Lecture – 45

Preparation for Homological Dimension

Gyanam Paramam Dhyeyam: Knowledge is supreme.

Okay. So, today I want to start a new topic Homological Dimension of modules. And I am not sure, how much homological algebra you know. Do you know what is homology? Do you know cohomology or, nothing? Do you know what are the ffunctors, what is the category, what is the functor. Okay. So I will not prove all the details, but I will give the complete definitions and results and the best reference as I said last time also should look at the TFR palm plate number 5. Homological methods in Commutative Algebra, that is available on the web site of the TFR. Okay. So, before I do it first I will say some words about why it is needed for us now. So remember we have defined local ring to be regular and we want to define arbitrary ring, arbitrary commutative ring regular, if localization at every prime ideal is regular local. And I want to define this but it is not clear form, so far whatever theory we have learned about regular local rings and if you have a regular local ring and if I have a prime ideal layer and if I localize at that prime ideal, then the local ring is regular or not it is not clear, and for that we need homological tools. So we want to basically prove that when we have a regular local ring, then all localization are also regular. This is what one want to prove. And this is not this doesn't follow from the theory, so far we have. So only I want to remind you that we have a very particular case of this we have proved it in one of the Lemma, so I will recall you what we approved. So if you have a polynomial ring in over a field in n variables, K field and suppose we have a prime ideal P here, P is Spec $K[X_1,\dots,X_n]$, and suppose height of P is m, they mean one of the Lemma what we have done is we have constructed m elements. So we have constructed in one of the Lemma, when we were preparing for Jacobian criterion. In one of the Lemma we have proved that there exist m elements, m polynomials in b search that when you localize at P, so in this ring $K[X_1,\dots,X_n]$, localize at P. Where this ideal P, localized P this is an ideal here, this is a local ring with this is the unique maximal ideal and this maximal ideal is generated by the images of this $f_1,\dots,f_m$. We have constructed $f_1,\dots,f_m$ like this. So this precisely means that, so this is the local ring of dimension height. The dimension of this ring equal to height of P, and this m is also height of P, so that means we have proved that this maximal ideal of this local ring is generated by m elements. So that precisely means it is regular, but this is a very, very special case of polynomial ring over a field, we have proved that, so this as proved that corollary that this ring is regular, regular means in general.

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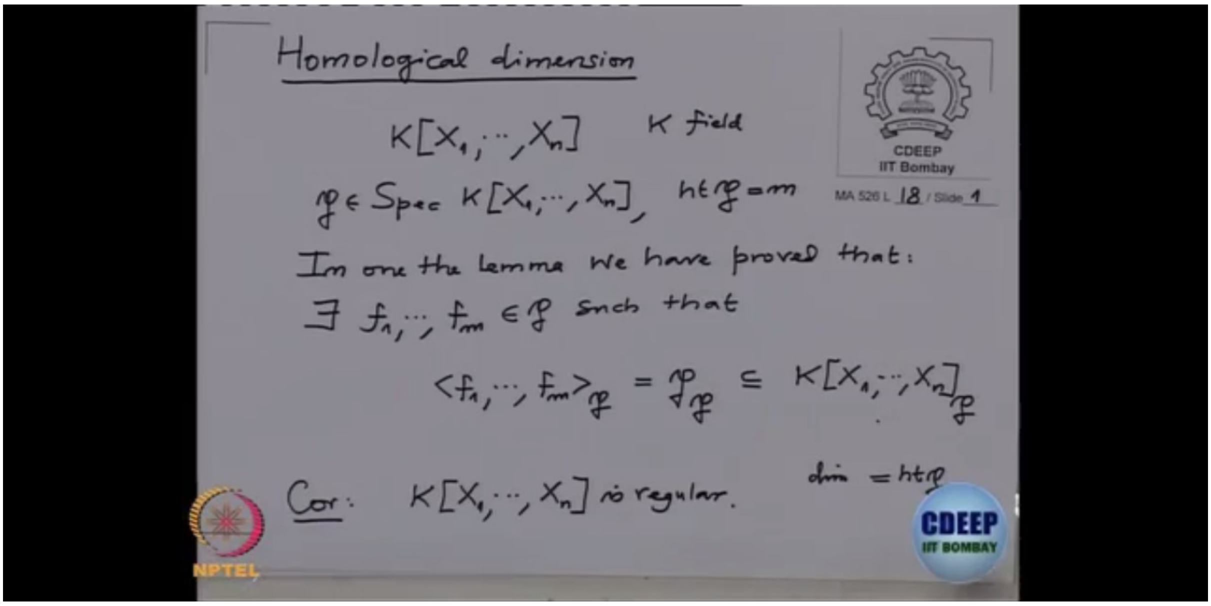

Let's us record, if a is A commutative ring, we say that A is regular if for every prime ideal P, A localized P is a regular local ring, that means height dimension of this ring equal to minimal number of generators for it's a maximal ideal and this is height P, that is the definition of the regular and we have proved that the polynomial ring over a field is regular and there we have constructed this polynomials, so even though the proof was not so trivial. It was, the construction was dependent on some field extensions and so on. So right now we are looking for a different kind of criterions. How do we decide a ring is regular or given local ring how do you decide it is regular, not so far what we have proved. So that was one comment, another comment I wanted to say that remember that recall we have proved also that if A is, (A, m) is regular local then it is normal and A is a normal domain. Regular implies domain that we have proved because we went to associated graded ring and then associated graded domain therefore original is domain, similarly for associated graded is integrally closed, then the original is integrally closed, that is how the prove of this integrally closed and this is also true, this is true only when dimension of the ring is 1. So this is the theorem and I will try to just tell you how does to prove this side. But I will only sketch and we should discuss details in the, our tutorial session about this prove. Okay. So the main step in this prove is the following, so we have one dimensional ring and you want to, we have given it is normal domain and you want to prove it is regular. That means you want to prove that the maximal ideal is principle so we need to prove m is principle, that we are proving this implication and the assumption dimension is 1. Okay. So, because the dimension is 1 by our dimension theorem we known there is one element which form a system of parameters. So that is since the dimension is 1 there exist a in m actually, such that this ideal generated by a, this as to be primary ideal, because its system of parameter means mod that it is finite ring, that means mod that is finite length, that means m is the only maximal ideal there, dimension is 1. So that will mean that the radical of this ideal is only m, because there is now maximal ideal, there is now prime ideal other than 0 and m. and this, a, is non-zero element, so it's actually radical as to be m. And now you prove that there is a b, and from here now there is next step is to prove that next show that m is principle, from here it is not too difficult to say we will see it in the tutorial.

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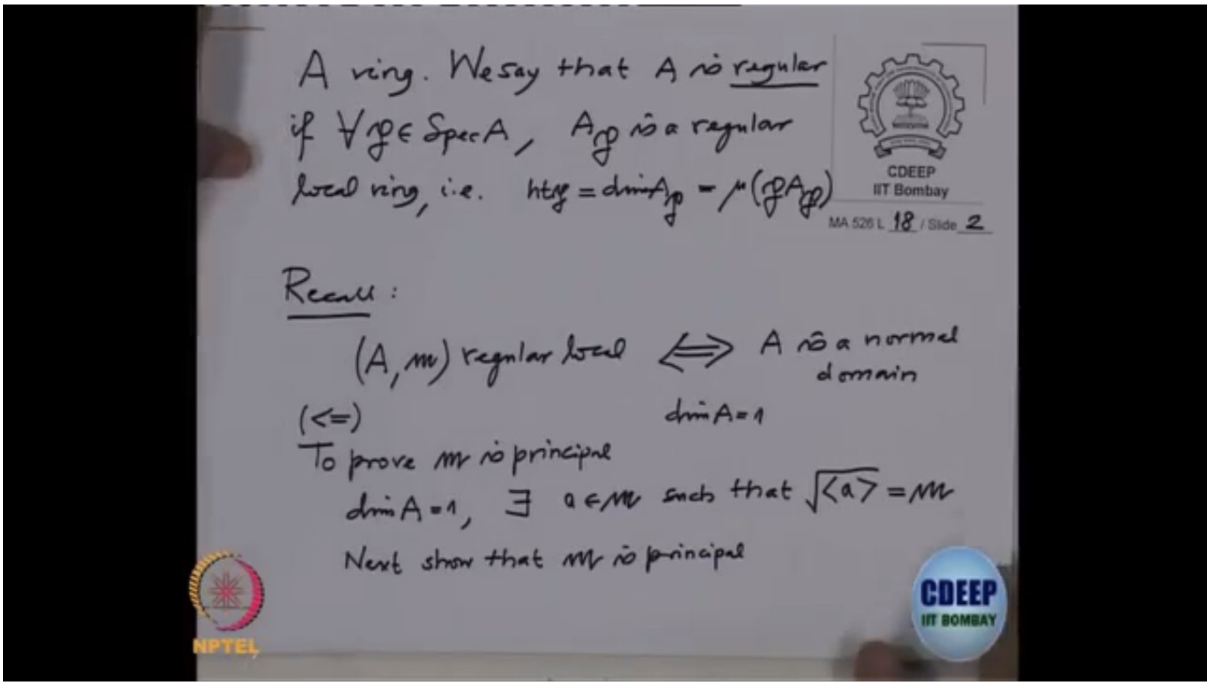

So now, what I want to recall few things, what is the homological dimension of a module? So does everybody know, so I will assume that everybody know what a category is, C category. So category as objects and so called Morphemes and module example of the category should keep in mind that we are going to apply all this abstract things to the category of A modules. So I will only tell you in this case and then you can formulate the statement in a general case. So here the objects are the modules fix A is our fixed commutative ring. A is a fixed commutative ring and objects are the modules, A-modules are the objects and we are denoting by V W ex cetera. And the morphemes are A leaner maps or so called module over morphemes on V to W and the set of all module morphemes are denoting by $Hom_A(V, W)$, so this is another module, right. This is also module because think of this, see this are all maps from V to W and with the special property. So this is the subset of W power V, you know the set theory notation. Set theory notation is, if I have said X and Y are sets and if I have a map, the set of all maps from X to Y maps that is also denoted by Hom X, Y, if you want you can write here sets and this you can think this is a sub set, this is precisely denoted by Y^X. This just a notation and this notation comes from, let me not say more, so this is notation so, with that notation this Hom V, W is a sub set of W^V, but W^V as an A module structure, think of this as a copies of W, this is a product, copies of W index by V. So this is same as product W, this product is over V, v in V. so therefore, this is a direct product of modules, if W is a module, so direct product of modules, so therefore this is the module and this is actually sub module. So where you prove is, if you have two modules homomorphism then the sub is a liner, the scalar multiplication is also a liner and so on. So that means your proving that this is a sub module of this. Okay. So this are the morphemes and now what do you want to do? We want to study this category, category of A-modules to get information about the ring A, not just one module the whole bunch of modules over the ring and we want to study this Hom properties of the modules and so on and so on. And from this we want to extract the definition, extract an information about A that is the basic principle of studying this categorical things. Okay.

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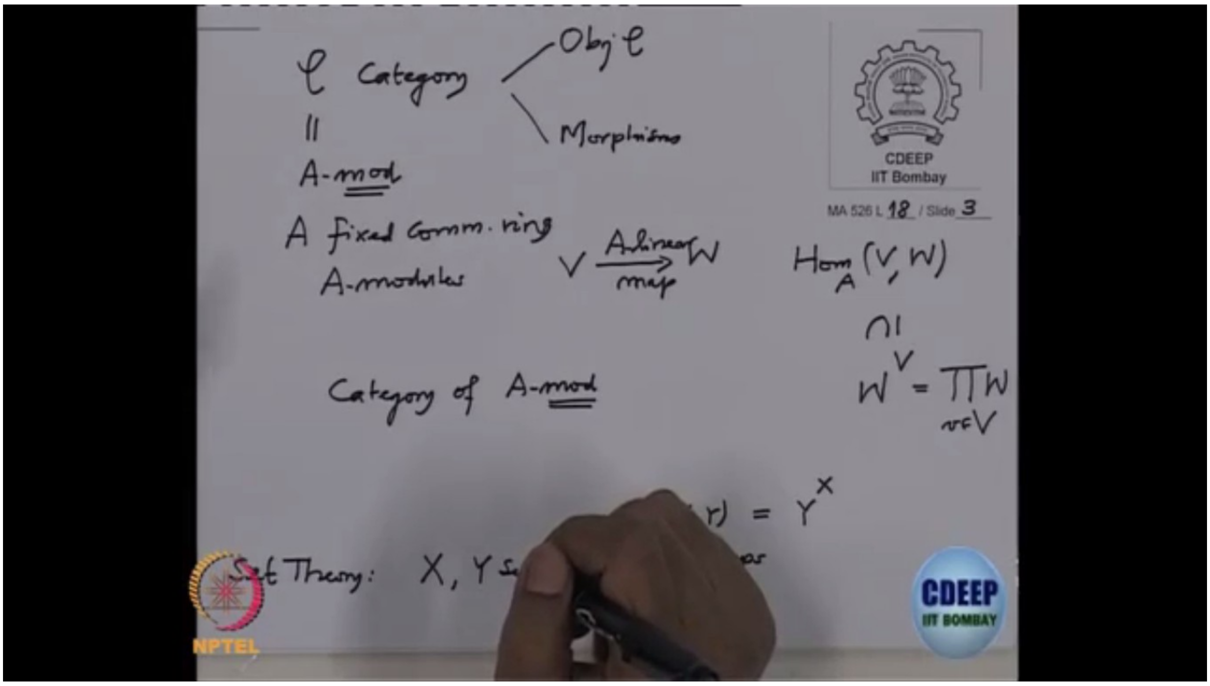

Now, so first of all what is the complex? Complex in this category A-modules and whatever we do it here it is not so difficult to see that, this will work in arbitrary category. Arbitrary category with reasonable properties and for example, category of sets, category of groups, category of vector spaces, category of A-modules, category of topological spaces, category of whatever subjects you did yesterday, difference manifolds and so on and so on. The category of algebraic verity and so many things, but we are concentrating on the A-module category, so the complex of A-modules it's a bunch of module of homomorphism and they are connected to each other like this, so you have a module V_0 , then V_1, V_2 and so on. And this ideals so V_{-1}, V_{-2} and so on, and there are maps here, this is d_0 , this is d_1 and this is d_{-1} , so -1 are really written up, it's easier to write the positive numbers and the negative numbers. So and this is d_1, d_2 and so on, right? So this is the sequence of A-modules and A-module homomorphism. Such that the components are 0 that means d_0 , compose d_1 is 0. This composition is, any two composition is 0. So this is usually abbreviated by V dot here and d dot here, such a thing is called a complex. And this is without bothering too much about this just write d composed d is 0. That means at any stage the composition is 0. Such a thing is called a complex of A-modules. Now you would have seen earlier also when do you say sequence is, okay, before I go on. So for some reason if this complex as only the positive and non-negative part or only the negative, non-positive part that we will see that happens you know in good situation. So then you call it homology or cohomology. So what is the homology? So on this part, non-negative side homology means, see this is a complex 0, therefore, what do we now for example from here we know that image of d_0 , is contained in the kernel of d_1 , that is because they composite it's here. At each stage that is, so this is a sub module, this is also sub module and if I take the quotient module this by this, if I take kernel of d_1 mod image of d_0 , this is some sub module, some residual class module it is an A-module, so this is called the homology of this complex H for homology and complex is this V dot and also it is written at which stage, so for example here, this is at this stage 1. Write so you can write down at any stage and the simple rule that then it is on the non-negative side you write the suffixes on the, suffixes, when it is on the negative side a non-positive side you write suffixes upper one. Simple because instead of writing minus suffixes it is better to write powers upper notation, what is called? Suffix and superscripts, right? So that's it. So that is the homology.

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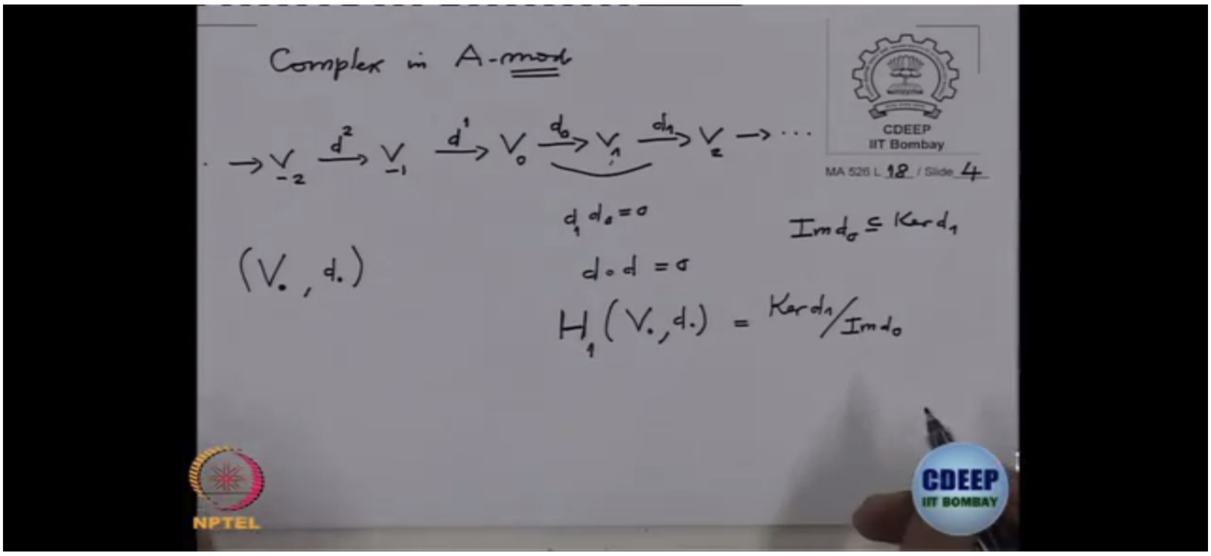

Now that is the complex, so you have lot of, for this complex you have lot of homologies that are attached to it, at each stage and similarly in the negative side also. So now I want to make a category out of the complexes. So that means what should we define now our objects are the complexes and we want to define morphism between the complexes, and I say morphism already the morphism also had some rules among them that composite of morphism is morphism. Also we want to add morphism and get another morphism. So to say that we want category to satisfy come property like additive category, that means morphism are not just sets but they are additive admin groups and so on, right. So that kind of property so in our case those things will be easier to check always, so we will not bother much about the general categories. Okay. So how do you define the morphism of complexes? Okay we have two complexes now, so let me call V and W, so one is $V.$ another W dot and this d is usually called boundary, what is it called? It's called a complex. So $d.$ and here also $d.$, actually there notation should be different, one way is to put V here and W here. Okay. Or just use the same letter but be aware what you are writing. Okay. So here it is, here it starts at V_n to V_{n+1} , this is d n and one before that is V_{n+1} , d_{n+1} , and remember because I have written like that obviously I have chosen the non-negative side, if I would have written on the other side then this will be different, right. So and W_{n+1} , W_n and W_{n+1} , so this is also d_{n+1} , d_n , at least, okay, once we write like this and so on. this side and this side also right, this have two complexes. Now, morphism of complex is a family of morphism A-module homomorphism like this. So this f_n , this is f_{n+1} , this is f_{n-1} . So morphism between this two complexes is f, it's not single f it is a family f_n . where f_n are A-module homomorphism from this. So that these diagrams are commutative and so morphism, morphism f of complexes f, f is f_n , which is or $f.$, $f.$ from $(V.,d.V)$ to $(W.,d.W)$, such that this diagram is commutative, so that means what at each stage if I take $d_n V$, then compose with f_{n+1} , that is same thing as this way $f_n, d_n W$, this is true for all n, then you call it a morphism of complexes. Now from this definition it is obvious that if I have three complexes and morphism between them then the composition of a, morphism of complexes again a morphism of complexes. Maybe just put a another bunch here and then you know, if this square is commutative, down square is commutative then the total together is commutative. So morphism composition make sense, also identity from the same complex to same complex there is obviously identity morphism, identity morphism obviously commute this, so identity is also there.

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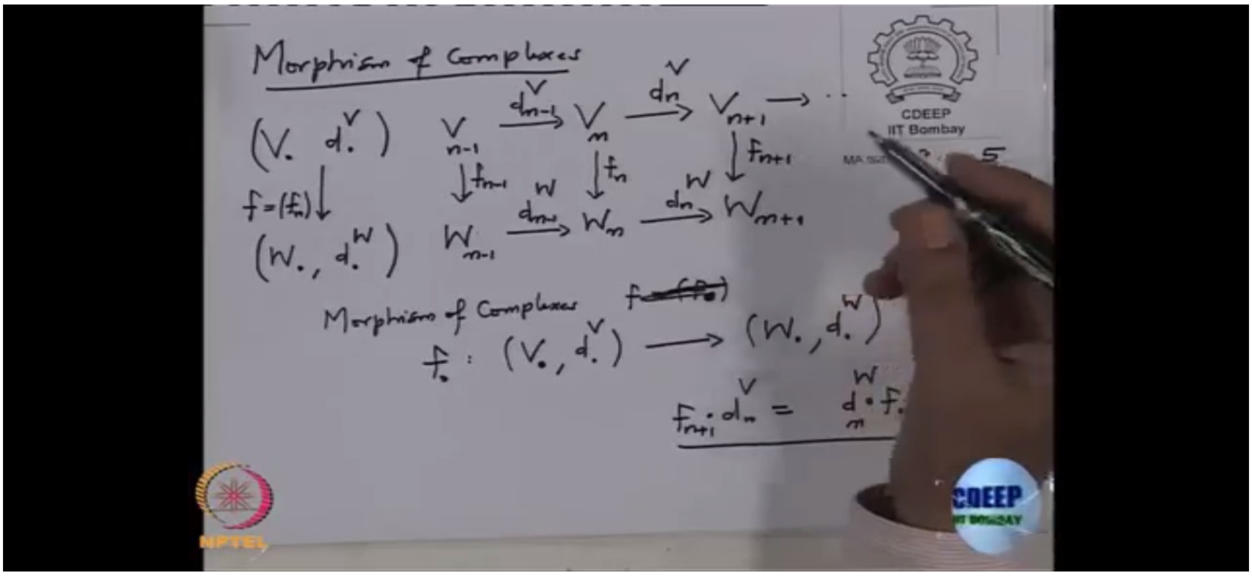

Therefore it now, it forms a category, now, okay. So what other things I need, okay. Alright. Now the main thing what we will need all the time is so called diagram chasings, so that will happen in the like this. So, now, suppose I have an sequence like this, now I will use the X as a complex $X.$ is one complex, $Y.$ is another complex and $Z.$ is another complex. And I will not, let us supress that d and shoot that d there. So when I say morphism that I understand from X to Y. Similarly from here to here we understand and if I want to put 0 here and 0 here. You are a way that when do you say module of the short exact sequence of modules. That means X, Y, Z are modules and this map is, this is f dot and this is g dot, this map is injective. So first think about modules, this is module, this is module, this is module, and this map is injective that means X is here, here X means, image of this equal to kernel of this, and here X kernel of this is image of this that means g is surjective. So g is a surjective and f is injective and kernel f image f equal to kernel g, then you call a short exact sequence of modules. Now, when do you say such a sequence is exact for complex in general that means at each, for each end its exact sequence of A-modules. See if this will give you, think of X as a vertical like that, Y as a vertical like that, this are verticals, so there are maps like that so this is called exact, is called exact, called short exact sequence of complexes, if for every end 0 X_n , this is f_n , Y_n to Z_n to g_n , this is exact. Now we at come down to the modules, short exact, short exact sequence of A-modules, so that means, so that is f_n is injective, g_n is surjective and image of f_n equal to kernel of g_n . Okay. So that is short exact sequence of complexes.

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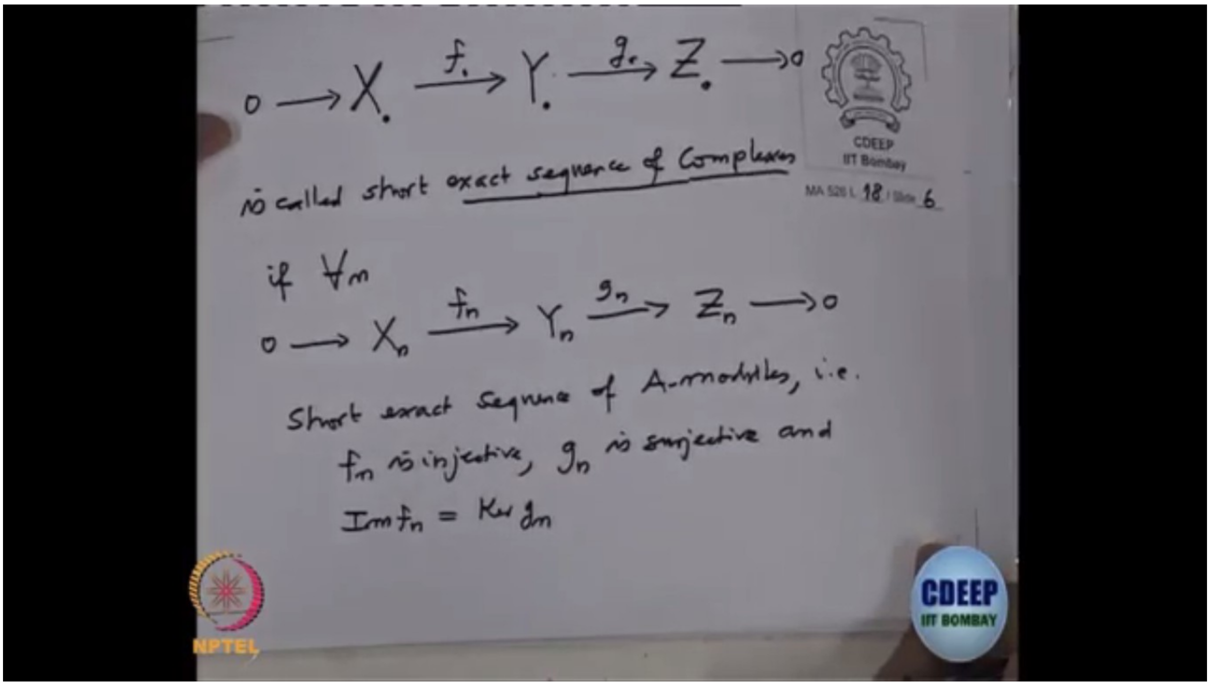

Now, I will write a proposition and then we will prove it orally.