this to $M \otimes_N A$ and , this part will remove an exact because tensor is right example. So this repetitively will be kept. So this when I can say this from $P_0 \otimes$ end to $M \otimes N$ that part will remain exact. Earlier it may not remain exact. So, when I take the Homology there, so $H_n(P. \otimes_A N)$. How do I calculate this at the each stage you look at cornel mod image. So, this is, if I take nH stage means I have to take NA. So this is at $P_n \otimes_N A$ to $P_{n-1} \otimes N$ and earlier N is P_{n+1}, tensor N and we are looking at cornel here mod image here. So this is cornel of dn tensor identity mod image of dn plus 1 tensor id, this is an etymology of this resolution. This has a special name. This is called $Tor_n(M,N)$, this is by definition $Tor_n(M,N)$. And now there are several things to check here, namely one, somebody may start with the different projective solution so we have to check that this definition doesn't depend upon the projective resolution. So I will say that, check that this definition does not depend on this projective resolution.

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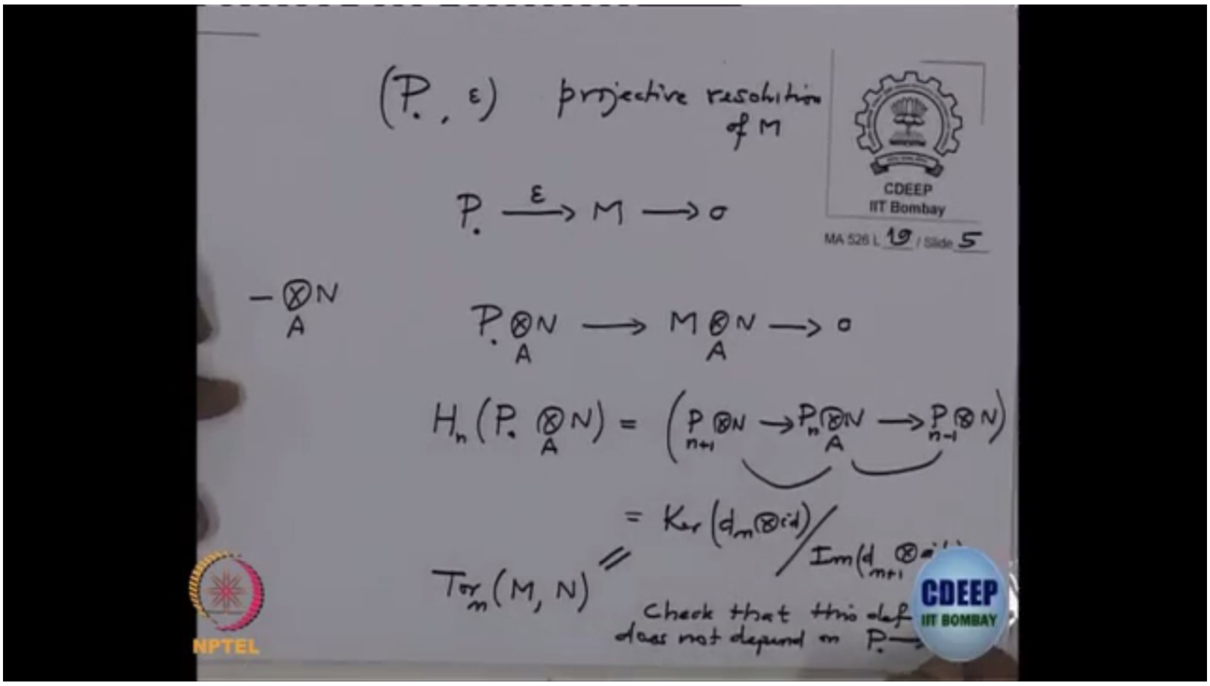

Any similarly when we apply now, the other functor that we are interested in the Hom. So when you apply Hom also we will get, and as several properties of this Tor and X. Now you apply Hom A dash N, on this is tensor functor was co-variant, this is contra variant. So the arrows will get reverse. So it will now start the other way. So now it will start with Hom is left exact, so this is 0 $H^n(Hom_A(M,N))$ and then $Hom_A(P.,N)$ and so on. In general this is, this should be $H^n(Hom_A(P.,N))$, now its long axis. So you get therefore the complex again and know we can again take Homology. But as we said when it is on one side of the 0 the suffix is N is denoted down or up. So, like Tor N these are denoted down. So here in that case I should denote it up. So that Homologies or called Homology, they are denoted by $Ext^n(M,N)$. So this is by definition. $H^n(Hom_A(P.,N))$. Again the same that is cornel by image. And again now here, this doesn't depend on P dot. So this definition does not depends on $P.$. To $M.$ So if you take suppose different projective relations then you get this modules and their A- modules.

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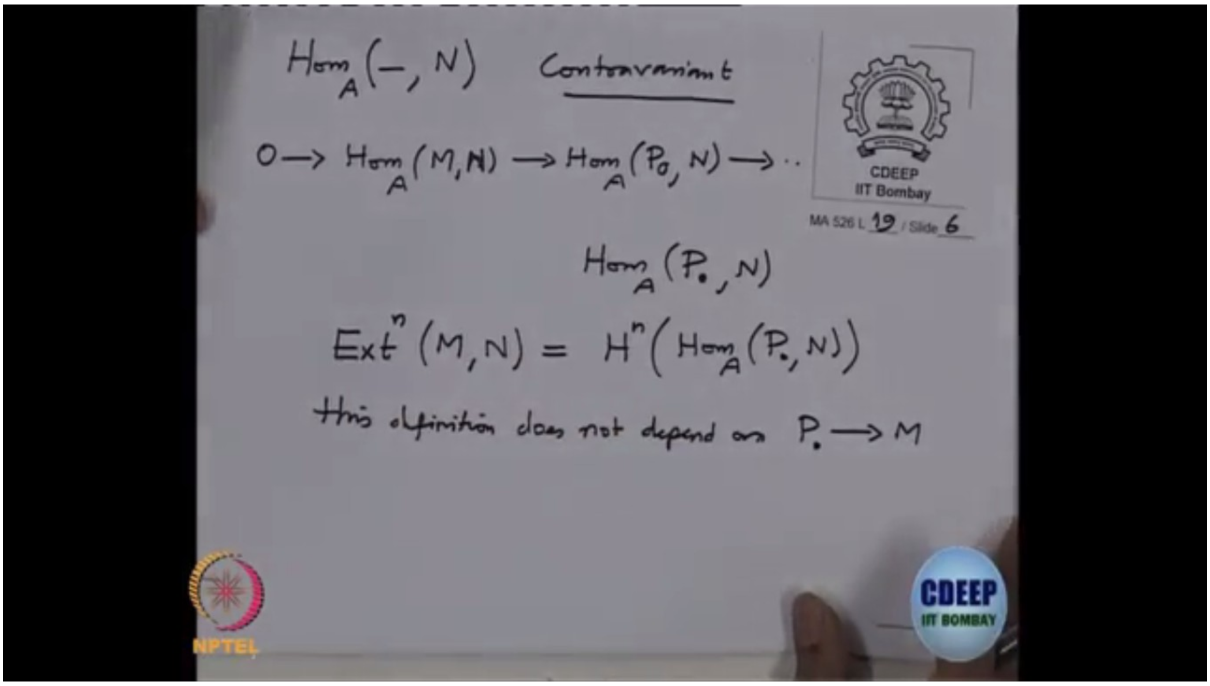

So, okay, now, let us summarise and then I will extract the properties of these module that we will need to use for our calculation. So for two modules M and N we have attached the first which it came from the tensor product that is $Tor_n(M, N)$ and these are also A-modules. And not only that , do you remember last time when we had a complex and then we wrote the Homologies it had a long exact sequence and connecting homeomorphisms. So this one, this Tor difference of Xs are connected to each other by the long exact sequence. And similarly we have this $Ext^n(M, N)$ these are also A-modules. And if M, N are finitely generated these are also finitely generated. Because they are coming out of how are Tor and Ext are coming out of either applying Hom or tensor product and taking the coefficients. Coefficient means a sub module and coefficient modules and all these if your basing is good, noetherian, then all of them are finitely generated. Therefore all these modules are faintly generated. So, one more thing. So here, remember we have taken projective resolution for M and tensor with N and to the Homologies. So one could also do this by taking a projective resolution of N and tensor with M but these two results will be same because of tensor products are commutative. Now, $M \otimes N$ is isomorphic to $N \otimes M$, so the resulting modules will be also the same isomorphic. But in this case we can not do that because $Hom(M, N)$ is not $Hom(N, M)$. So either you resolve all these M projective and then take the homologies or there is a other way which is more difficult than this projective modules. Mainly take the injective to the illusion of N and then Hom and then do the operation. But you know, for a module to prove there is a projective resolution is much more relatively easier than proving that every module has a injective resolution. Because the structure of injective module is much more complicated then the projective modules.