# Lecture No.43

# Proof of Jacobian Criterion

Gyanam Paramam Dhyeyam: Knowledge is Supreme.

Today, I will continue and prove Jacobian Critrerion, assuming the lemmas we have proved. And I will recall those two lemmas when we actually need it. So let me just recall what do we want to prove and what are the ingredients. So today the goal is and then we will state precisely, then we want to conclude or calculate the singular locus of affine algebra. A, which is $K[X_1,\ldots,X_n]$ modulus from ideal. Modular ideal and finitely generated, therefore it is generated by some n polynomials. And these ideal a we will write in it. And what does that mean by singular locus, that means all those prime ideals in this affine algebra, so there the localization, there are regular locations. So prime ideals will look like $\frac{P}{a}$ where a is an ideal in the polynomial ring. And this P is a prime ideal which contains a. So this P is actually prime ideal in the polynomial a, and we are looking that, every prime ideal in a looks like this. So that is a notation. And we want to know when exactly the local ring A localized this $\frac{P}{a}$, is regular. And such a criterion, you see in general it is not easy for arbitrary commutative ring to decide what are exactly the prime ideals there it is regular. But for a fine case it's easier in terms of the equations. So that is what we want to know. And this is our goal, there so.

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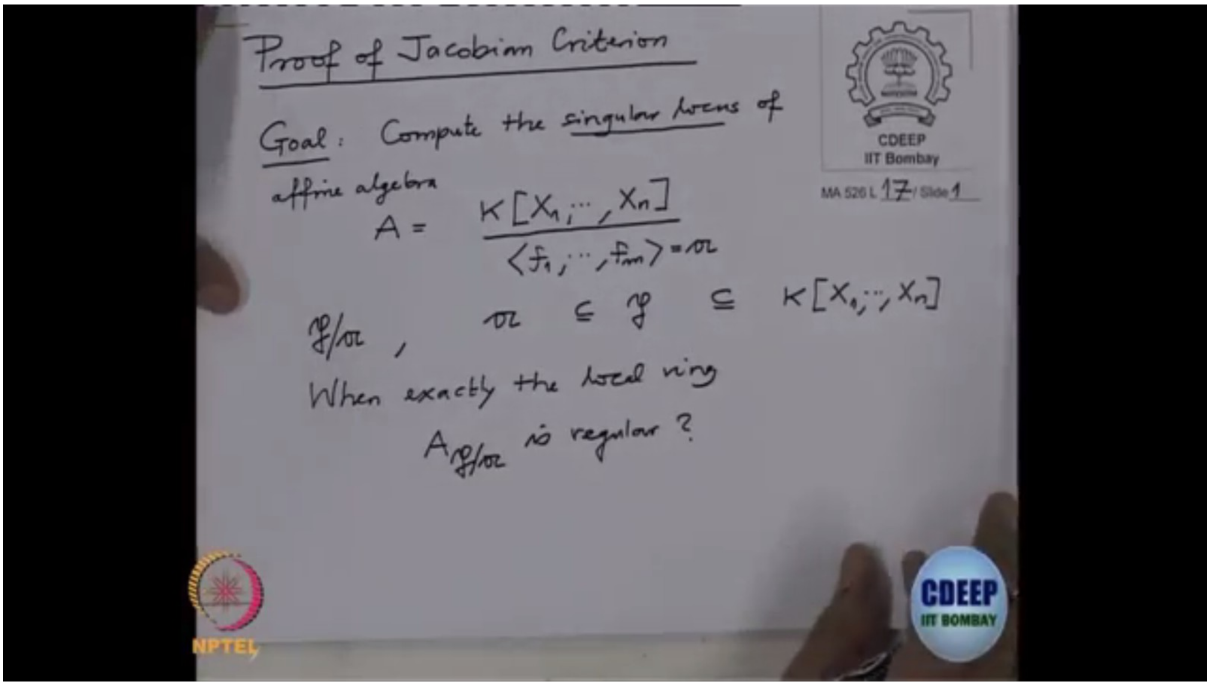


What are the ingredients in the proof that we will need. So the answer will involves, first of all it involves irreducible components of Spec A which contain is given prime ideal $\frac{P}{a}$ . So this is, in purely algebraic language, so this means, its minimal primes of the ring a. That means minimal primes a containing q, containing P. These are in the spectrum of $K[X_1,\ldots,X_n]$ . So in

$K[X_1,\ldots,X_n]$ this P may not be minimal over a. So locate the minimal prime, so a which are minimal over b. These correspond to irreducible components. And what else? So this is one and the matrix says the rank. So the matrix says we got it from the defining equations of a. So the rank of, the matrix says and how do the matrix arrive, that is by differentiating. So I will just write rank of matrix like this. Some polynomial $g_{ij}$ , their entries are in $g_{ij}$ and their entries in, actually this a matrix of some order. I will just write here n and k. Entries are in the polynomial ring mod of prime ideal. And that you can think because this is an integral domain, you could also think that these matrices are containing the matrices with entries in the coefficient field. So this is subset of $M_n(k)$ in the coefficient field of this. So therefore the usual linear algebra it should help us to conclude this rank. That's what last time also in the two lemmas this was seen.

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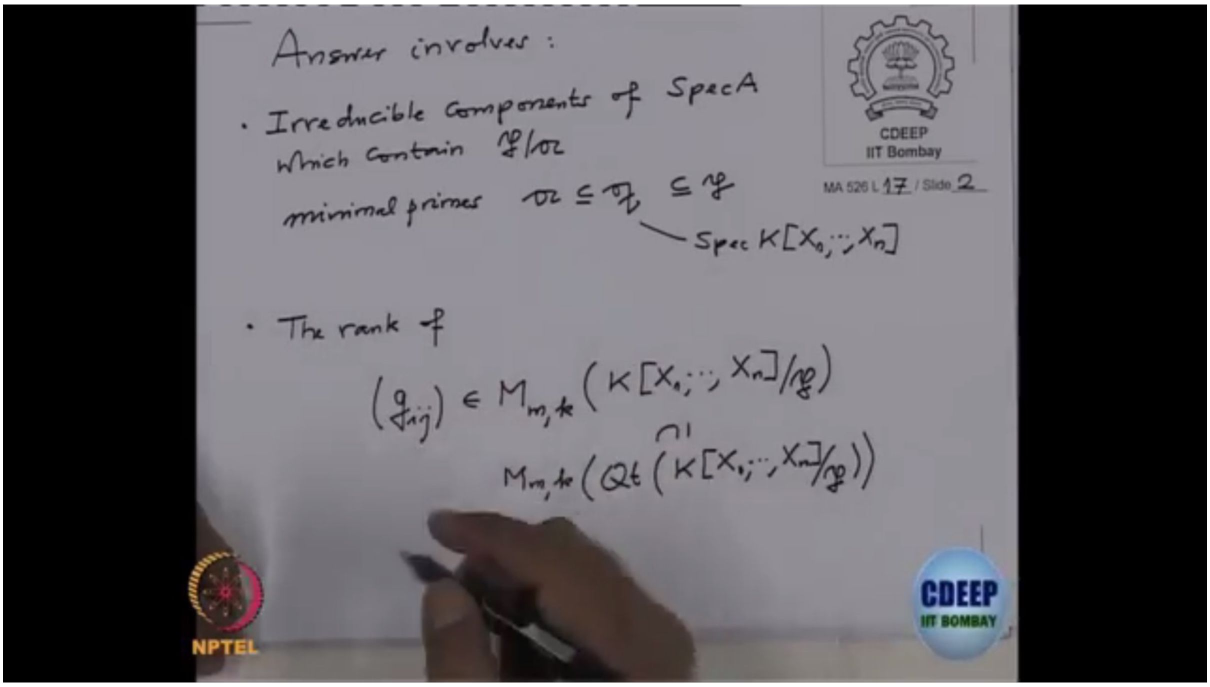


Okay, so to be preside now let us now state what we want to prove. So the theorem we want to prove is, this is called Jacobian Criterion because it involves the Jacobian matrix. Okay, as usual earlier notation $f_1,\ldots,f_m$ are the generators for the given ideal a, and also we have given a prime ideal P which is everything is happening in the polynomial ring in n variables or if we will keep. K is a field. Okay, and q is a minimal prime, between these q is a minimal prime over a. That is a situation. Okay, so and what are we denoting, I want to denote now, if I take this matrix $\frac{df_i}{df_j}$ this differentiate this polynomials with respect to $f_j$ and read the mod P. So this is now a matrix $m \times n$ matrix. n are the rows and js are the columns. Sometimes it might happen that I would have interchange rows and columns. So that you have to watch out. If you have trouble in one way try the other way. So but mostly it should be correct. This matrix I want to denote because every time it is too much to write like that. So J, J for Jacobian $f_1,\ldots,f_n$ and semicolon P. This is called Jacobian matrix of this equations at P. So Jacobian matrix of $f_1,\ldots,f_n$ at P. So this is a matrix or the ring

$\frac{K[X_1,\ldots,X_n]}{P}$ but that is an integral domain therefore he could also think it's a ,matrix or the entry is in its coefficient field. Okay, so the first assumption is A, the rank of this matrix as a bound. So the rank of $J(f_1,\ldots,f_n;P)$ this cannot be more than height of q. So as I said earlier it involves the rank and also it involve the irreducible components. These irreducible components is coming from this q. Okay, and now this inequality you could also write in terms ranks or nullity. Actually I don't like to use the word nullity for many reasons but that is quite popular for engineers. So this, if you use nullity then inequality will get reverse. So nullity of this will be bigger equal to dimension of every irreducible component of Spec A which contains prime $\frac{P}{a}$. You see, remember this q is not only one, there may be many qs and we are making this statement. So that means this is less than equal to every minimal prime side and when you take the equality it will become every. Okay, that is the statement a. Second b, and talk about when do equality hold here. If equality holds in a then it is regular. Then A localized $\frac{P}{a}$ is regular. Regular local been. It's a local already, regular local. That is when equality holds. Now where this part c statement will be when the converse of this b holds. That means when exactly the equality holds.

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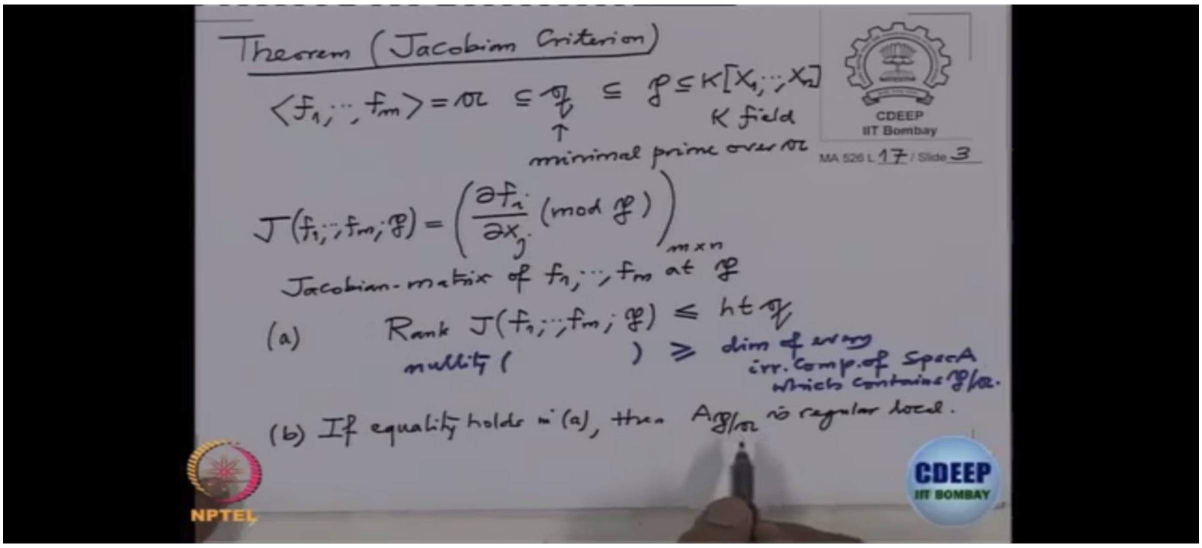


So c apparent says, c part says, if. Actually one would like to, wanted to write here if and only if. And would like to have equality holds only if and only if the ring is regular local but that doesn't always happened. It depends on the base field. So with that the base field is perfect or not. So base field perfect or not. If the coefficient field, base field is perfect or not. If the coefficient field of this ring $\frac{K[X_1,\ldots,X_n]}{P}$ is separable or K, then converse holds. Then the converse in b holds. So this will particularly, will happen in two cases, when the base field perfect. But zero fields are always perfect and characteristic P case. So, I will just recall what is perfect. So when, this will happen in these two cases, K perfect. Remember this field is not algebraic over K in general. So separable should mean that there exit separate in transcendence basis. That means the algebraic part of that after you attached transcendence basis the algebraic part is separable extension. And that means algebraic extension is equal to separable if every element has the minimal polynomial doesn't have repeated roots. Okay, perfect means K to, you have natural map from K to K namely any a going to a power p, where p now characteristic but it's almost like a characteristic but p is called a characteristic exponents. Characteristic exponent, that is p, it is 1 if characteristic of K is 0. And it is p when characteristic of K is p positive. So when this map, this map, it's clearly them because of the definition of the characteristic exponent. And when this map is so subjective, then it is called perfect. That is the definition actually, so subjective. That simply mean that every element in K has a peak root, where p is a characteristic exponent. Okay, so for example, finite fields, because in case of finite fields this might be always injective and subjective up to residual fields. Okay, or it can also happen without base in build, the equality in a, inequality in, so this equality can also happen without a, k being perfect. For example, it will happen in the case this is about the p now, if P is actually a close point, close rational point. So that is p is $m_a$ which is generated by $X_1 - a_1, \ldots, X_n - a_n$ , where a is a1 to an this is actually in $K^n$ . So, these are very special maximal ideals. These are the maximal ideals where the residual fields are K itself. It's not changing. So the residual fields not changed. And why in this case because let us write, this is very important, I want to calculate this. So in this case I want to write down the Jacobian matrix, what it means, right. So the Jacobian matrix in this case is what, so we have to differentiate the generators and read their mod P. Now I will write ma but this what, this is you are putting $X_i$ is equal to $a_i$ . So that mean this is $\frac{df_i}{dx_j}$ evaluated at that a. So this is what exactly you learn in analysis. When you expand did that you use such an expansion because you need and you valued at a point. There was no prime ideal, there only the points were there. So therefore in this case the Jacobian matrix $f_1, \ldots, f_m$ at P this is the Jacobian matrix of $f_1, \ldots, f_m$ at that point a. This is more familiar $f_j$ at, so this is also called functional matrix at a. In analysis people they call it alpha functional matrix . Okay, and therefore also one more command that because in this case the residual field will not change. This field is K itself. So therefore there is no question of separable. Okay, so also these matrix, in this case this matrix is also denoted by, in analysis it is denoted by a differential of f the map f at a. Where f you think of it's $f_1, \ldots, f_n$ and think of this a map from $k^m$ to $k^n$ . This is valid in real number or complex number. Think of this as $k^m$ to $k^n$ actually. And then we will differentiate and take the total differential and evaluated point, the total differential at a. If given by this matrix.

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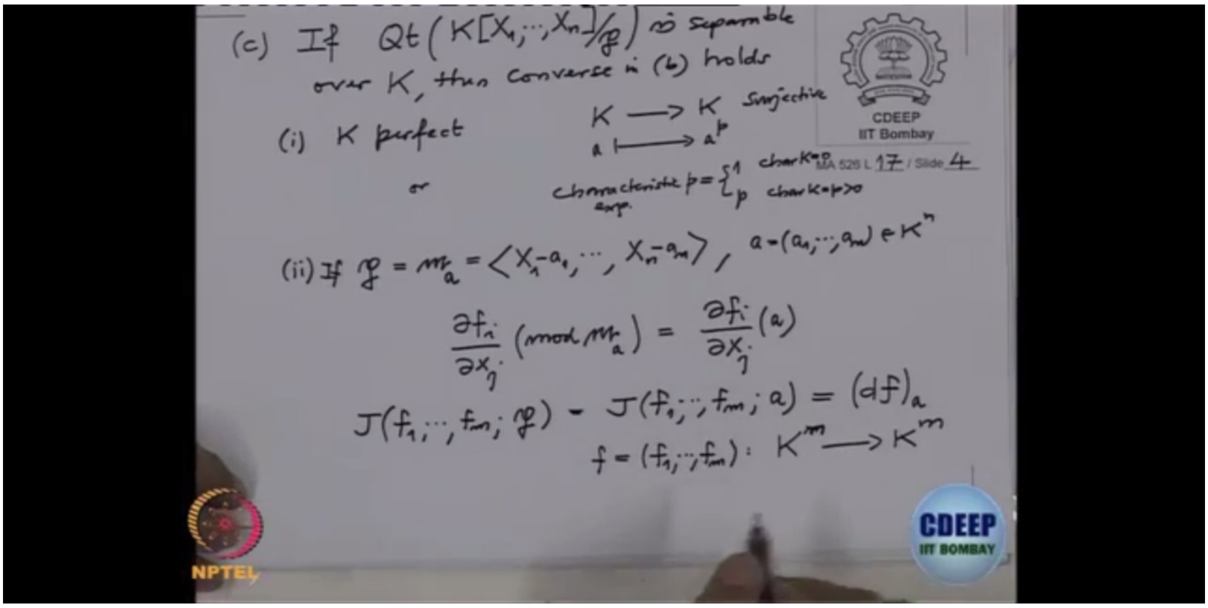


Okay, that was it. Yeah, equality in A holds then the ring is regular. No, no, the height comes only the equality here also that will, you see it will also be clear because it is the proof that this P will not contain more than one minimal prime. Then the question will not come, right. That will happen, because if you remember an example, last time I gave an example if a prime ideal P lies on two irreducible components then it cannot be non singular. It has to be singular, right. Because of that this will not happen. Okay, so now let us. Okay, therefore in that notation, I have not completed the sentence. If you take the differential of this map at a. This map is the kernel of this map. See this map is the kernel of this map, it's a linear map and the kernel of this map is called tangent space at a. So the point is singular if the dimension of this is more than the lower one. Do you remember we are saying about rank is less equal to height. But that inequality when you use instead of rank nullity, then the equality will get reverse. Just now I said, the dimension of the kernel is rank , so this kernel is a tangent space. So the tangent space will have more dimension than the lower one.

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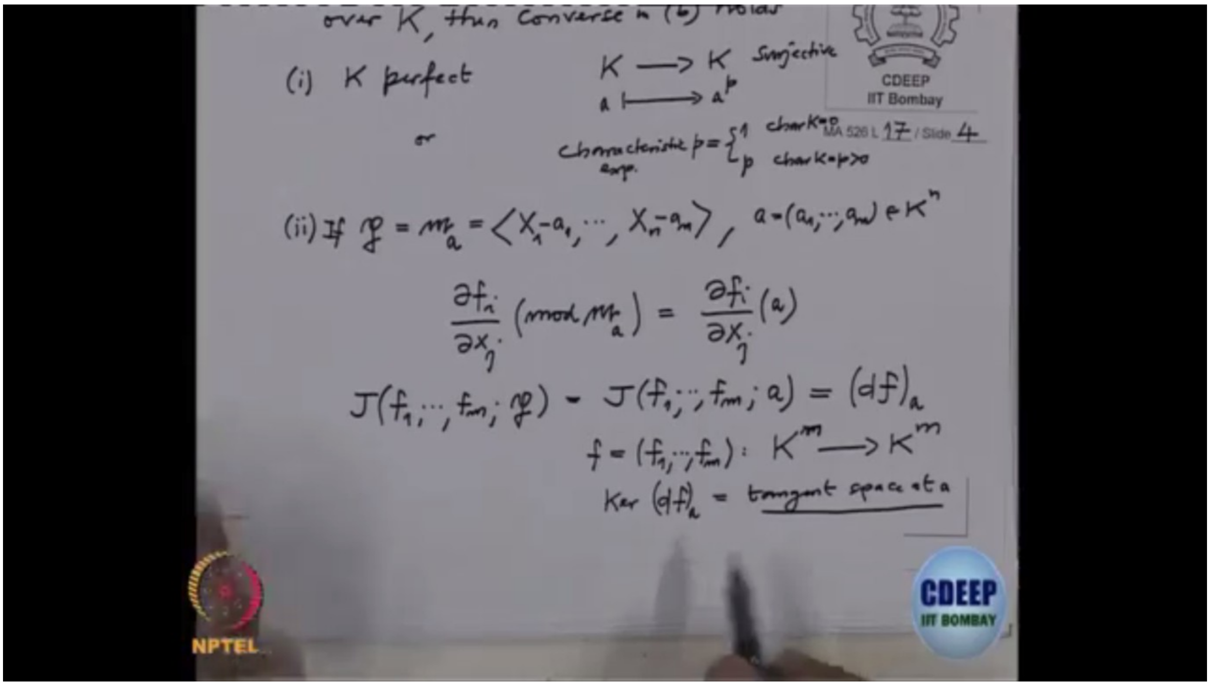


Okay, so that's it. Now let us prove, so proof of Jacobian Critrerion. And then we will work out some examples, concrete example. Okay, so a and P is here. And first I want to choose, you know, the statements a, b, c are about the given minimal prime which is contained in a and P. But now I choose q0 here. So q0 is minimal of all the primes which are in between here. Where this $q_0$ is the minimal among all primes in between. Actually minimal among, all minimal primes . And we look at all minimal primes of a. So among them you will choose the minimal one. So that means it exactly give the irreducible components otherwise, you see there are embedded primes.

So I am taking the $q_0$ is a minimal one. But we want to prove about q. So $q_0$ contained in q. Now if I prove. See we wanted to prove the inequality for q. But in fact I want to reduce to the case for proving to $q_0$ . So assume, suppose I have proved for $q_0$ then I want to say that I can deduce for q. So assume that we have proved theorem for $q_0$ . Okay, I want to claim that I want to prove it for q. So for example, so let us look at a. So what do I want to prove? I want to, see this rank has nothing to do at q. So this rank is same, so I want to prove that it is less equal to height q. But if I have taken smaller one that height is already is more, so that , right. So that implies inequality in a for q as well.

Also note that, note if $q_0$ is properly contained in q if this, then what it means that, see this q is minimal prime, this is also minimal prime. That means this embedded one. So that means, then a, P cannot be regular. Because we have seen that these are the two primes and $\frac{P}{a}$ , this is passing by both of them. Therefore that ring cannot be regular this is what we have seen in the example. So therefore in this case A localized $\frac{P}{a}$ is not regular. Okay, so what about b, now b is, if equality holds. So if this is not regular therefore by b, b for $q_0$ this, it's not regular so inequality cannot hold. So by b for $q_0$ inequality in a is strict for $q_0$. And once it is for $q_0$ it will be strict for and hence for q. And the converse of b, so finally converse of b is trivially true. A localized P mod a is not regular. So all these comments just say that to prove the theorem I will assume this the given q is minimal among the associated primes of A.

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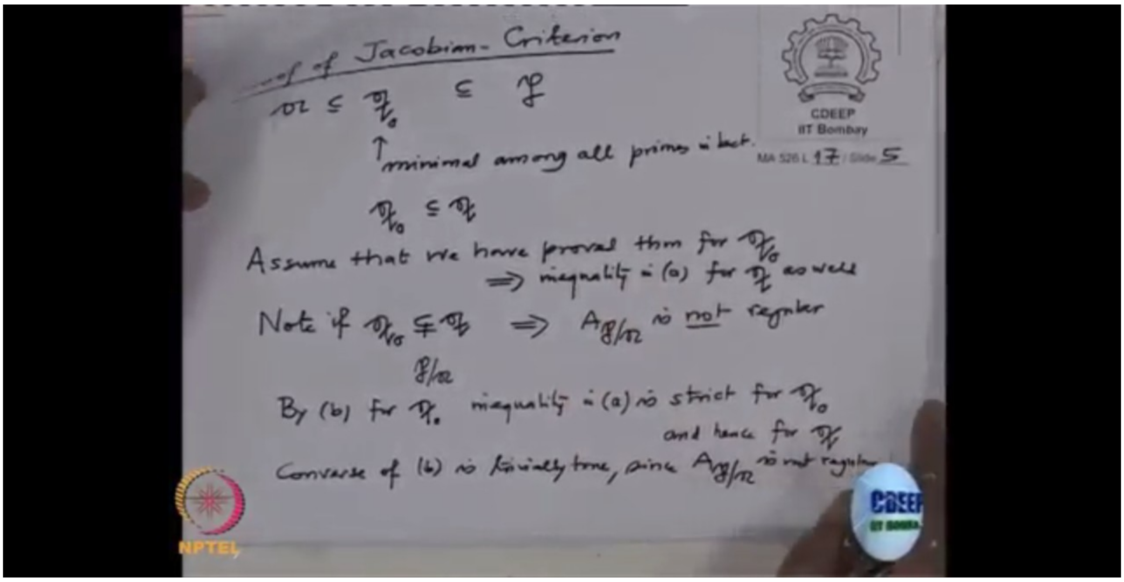


Okay, so now therefore we concentrate on the case $q_0$ is equal to q. So assume $q_0$. This really simple, I might have goofed it a little bit but when you think about it just say that if $q_0$ is smaller than, $q_0$ is strictly smaller than q then things are easier. Okay, now what do you want to prove in this case? We are looking at the ring $K[X_1,\dots,X_n]$ mod the ideal and then we are localizing at P. And we are debating whether this ring, which is a local ring, whether is it regular or not or when it is regular. So the maximal ideal here is the P. So the maximal ideal is, strictly speaking the maximal ideal is $\frac{P}{a}$ and then localized P. This is the maximal ideal. So this I am going to denote by m. We are interested in when Rm is a regular local ring.

Okay, so this is so. And what is the residual field. Residual field is R by n which is L, which is you take $\frac{K[X_1,\dots,X_n]}{P}$ and then the coefficient field of this. That is the residual field. Okay, therefore what we know always the dimension of m by m square as a vector space or L where dimension is. This is embedding dimension of R. This is always big or equal to the dimension of R. And we are debating when will the equality. Okay, so equality here if and only R is regular but what is the dimension? That we know this is dimension is height p minus height of q, this is the dimension. You remember that. So I want to recall here. Remember that suppose I have. I said. Suppose I have affine algebra A. Affine K algebra. And suppose I have chain of prime ideals of length n. And I assume that this is a maximal chain. Maximal means you cannot insert any more prime ideals in between so that. So for example chain of primes in A of primes in A then these n must be equal to the Krull dimension of $\frac{A}{P_0}$. These we would have deduce from normalization lemma long ring, right. So if you this then you also get this equality here. So that is the dimension. Now therefore we are debating when will the equality be here. That means when will this be equal to this. So we want to analyze now when exactly Vector Spec Dimension of $\frac{m}{m^2}$ is equal to height p minus of height of q. This is what we are looking for.

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