Lecture – 46

Complexes of Modules and Homology

GyanamParamamDhyeyam: Knowledge is supreme.

So, the proposition is, and this is used many, many times not only in commutative algebra but every subject. It's like a machine. So, let $0 \to X. \to Y. \to Z. \to 0$, let us say this is f and g, be an exact sequence of complexes. Okay. Then, the sequence, which sequence? Now, I write the sequence. So, $H_n(X.)$, what does this mean? This means, when you look at this complex in nth level and nth place, this is homology, that means, this is kernel by image. If for some reason, if you look at this n to be negative side, then one should write it here. But don't bother about it, only think at which position the kernel by image, that is called nth homology of this complex. Then, $H_n(Y.)$, and then $H_n(Z.)$, and there's a map here which is induced by this f, so that is also denoted by $H_n(f.)$. We will define this, $H_n(f.)$. And here, $H_n(g.)$. Right. So, at the next level, that means, if we chase this position from n to $n-1$, then we have this also know, $H_{n-1}(X.)$, $H_{n-1}(Y.)$ and so on. And now, so this is $H_{n-1}(f.)$, this is $H_{n-1}(g.)$, and this side also there is. And now, so we will define these maps and what is called connecting homomorphism. So, these pieces are connected by the maps here, those are called delta, delta n. This is called connecting homomorphisms. Okay. What do we want to define? We want to define this $H_n(f.)$, $H_n(g.)$, δ_n so that this long exact sequence that we got is exact. That means, that each stage there, kernel is image is exact sequence of A-modules. And ultimately, we will measure the homological dimension by how long is this sequence, know, that with the major. And then we will define what is homological dimension of a module, and then we will characterize the regular local rings by looking at the supremum of the homological dimensions of the modules over that ring. So, right now, our problem is to define $H_n(f.)$, $H_n(g.)$, $H_n(f.)$ that means, similarly, we will define these also and the connecting homomorphism. Okay. So, how are we going to define this? The best is to draw a diagram. So, obviously, this is what, this is by definition, it is at nth stage, that means, X_n is involved. So, this will be X_n, so the kernel will be inside X_n and the image is also inside X_n, so this is some sub-module of X_n divided by some other some module of X_n, right?

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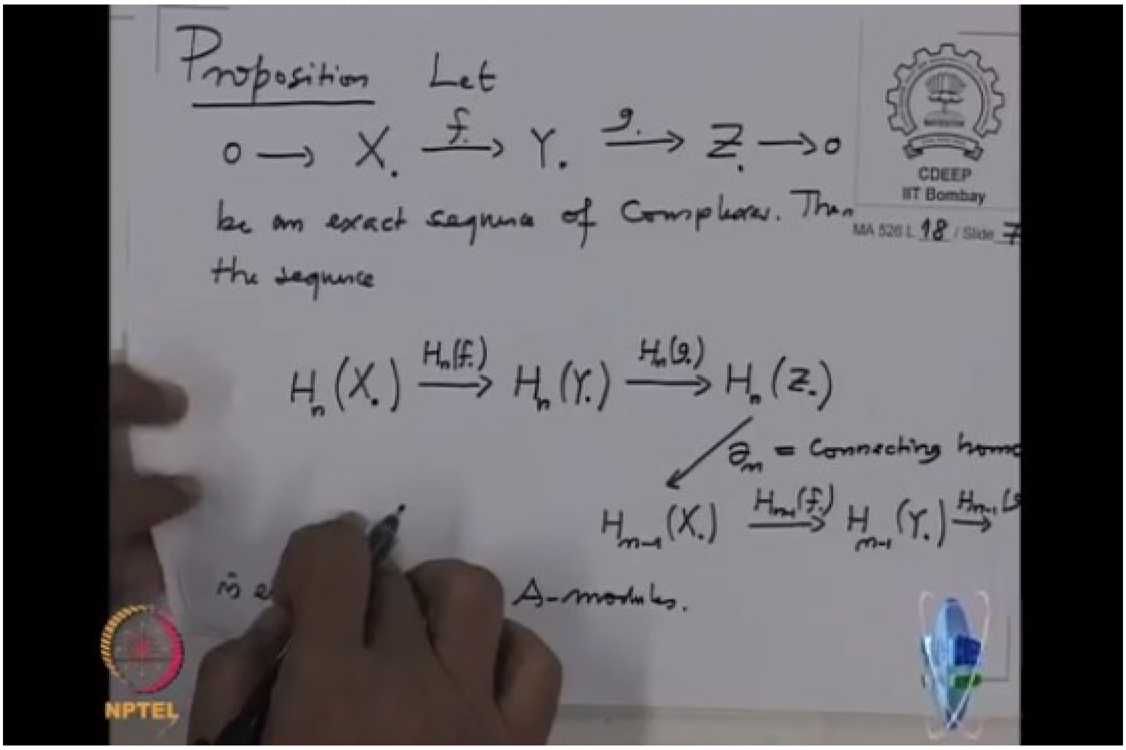

And, where are the sub-modules are coming from? They are coming from the Kernel's of a kernels and images of this f_n s. Okay. So, we have come down to this level. So, 0, go to nth stage means, we have to read this given complexes at nth stage. So, we have, because the complexes sequence is exact, we have this X_n , Y_n that is f_n , and then Z_n that is gn, and this is zero. And remember, we want to define from homology at this stage means, this X_{n-1} here, X_{n-2} , this is d, so d_n , this is d_{n-1} . Oh, for some reason I've chased to the lower, it doesn't matter. Okay. This is Y_{n-2} , this is Z_{n-1} and of course, this Z_{n-2} here. This is exact here, this is exact here, and maybe here also I'll write one term, Y_{n+1} , this is Z_{n+1} to zero. This is d_n , d_{n-1} . Such a thing is there, such a thing is given to us. And what are we trying to define? We are trying to define a map from $H_n(X.)$, this to $H_n(Y.)$. This is what we are trying to define. So take any element here, how do any element look like? This is, obviously, the Kernel by image. Kernel of whom? It is this stage, right? So, here, Kernel of d_n mod Kernel of d_{n+1} , image of d_{n+1} . So, this is Kernel of dn divided by Kernel of d_{n+1} , by definition, this is by definition. Okay. So, take any element here. Any element here is a class, residue class of some element in the kernel d_n . So, therefore, then element I will denote by, say, what we denote it by, $\overline{z}$ or z, or $\overline{x}$. And we want to define this map, right? So, what is obvious choice? I just take this x, so that is, this is in here, this x is here. It's in the Kernel of d_n , in particular it is in X_n . So, it goes zero here because it's in the Kernel of d_n . So, obviously, what do we try, that is, $f_n(x)$ we apply f into that and take the bar of that. But first, you have to check that this f_n is indeed in the Kernel of this, and we'll take the image there. So, this way it goes to zero , so therefore, this way it will go to zero. This way it goes to zero mean, this $f_n(x)$ belongs to the Kernel of d_n . So, therefore, this makes sense, because, this is by definition, this is the Kernel of d_n divided by image d_{n+1} . Dn is dnY really, right? So, what we observe that, this X_n is if within the Kernel of this, then fn of that is in the Kernel of this, so therefore, it makes sense to talk about the residue class image of that Kernel by

d_n . So that is the map. And now, we have to check that this is well defined. That means, if I the different representative class, then they are same there, but that is, I'll leave it for you to check that this is well defined. So, similarly, you would have defined from here to $H_n(Z.)$. So, this is the easy part. And check that it is exact, means, you have to check that if somebody is the Kernel of this equal to the image of this. But that will again, follow from the fact that, this each part is exact, each layer is exact. So, from that, you check that way. But the more, little bit trick y is defining that connecting homomorphism.

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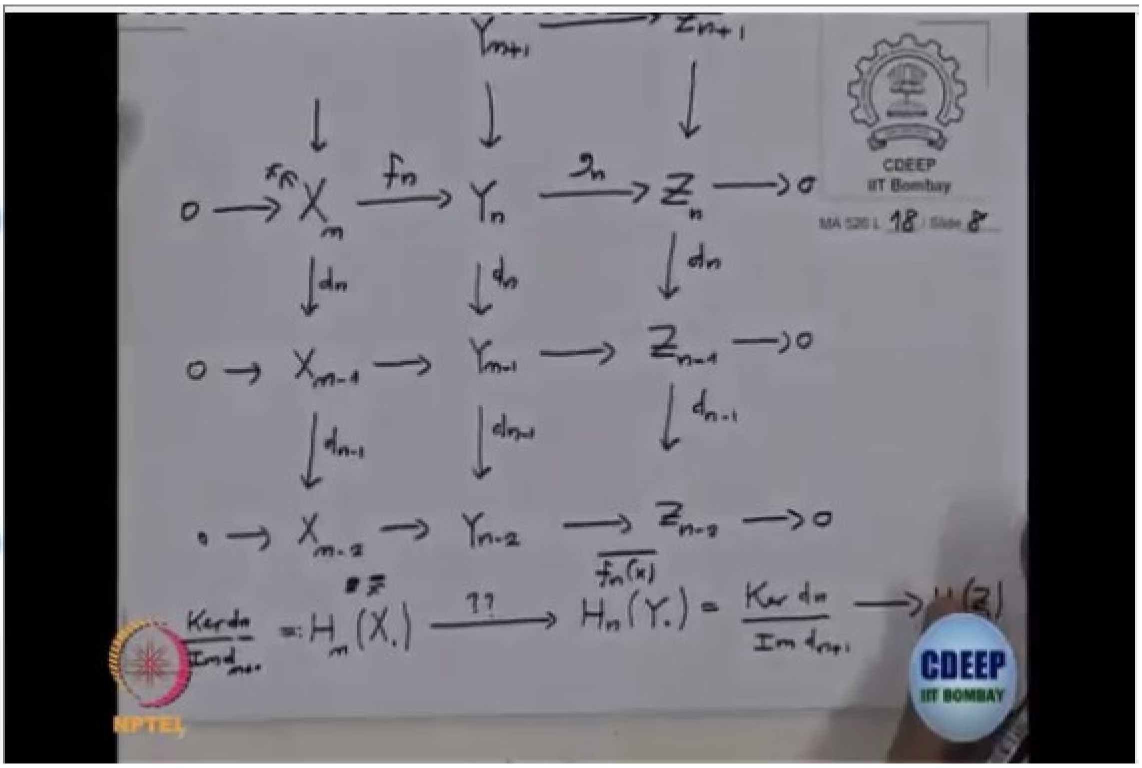

So, connecting homomorphism means, now what do we want to define? We want to define from $H_n(Z.)$ to $H_{n-1}(X.)$. This is what we want to define. That means what, that means, where is this $H_n(Z.)$, that is somebody here. And from here, we want to go down to here, right? So, we will have to chase the diagram. So, what do we do? Okay. So, you take an element in here, that means, it is here, right? Here means, it is in the Kernel of this module image of this, so we have taken an element z here. And what are we looking for? We are looking for somebody here, we are looking for somebody here, then it will be in $H_{n-1}(X.)$. So, this is Kernel here. So, this is rejected, so you choose somebody here which goes to there. Then, because it goes to zero here, that one will go to zero here. But then, this one go to zero here, that mean, that image here is going to zero here, that means, that image here is coming from somebody here. And that is the required pre-image we took, right? Z is coming from here y, this y goes to zero here, so therefore, that is the image y here which is, let's call it y' , that goes to zero here. But this is exact, so therefore, this y' will come from here. And that is x' . Now you check that this x' is in the Kernel of this. But that, to check that, again, you chase it, what do you want to do it, and then you take the image of that, and that is the required map. So, this one $\bar{z}$ going that $\overline{x'}$. And we have check several things that all these definitions, they don't depend on the pre-image of the-- We are choosing x, we are taking the pre-image of z but it may have many pre-images. Okay. I will define that, then we have to check that this resulting long sequence is exact, this resulting thing is exact. That is also we can diagram chasing, because we want to prove at this stage, Kernel equal to image, and so that is also, then you take an element here goes to Kernel and prove that it is coming from here, so that you have to chase that diagram. That is called diagram chasing. Okay. So, that is complexes, okay. Okay, now, and again, if you have—So, the functoriality. Okay. So, let us first give some examples of functors. So, what is a functor? A functor is the analog of the function in set theory or a map in set theory, so that is set to set. Now, we don't have set, we have a category. The category has objects and morphisms. So, c is one category, and c prime is another category. And then, we are looking for some associations. Now, this association should compatible with the objects as well as morphisms. So, first of all, so this is usually denoted by script letter's f, f for functor. So, each object here, suppose XA is, not A, V is one, think of this as modules category. So, I will denote object by V, W et cetera. So, if V is one object here, it should associate object in this c prime, so that is $f(V)$. So, this is the association, in such a way that whenever I have a morphism here, the corresponding thing, corresponding objects which are associated under f, so they will have morphism between them. The question is, whether direction is kept or direction is reverse, and both the things are possible. So, if the direction is kept, that is called covariant. We will see an example. And if the direction is reverse, it is contravariant. And in such way that if I have a composition, then we have the composition whether you take a composition before and then apply f or apply f and take the composition, this is all should be same. So that is, all this together is f is called a functor. So, now two, three examples we will see. So, examples, concrete examples, so you take our category A-modules and the category of sets, let us take, where the objects are sets and the morphisms are maps. Here, the objects are modules, A-modules, and the morphisms are A-module homomorphisms. And I have functor here, namely f, namely this is a forgetful functor. You take a module and forget that it has A-module structure, think of it as a set only. So, this is associated to that. And then, when we have a module homomorphism, you forget that it's a module homomorphism, it's a map also. So this is true from any category to the category of sets where the objects were sets. Or you could have also done category of A-module to category of abelian groups. You forget the module structure but keep the abelian group structure, right? So, that's it. And note that all these examples are under covariant because the direction of the morphism is not changing.

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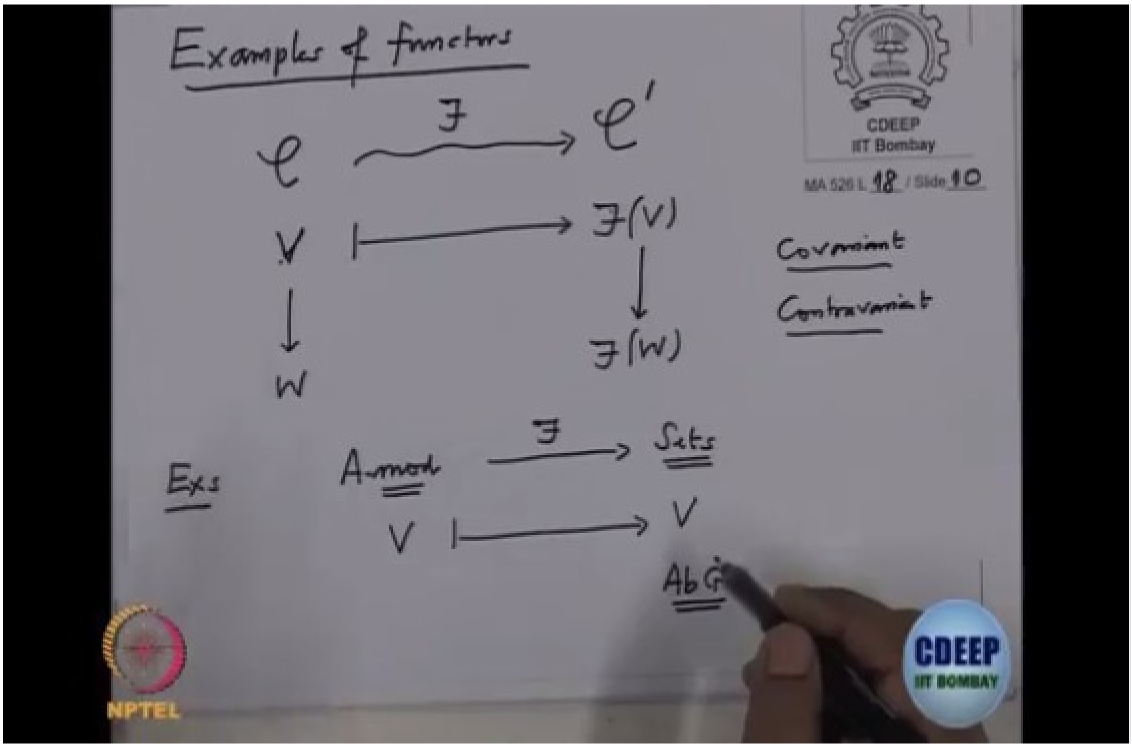

Whereas, now one example we're seeing, now category of rings, commutative rings. Don't take arbitrary rings, commutative rings. So, this is a category so that morphisms are ring homomorphisms. And note that ring homomorphisms for us, ring homomorphism means. It should carry identity element to identity element. That is not automatic, and many books don't assume this, but do assume under ring morphism one will show one. From here, we have defined a functor from this category to the category of topological spaces, and this was Spec. Any ring A map to Spec of A. Remember, the spec A is topological space with the Zariski topology. The prime ideal, set of all prime ideals, and there we have put a topology on that, and with that topology, this spectrum is a topological space, and the close sets were the $V(A)$ s. These are the close sets. This $V(f)$ is by definition, all those prime ideals p such that A is contain in p. And then, we have seen that, or you have seen in other courses that this collection forms, they satisfy properties of the closed sets in a topological space. And therefore, it is a topology, that topology is called Zariski topology. Now here, we have also seen that if we have ring homomorphism from A to B, then the map on the spectrum level is the other direction, because, how do you define it? You take a prime ideal q in B and then contract it, q intersection A, this is the math. So, the direction changes. And we have not checked in our course, but you would have done or maybe it's easier to check that if this is continuous map, this is continuous. If you take a Zariski topology here, the Zariski topology, then this is continuous from. So, this way, the gain is, we are going to study these rings, and sometimes, compare it here, the spectrum. For example, if you want to prove two rings are isomorphic or not. If they are isomorphic, then the corresponding topological spaces will be homeomorphic. So, for some reason, if you have information that one ring has only one prime ideal and the other ring has two prime ideals or more, share only the cardinalities, then they cannot be homeomorphic, therefore, the rings cannot be isomorphic. Like that. And also, you might have studied another functor which is also very useful, that is, from the topological spaces to groups, category of groups. So there is a nice functor called ϕ_1 , so in topological space x, you attach a group called first fundamental group ϕ one, the first fundamental group that is defined by using homotopy et cetera. And then, that is group but unfortunately, it may not be abelian group and so on, so that is usually one gets information. For example, if you want to prove some topological spaces are homeomorphic or not, this ϕ one is used. And not only ϕ_1 but there is a bunch of ϕ_n s, they are a lot of functor.

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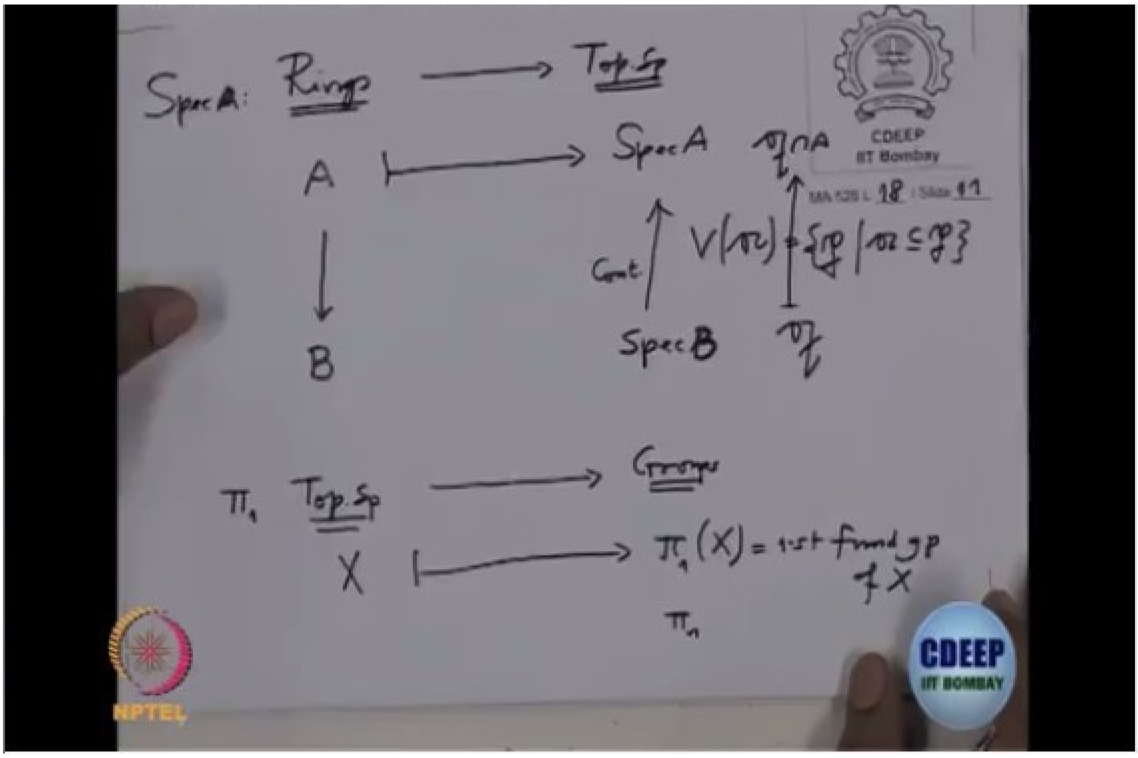

This study became more and more, in the late 20th century more and more common. Okay. So, another important functor for us, there are two funtors and probably, you would have studied their properties. Now, get back to our, A is our fixed commutative ring, and we have the category of A-modules. This is a functor from category of A-modules to category A-modules. So that means what? Given a module, I want to associate a new module. So, for example, I would have fix a module, some fix W, fix A-module, then you can do it $Hom_A(V, W)$. So, this obviously, W is fix and this V is varying, right? So, this functor is denoted by $Hom_A(-, W)$. Or why did you prefer one, so you could have also done the other way. So, W going to, now I'm varying W, same $Hom_A(V, W)$, Hom, this is, we are fixing the first module V and the other is varying. Now, I don't know whether you have studied this, whether the home is, what kind of functor it is, right exact, left exact, covariant, contravariant. Or that this, you have done this in your earlier courses? So I will assume that you know about the Hom functors. So, Hom functors. Also, we could have also done the tensor functors, so the tensor products. So, similarly, you can think, you can fix one module and vary the other one, right? So, the functors this tensor over A to W or V tensor over A-. These are the functors from same category to same category. And the good thing about the tensor functors are, they are isomorphic because they don't change. So, this is better and that's it.

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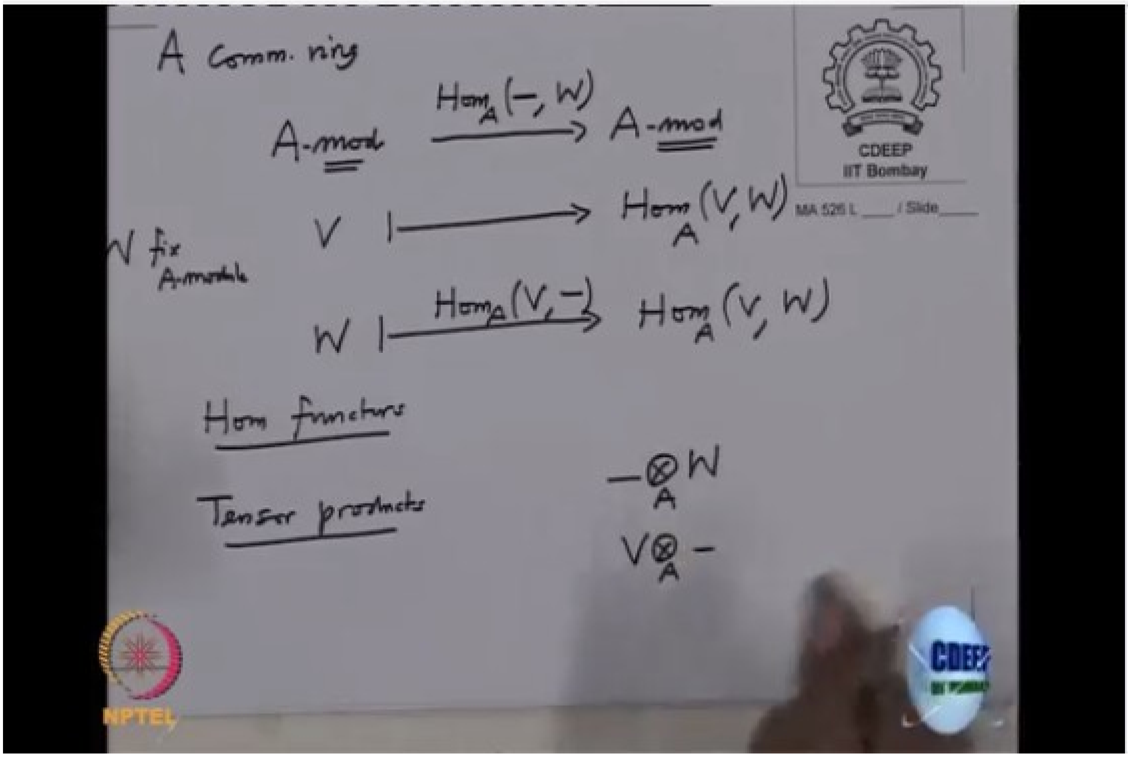

Okay, now with this, also you have noted that, when do you say a functor is exact? When it carries exact sequences to the exact sequences. So, f is called exact if whenever you have exact sequence V', V, V'' is exact, that should imply $f(V')$, $f(V)$, $f(V'')$ is exact. And I'm saying this f is from our situation A-mod to A-mod, category of A-modules. Only this is f prime, this is f, this is f of f', this is f of f. So, that's it. Now, we have seen that this $Hom_A(V,-)$, this is the functor from A-modules to A-modules. Is it covariant or contravariant? This is covariant, right? Because it map W to this, and if you have a map from W to W', you should have a map from-- And, what is this map? This map is you take a morphism f from V to W and then compose with this, so you will get in the same direction. So, this is covariant. And we have seen that this is may not be exact, this is not an exact functor, but it is which exact? Left exact. So, when do you say a functor is left exact, that means, only if you have s short exact sequence, then the exact names will be written only at the left side, therefore, it is called a left exact. This is more general definition, more general vocabulary there in homological algebra, but we don't need it. So, now the question is, when is it exact? When is this functor exact? So that will give a condition on V, for what condition on V there so that this functor is exact. Similarly, the other one, $Hom_A(-,W)$, this is now contravariant, because it will reverse the arrow. So, this is contravariant. Actually, the study will be similar, know, one will be the doable of the other.

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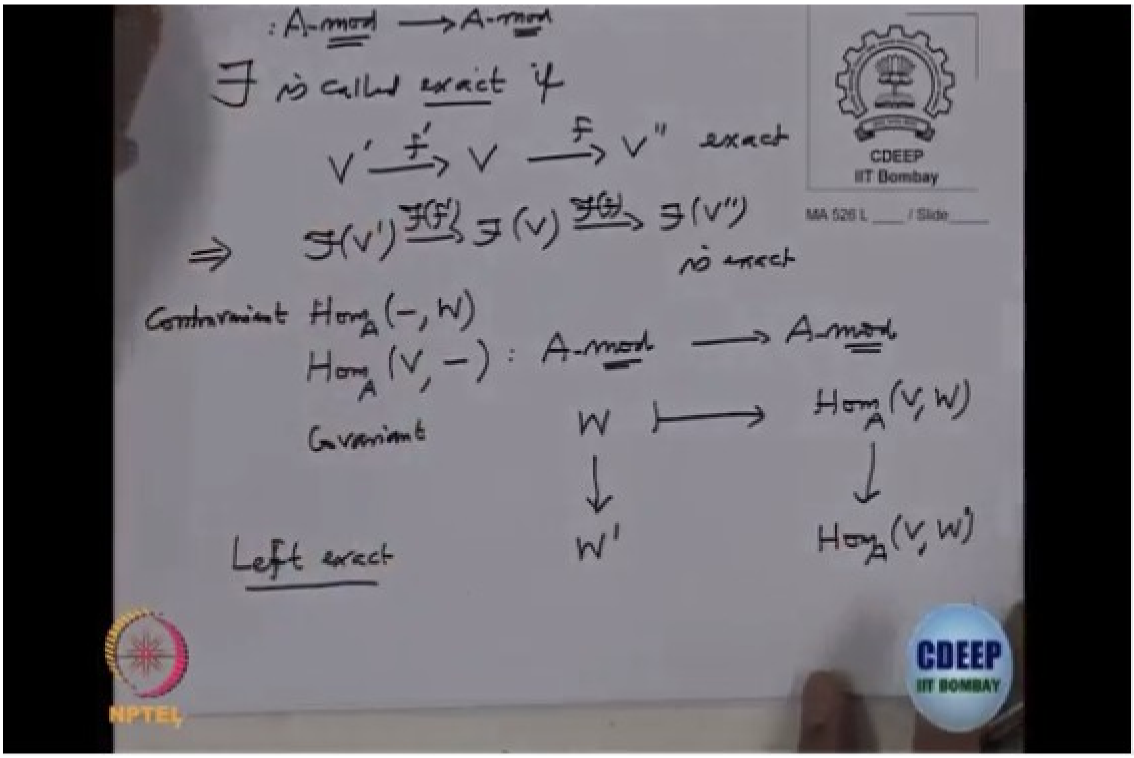

So, now I will define an A-module. So, definition. An A-module P, for an A-module, for an A-module P the following are equivalent. So, one of them is $Hom_A(P, -)$, this is an exact functor. Two, when we have a diagram like this, V to V'' to zero, that means this rejective map. So, this is ϕ, and if you have given a map P to this f, we should be able to lift it. So, given ϕ, so rejective, and f A-module homomorphisms, there exist $\bar{f}$ here. From P to V A-module homomorphism such that this composition is f. $\bar{f}$ compose ϕ is f. If these equivalent conditions are satisfied, then one call the module P to be projective. P is called projective A-module. And, please check that these two conditions are equivalent, that is not so difficult. You just have to say, understand when the sequence is functor is exact and this, so it is similar.

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Def For an ~~An~~ A-module P TFAE:

(1) $\mathrm{Hom}_A(P, -)$ is exact functor.

(2) Given φ surjective and f A-module homo

$$\begin{tikzcd} & P \arrow[dl, dashed, "\bar{f}"'] \arrow[d, "f"] & \\ V \arrow[r, "\varphi"] & V'' \arrow[r] & 0 \end{tikzcd}$$

$\exists\ \bar{f} : P \to V$ A-module homo.

s.t. $\varphi \cdot \bar{f} = f$

P is called <u>projective</u> A-module