Lecture - 42

Jacobian Matrix and its Rank

(Contd)

GyanamParamamDhyeyam: Knowledge is Supreme.

So next we want to prove the rank is M.

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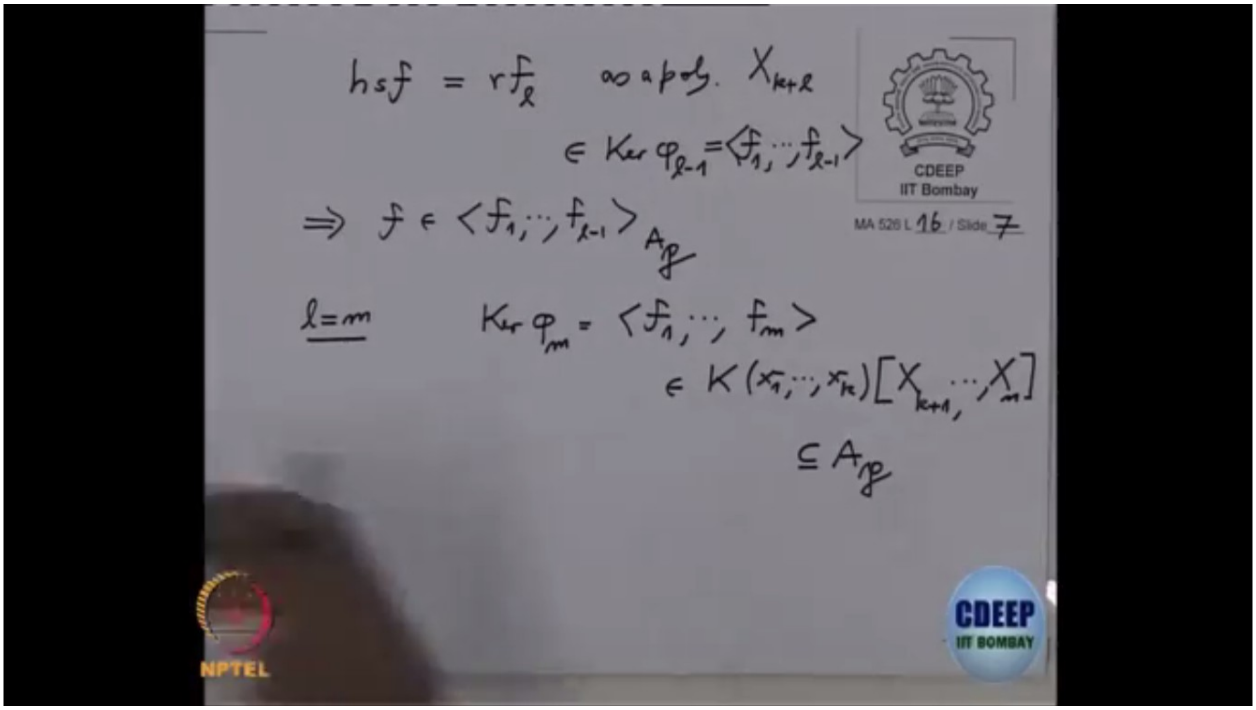

So to prove that I will consider the last, so you have these matrix now. $\frac{df_i}{dx_j}$ and we are reading this mod P, this matrix. And we want to decide what is the rank. So first all, this is a matrix, I will take the. This has how many, the columns are more, j is the column index and I is the row index. The columns are more. j is the column index and I is the row index. And there are how many fis there are m fis. So their m rows and more columns actually. So I have to choose the columns to show that the rank is m, I choose now the last m columns. So choose last m columns. That mean each variable involves there, so we have to differentiate with respect to which variables. That means $X_{k+1}, \dots, X_{k+m}$. these are the variables we need to differentiate. Now I want to claim that this is actually a lower triangular matrix with diagonal entries. So these last columns, so the matrix now is m cross m matrix. Let us write it, so that it will be easier to understand. It starts with f_1 then starts with $\frac{df_1}{dX_{k+m}}$, which is of course m. This is a first row and second row is $\frac{df_2}{dX_{k+1}} \frac{df_2}{dX_{k+2}}$ and soon. This one is df_m differentiating with dX_{k+m} and this is df_m by dX_{k+1}. This is that matrix. I claim this is lower rectangular. So that means these, first of all the diagonal entries are not in p. That we have chosen, right. Because we have chosen the third claim in the separability case.

The third claim we have chosen this. And that mean this, these are the diagonal entries. So the diagonal entries are not there, so what were the earlier entries, for example this one, yes. See actually they are in the kernel. So when you-- this variable doesn't appear there. So they are actually lower rectangular. So therefore, the rank is correct.

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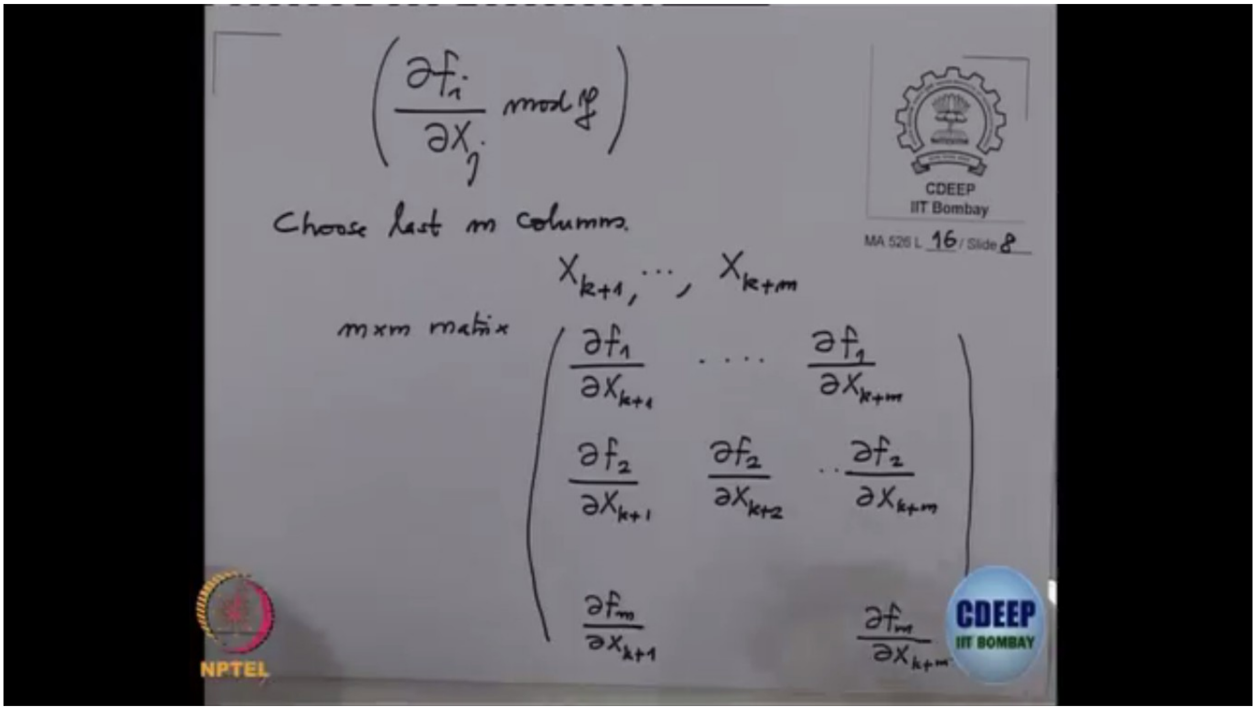

So all these happen in this lemma. It's not a magic, it all happened because of the following reason, this ring, if you take polynomial ring in any number of variables and localize that some prime ideal P. This ring is regular. This is regular local because we see this is of dimension how much height P, height P was m and there we found $f_1, \ldots, f_m$ so that locally P generated. And then, so therefore $f_1, \ldots, f_m$ must be the minimal number of generators or the maximal ideal in the local ring. So by definition this ring is therefore regular. So all these happened because this ring is regular. And this one wants to abstract this. That is the ultimate generation of this and also ultimately I want to get rid of this coordinates. So that also, okay. So this was one lemma which we need, the other lemma is the following. This was the Lemma 2 and then we will patch. So Lemma 2 and all these calculation will also become must if you make some definition abstract. So for example, there is a concept of k-litedifferentials. If you go to modules and we learn the k-lite differential modules and their universal properties then all these coordinates and all will go away. But and the cost of some abstracting. Okay, that I will come after when have enough room for . Okay, so what is the Lemma 2 I want? Lemma 2 is, suppose I have now ideal in a polynomial ring in n variables and the generator of that. So suppose, I have a ideally which is generated by m polynomials $f_1, \ldots, f_m$ in the polynomial ring in n variables. In $K[X_1, \ldots, X_m]$. And of course, K is a field and suppose we have a prime ideal P to ease up. Now let's call this as A again. Suppose I have a prime ideal P in A which contains this A. And again I am putting L to be the root A, first of all, the $\frac{A}{P}$ and coefficient field. This is same thing as, you take the localization first and take the residual field. Take this. So whether you take their localization first and then take the residual field or take the and then taken the coefficient field. These two concepts are same. Okay, then under this situation what do I want to prove that, rank of Jacobian Matrix, rank of again I will form the Jacobian Matrix, so dfi by dxkj mod P. This rank is small or equal to. You take the ideal, so in a local ring out, take the ideal and take P square divided by P square. And these we are taking in a local ring. I will explain you how did it comes out. See this, see you want to, we have this local ring P, A_P is the maximal ideal. So you want to test, so when you want to test for example, what is the dimension of, hold on this for a moment. Suppose you have a local being A localized at m. Am is a local ring. If it is local then one who always want to conclude this m by m square and the dimension of that as a K vector space. This is embedding dimension. You always want to compare embedding dimension with the dimension to test regularity. So if I want to test a by A, a by the ideal then I have to localize and take. So I will have to take a by-- When I localize it A_P , this is a local ring with P as a maximal ideal. But in this A, P^2 may not be contained. So when I want to go mod.

For example, here also if I have an ideal A here and I want to check mod m^2 . m^2 may not be contained in A. So I have to add in the numerator and then take the portion. That is where P square comes. So that is the reason one. Another reason now, now it may be better appreciated if will recall you the notation I was saying. So these, for example, matrix reading mod P and taking the rank, so this also will better understood from this notation. For example, I was telling you sometime back that if I have a Spec here and a prime ideal P here. And we are considering the topology off this. So physiologically one feel better to write a notation X for a topologic side. Similarly for a point small x, right. So this small x will correspond to this P. And these correspond to this, right. Okay, so imagine this is at topologic side, this is algebraic side. Now from P what you do, you can go mod that is a residual class $\frac{A}{P}$. $\frac{A}{P}$ is an integral domain. So therefore the coefficient field is this. This is what we are considering here. So when you go from A to $\frac{A}{P}$ you are reading mod P and then considering another level in the field, that's why these matrixes are done. Partial differentiate, read mod P and consider it as a matrix over the coefficient field. So we are considering these maps. So given any element f here, we are reading it here, f mod and then coefficient P here. So this, because these corresponds to these X, so let us write x here just to remember. This is a transition from one subject to the other subject. So when you, so this-- in algebra it denote $\bar{f}$. But when we, this is only a nation. So I may write it $f(x)$, it is only a notation. And then this goes directly to $f(x)$ because this is the inclusion math. So the notation is, I want to propose that this math is denoted by like $f(x)$. So $f(x)$ is 0, what should it mean? This should mean, this is equivalent to saying that this $\bar{f}$ is 0. But $\bar{f}$ 0 means f is in P. So this get translated like that. So this is very useful. So therefore these matrix is like evaluation and then we are doing this extension. This is preciously what we do in, when you have a finite take algebras. You evaluated points in K, then a polynomial, you evaluated $f(a_1,\ldots,a_n)$. That is exactly saying read this polynomial F mod the ideal, maximal ideal corresponding to the point $a_1,\ldots,a_n$. So then it becomes neater and also less confusing. There it is clear that what we are doing. So similarly if you translate now these V of the ideal a, this will get translated to what, these are the ideal's Ps but then if you were translated it is P, this is X, this is small x. Write and when does it belong? How do we test these belongs here when P contains a, a is containing P is the definition of this, right?

And how do we test it, that means, every polynomial, I will say it polynomial but it's a ring, right. So every f belongs to P. And when then A belong to P when $f(X)$ is g. So it is a same feeling as like maximal ideals. So that becomes better and better conceptually. All right. So, we want to prove now the rank of this matrix is bounded by this. What this? This is a vector space . So I have to write a dimension here. Dimension of L, that is why we have to go mod P^2. Now we want to vector space that means P should un-highlighted. So, if the extension is separable, then the equality will hold here. So, the continuation of this lemma statement, the last part of the lemma is, if L over K is separable, then equality holds in a. So that mean, the rank is equal to the dimension. So, proof. So, first of all, we have to know how many cross, how many matrixes is this. This is $m\times n$ matrix, right? These are m polynomials and n variables. So, this matrix $\frac{df_i}{dx_j}$, mod p, so let me write it of p, that means reading mod p. So, this is actually a matrix in $m\times n$ matrix with entries in a field L. So, this is $M_{m\times n}(L)$, L is the coefficient field of $\frac{A}{P}$. And we want to check some inequality about vector space dimension. See, rank of a matrix is equal to the dimension of the image phase, and matrix we should think, it's linear map from where to where? If I have $m\times n$ matrix, what do you think, it's a map from where to where? The images or the columns? So, images go in M, right? So, therefore, this if you want to think as a linear map, it should be from where to where? L^n to L^m, right? And we want to prove the rank means the dimension of image phase is less equal to somebody else, less equal to some dimension. Okay. So, that vector space is what? That is p, so we have a polynomial ring and localize at p, A localize at p, and the maximal ideal, it is pA_p, and then, mod pA p^2A_p. This is a vector space of how much dimension? This is $\frac{A}{P}$, right, $\frac{A}{P}$ and then localize. But $\frac{A}{P}$ has dimension, how much? Whatever, okay. So, right now we are not bothered. So, I'm going to construct two linear maps, one is from Lm to this $\frac{p}{p^2}$ and localize that p, that is one. And the other is from A to L^n. This is this, this is this. We are going to check two linear maps. Okay. What are the linear maps? Okay, the first one first, this one. So you take any tuple g one to gm, and just read them in this. See, they are polynomials, where they are, see this, just now I said know, the L is coefficient field of $\frac{A}{P}$, or which is also a residue field of A_p. So, each one of them is a polynomial here, right? So, each one of them, so I just want to read them, so you think of these guys as a residues here in a local ring. And now, where do you read them? Map it to, we want somebody here. So, this p is generated by $f_1,\ldots,f_m$, A is generated by $f_1,\ldots,f_m$. Okay. So, read, look at g_jf_j, look at this linear combination, and read this mod p square, mod p square, this mod pA_{p^2} . This is what the denominator we want, right? See, what do you want? You want somebody mod this, right, p in a local ring mod maximal ideal square. That is what we want. So you have this, given this, just multiply by the gifj f one to fm are given to you, that is generates the ideal which is contain in p. So, therefore, all these guys are in p, this element, this sum is in p. So, it is indeed in the numerator, and we have to read mod, square of that. So it makes sense. Okay, so this is clearly linear map, L linear map and I want to check its image. What is the image of this map? Image of ϕ , I say, obviously image of ϕ is, this image of ϕ is contain in this ideal generator would be $f_1, \ldots, f_m$. So that means, image of ϕ is precisely, if I take ideal A, add it to p^2 , mod p^2 and localize at p, this is the image.

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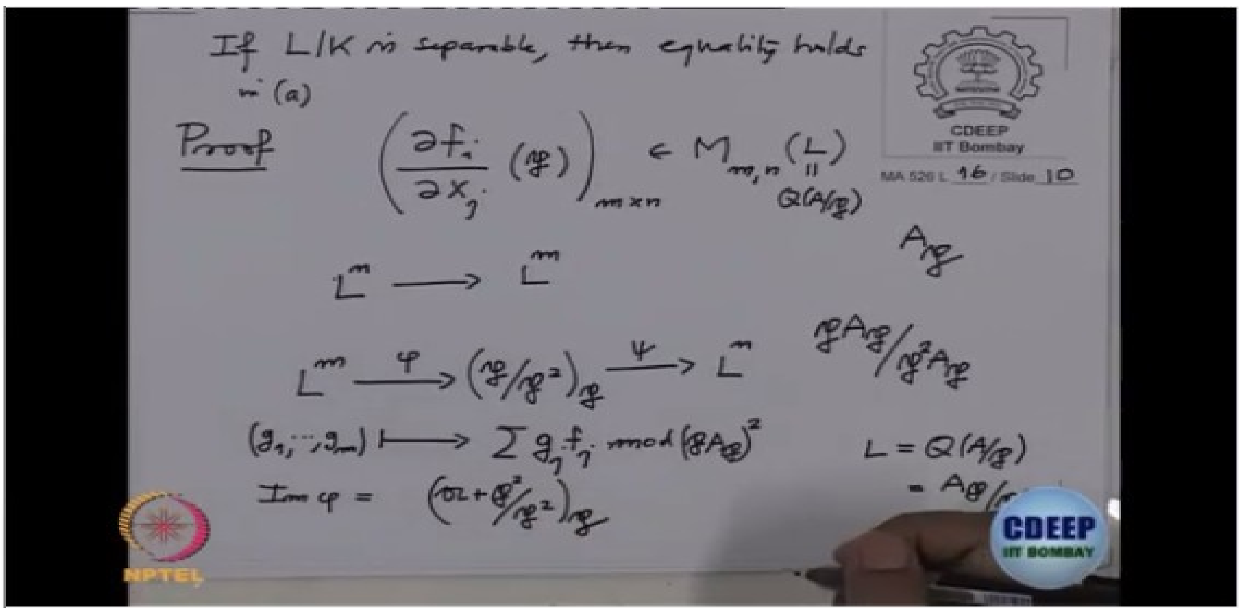

That is clear, because we are reading mod p^2 , pA_{p^2} . And this one, not only here it is actually here in ideal A, therefore, this is indeed this one. Now, for the other one, defining psi, to define psi, I want to define, actually I want to define $\frac{p}{p^2}$, localize at p to L^n . This I want to define, right? This is what I was looking for, this ψ . Okay. So, suppose, I take f in p^2 , it should go to zero here, right. If I have to define, this has to zero there, okay. So, what does this mean? It's in p^2 means, this f we should look like square of somebody, right? So, it should belong to, so that means, I need somebody not in p and some guys are in p square, so that means, I need gjhjs are in p and some h which is not in p, and hf should be like this, this sum. See, suppose I have such an equation, what will it mean when read in a localization? That mean this h go down and if gjhjs are in p, therefore, this part is in the square. So that is what one wants. Okay. Suppose, I have such an equation with this gjhjs in p and h not in p, and I differentiate, what do I get? So that means, I will get, change rule. So, $h\frac{df}{dx_i}+f\frac{dh}{dx_i}$ equal to summation, this summation is over j, let's say j equal to 1 to r, $g_j \frac{dh_j}{dx_i} + h_j \frac{dg_j}{dx_i}$, such a thing I will get. And where will be this, this is in any case, it will be in p always because g is in pA and h is in p. So, this side is always in p. So, this is in p. So, doesn't that mean that, now when I read, if I have to read mod p square, that mean this term will go. So that will mean that, and this will become zero, this will become in pA_p . So when you read, now read in pA_p , mod $p^2 A_p$ in this, this is m by m square in a local ring. So I want read this equation here. So, what do I get? This will get, this will mean that df by dxi this will be in pA_p . And this will allow us to define this map. So, what is the map now? Take any f and map it to-- of course, when I say f, f is image here, $\frac{f}{p}$, this one, map it to $\frac{df}{dx_i}$ and read this mod by A_p . This $\frac{df}{dx_n}$, and this we are reading pA_p . So you get an element in L^n . And so now this map, see I calculated it here and showed you. I could have directly tell you this map. But that's not-- So, once check that this map is L linear, so this is L linear.

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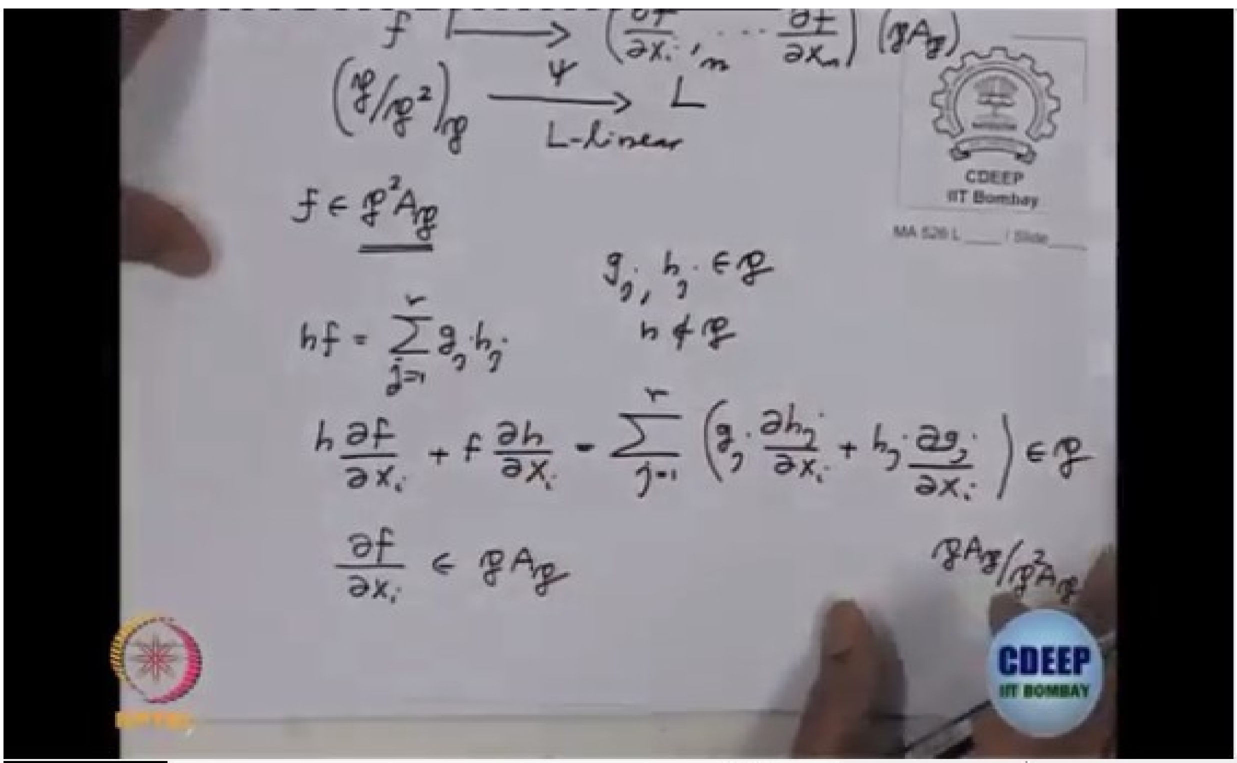

And therefore, how do we tie up with our problem? Our problem is now, therefore, I compose. So, ψ compose ϕ . Remember, ϕ is from L^m . This is from L^m and to L^n . And now, the Jacobian matrix is precise, so what we need to prove is the rank. Rank is therefore, so this map is precisely the Jacobian matrix like, and therefore, we want to compute the rank, rank of a composite. So, rank of ψ compose ϕ , this will be less equal to rank of the image, image of the first one less equal to. This is clear, because this is a, think of this as a linear map from the vector space image. So, this is clear. And this one now, you check it is the rank what we want. This one is the number on RHS, and this is what the number we were looking for. So, if you want to prove equality here, all that we have to prove is that this image is everybody, not image. If we want to prove equality here, then we want to prove that, what will we need to prove, that the rank is m. The rank of this is m. And that was, again, under the separability assumption. So earlier lemma also-- So, just this, this will tie up.