Lecture – 41

Jacobian Matrix and its Rank

Gyanam Paramam Dhyeyam: Knowledge is supreme.

We are going to continue the proof of Jacobian criterion and for this proof I will recommend you to see book by Balwant Singh. The title of the book is Basic Commutative Algebra its by World Scientific. There it is proved much more general then what I am doing but its there in every. SO the basic problem is one you set up correct notation and correct definition things will become clearer. So I will go little slowly today. So last time I was trying to do a Lemma , let us equal that, and we've not finished a proof yet. So the Lemma 1, which I will need for the main proof of Jacobian criterion, so that K field and we are working in polynomial ring $K[X_1, \ldots, X_n]$ and this are denoted by A. And we have a prime ideal p here, p is a prime ideal in A of height M, height of p is M. Therefore by our dimension arguments we know that dimension $\frac{A}{p}$ this is equal to the dimension of , so n minus height of p. The height plus dimension is equal to dimension of the ring. So this n minus m. And let us put this number as K. So that also shows that if I take the transcendence degree of the coefficient field of $\frac{A}{p}$ over K, so transcendence degree of the coefficient field of $\frac{A}{p}$ over K this is the dimension. So this is K. So that means if I call this L to be the coefficient field of $\frac{A}{p}$ this as transcendence degree K or L. L over K transcendence degree is K. Okay. That means, I can also-- So, first let me write the statement then we will-- so this is the situation then what do I want to do. I want to prove two statements, a that I want to find m elements., $f_1, \ldots, f_m \in p$ such that, if I go to localization A_p in Ap the ideal pA_p is generated by this f1 to fm. This happens in the ring A localize at p.

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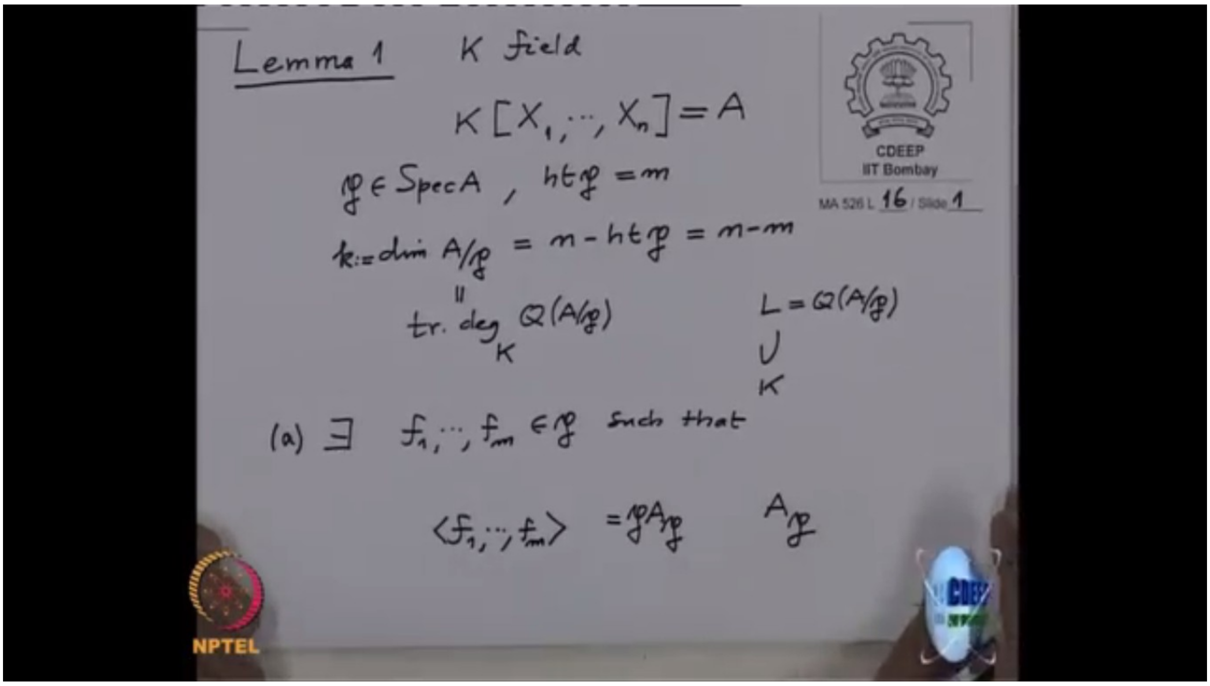

That is the first part and second part B, I want to prove that if this L over K is separable then if I look at the partial derivate of f_i with respective to X_j and read mod p, now this is think of the other matrix, this matrix has row are numbered by i, columns are numbered by j, so row numbers are m. So this m cross n matrix. What are the entries the entries are in, so this in $M_{m\times n}(\frac{A}{p})$. So this the matrix, again, this is matrix in L, it entering in L. So rank of this matrix that is the statement. The rank of this matrix, $\frac{df_i}{dX_j}$ mod m mod p, this is prissily m. That is under the assumption re-separable. And let us we call separable one way to think separable is their exist a transcendence basis so that the algebraic part is separable. So, that is definition of separable. Arbitrary field exchange and it's called separable if their existing transcendence base is, so that the algebraic part of this is separable extension. And separable algebraic means every polynomial the minimal polynomial of every element is separable polynomial. Separable polynomial means it has distinct zeros. No repeated zeros. No multiple zeros. Okay. So, proof . We are proving first we want to choose f1 to fm. So that locally they generate p. Okay. So first of all we may assume, so I am denoting $\frac{A}{p}$, this is in small letters now. K small x1 to xn, it's the coefficient of the polynomial ring, so it is in small letters. And then our, this is the coefficient field of this is L. L is the coefficient field. So this is a coefficient field. This is $Q(\frac{A}{p})$. And there is K here. Okay. And now, this is therefore this L over K is generated by this small xa, so this L is capital $K(x_1,\ldots,x_n)$. Because it's a coefficient of this integral domain. Therefore this $x_1,\ldots,x_n$ I can choose, so choose we may assume first the transcendence degree is k that we know. So, first $x_1,\ldots,x_k$ is a transcendence basis of L over K, because any generating set of a field extension will contain a transcendence base.

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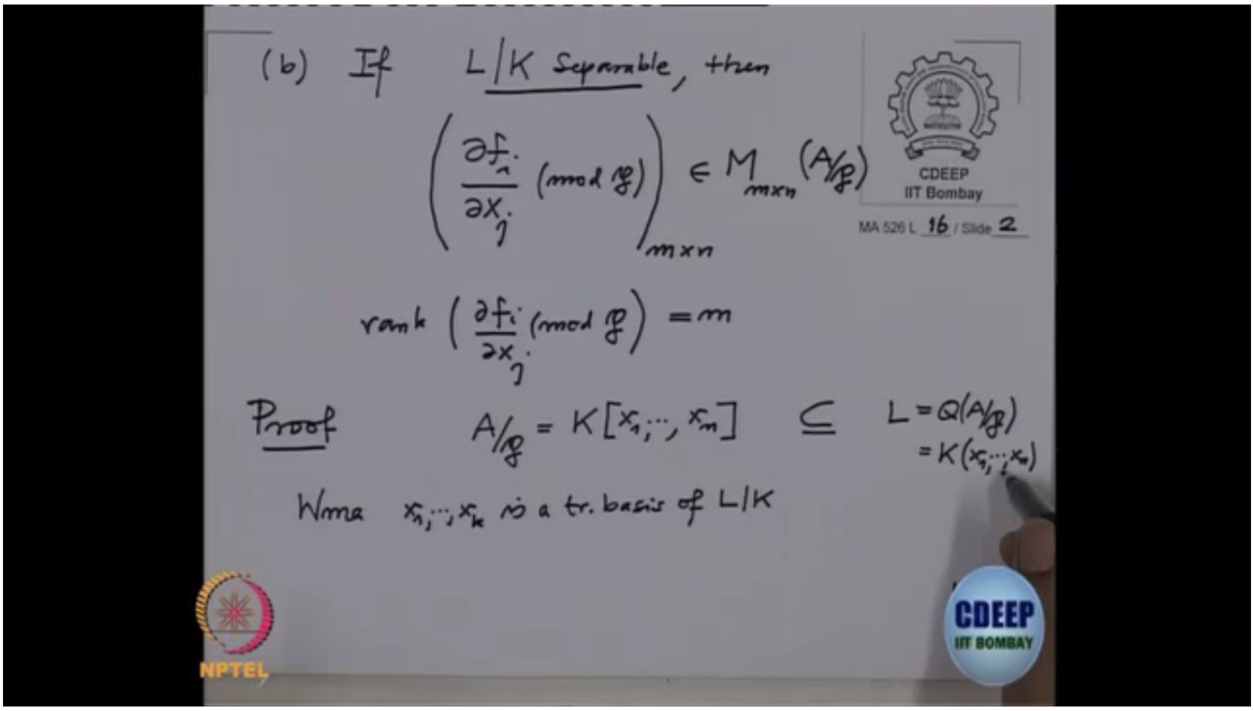

It's like a vector space. Every generating set of a vector space contain sub-basis. The same, every generating set of a field extension we contain a transcendence basis. So I will renumber them and assume that $x_1, \ldots, x_n$ is a transcendence basis of L over K. All right. So, again let me draw a diagram, so $\frac{A}{p}$ is here, L is here this $K[x_1, \ldots, x_n]$, and there is a L not here which is k small $x_1, \ldots, x_k$, this part is algebraic now and this part is transcendence part. So that's the situation now. This is coefficient field of $\frac{A}{p}$. And now I claimed last time something and one of the trouble was, it was the notation et cetera was little bit clumsier. So I have to improve that. Okay. So, what is that claim? So take any integer L between 0 and m, remember that $k+m=n$. So take any l in-between 0 and m and then I have defined a map from L not and then the polynomial variables. So, which variables, already L not, this $x_1, \ldots, x_k$ already they are algebraic independent. So they are also like variables. So I take the next ones. So that $x_{k+1}, \ldots, x_{k+l}$. I cannot take more than m because otherwise it goes. So, look at this polynomial ring and from this polynomial ring to L I am defining a map pl. So I just have to assign where do the variables go.

So take any X_i variable, and mapped to small x_i . And want to describe the image and cornel of this algebraic modules. That will also lead us to how to choose the polynomial to $f_1, \ldots, f_m$, so that p is locally generated by $f_1, \ldots, f_m$. Okay. So, the claim is a fallowing. They are three parts in the claim, claim is image of pl is field generated by $K(x_1, \ldots, x_{k+l})$. So this field I want to denote by L_l . That is the one part. What is the Ker of pl is generated by the polynomials $f_n, \ldots, f_l$. Their this f_i is actually, f_i belong to polynomials up to ith variable, $K[X_1, \ldots, X_{k+i}]$ and they also belong to p, intersection p.

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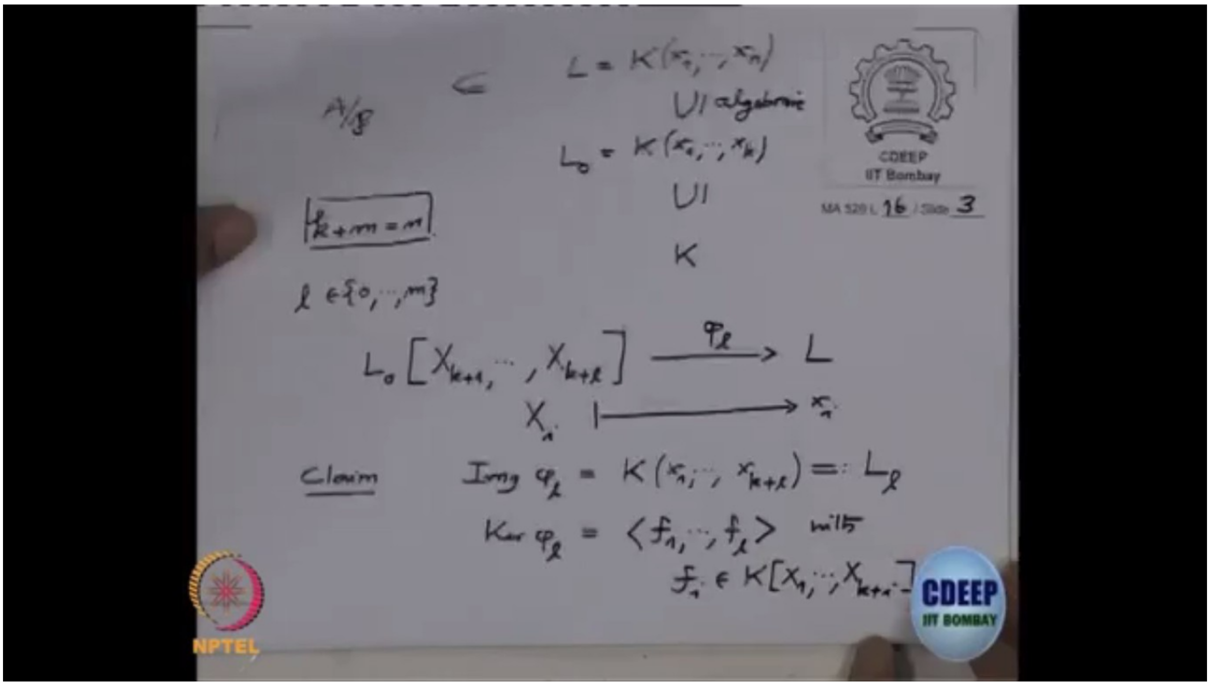

These are two parts and the third part is under the assumption if L or l not is separable. So this is same thing saying L over K is separable. This is the algebraic part of the L over K. So if that is separable given then if I take the partial derivative of f_i with respect to the last x, dX_{k+i} , this cannot belong to p. So that is a claim. Claim is very simple and if you do one by one I am going to prove claim by induction on l.

So proof of the claim, this is by induction on l. So remember l is that, if l is 0, there is nothing to prove. Because what you want to prove. Let's see here, what do you want to prove. If l is 0, you want to prove that the image is $l=0$, L_0 is transcendence part. Then p is, zero and so on so it's nothing to prove. Okay. So we may assume, l is bigger than 0 and already we have found a polynomial of $l-1$, already we've obtained $f_1,\ldots,f_{l-1}$ with the property we wanted. Image of l , this one, kernel of l is generated by $f_1,\ldots,f_l$. And their all in p. So already we have found this. Now I want to do the next step. Next step is what now? I want to choose one more polynomial. So just look at the element x_{k+i} , this is an element in L and this one is algebraic power of L_0 , And remember we've already have this L_{l+1} .Which is by definition it was the image of $l-1$, this is image of ϕ_{l-1} , remember if $l-1$ is a map from the polynomial ring over l 0, from $k+1$, up to $k+l-1$ inside l. So this already found, so this contains L_0 , so already this is algebraic, therefore this is algebraic, therefore this x_{k+l} , will have a minimal polynomial over L_{l-1} . So $k+1$ is algebraic over L_{l-1} , so it will satisfy a minimal polynomial, every disable polynomial, so that I am going to denote by g. So g is by definition irreducible polynomial of, so some times one denote like this $Irr(x_{k+l},L_{l-1})$. So this is a polynomial in one variable over this field. So this belongs to L_{l-1} and the variable I am we are writing it capital x_{k+l} . So the g evaluated at small x_{k+1} , l is 0, and anybody any other polynomial, which when you plug it in the small x_{k+l} it becomes 0, then g divides that, that is the property of the individual point. Okay. So this and this is the polynomial over L_{l-1} and this is the image of this field map.

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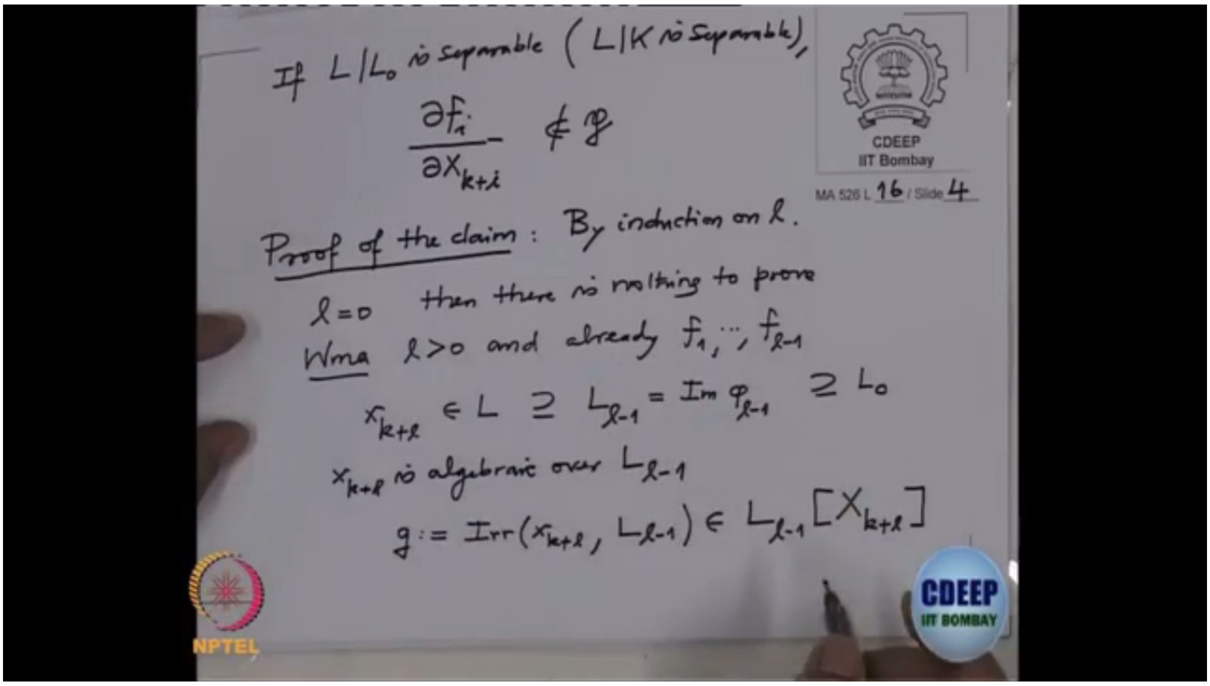

So therefore this polynomial g is will definitely look like, the sum polynomial f_l and then it's polynomial over with over L_{l-1}, that means it can, it as this x_{k+l-1}, and the variable. So x_{k+l}, and it will be also, it's polynomial over this so this one is a field generated by $x_1, \ldots, x_{k+l-1}$, so that divided will be h common denominator will be $x_1, \ldots, x_k$. So this is our g will look like, where, because this h is in the denominator, the polynomial $h(x_1, \ldots, x_k)$, this polynomial cannot be in P, because you know everything is happening in l, l is the coefficient field of $\frac{A}{p}$, where P is. So therefore this denominator cannot be zero. So therefore that means the polynomial h will not be in ideal P. So therefore all these when you and the numerator f_l, this is actually polynomial in $K[X_1, \ldots, X_k]$, this is l,..., k + l intersection P. This is what we have, because when I put X_{k+l} here it becomes 0. So therefore it is in p. everything is happening in the coefficient field of $\frac{A}{p}$, remember that, so, okay. Now if so what do you want to prove, here we want to prove two things that are mainly, we want to prove that this f_l what we found here, along with $f_1, \ldots, f_{l-1}$ and f_l, the generate P locally at P, that was one. And second was the derivative of, f_i by differentiation is $2X_{k+i}$, this cannot be P. This is also you have to prove. Okay. All right. So now if you assume it is separable, if you assume separating transcendence base. First I am proving derivative cannot be 0. But that was under the assumption that separable extension, right? I want to first prove this the last part of the claim. So that we have a assumption it is separable extension. So, because L over L_0 is separable then L_l over L_{l-1} this is also separable, because this are in between fields. Okay. That means, that the errdisable polynomial of any element here, over this the derivative of that will not be 0, the route will not be repeated. So that will mean that if I take the differentiation of g, dg with respect to this one, dx_{k+l} , and he velvet at where it is 0, that is x_{k+l} , this is not 0, because the g is a errdisable polynomial of x_{k+l} , over this field L_{l-1} . But what does this mean? These mean that, so that these mean that if I differentiate, see when you differentiate this and it is so this means that $\frac{df_l}{dx_{k+l}}$, this one is not in P. if suppose somebody is in P that means you have to put all this small, small x size and it should become zero, but it doesn't become zero that means it is not in P, see remember we are working in the field L, where L is the portion full of $\frac{A}{p}$. So if you want to test somebody in P are not you had to say it is 0 in l, but it's not 0 in l, therefore it is, it is not in it. Okay. So that first of all shows that, first note that is f_l is already belongs to kernel f_l , this is clear because how do I test somebody is in the kernel f_l , I have to replace all the variable by small x size. But when I put only one variable here is this. When I put this equal to x_{k+l} , that is like putting $g(x_{k+l})$ is a route of g, therefore that is 0, therefore this belong to this, it's clear, all right.

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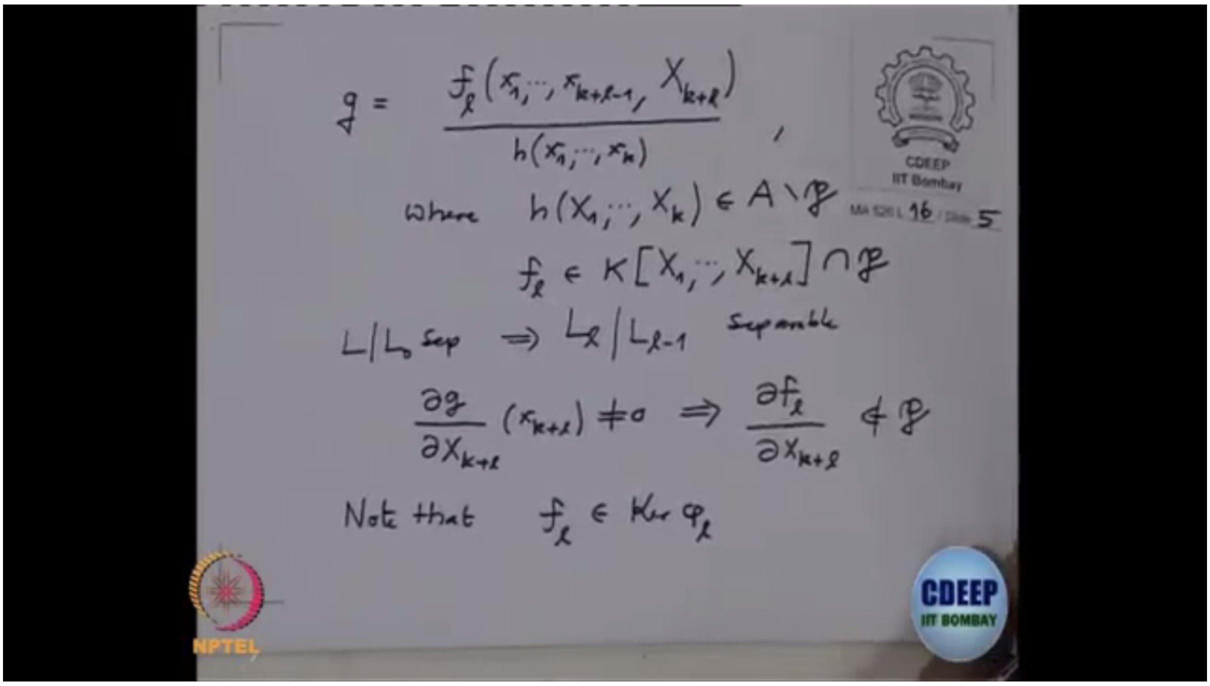

So this shows first of all that if I take the ideal generated by $f_1, \ldots, f_l$ that is contained in that is, see I want to prove this is equal to kernel of f_l this is what we want to prove equality here. This is the first part of the claim, right? We want to choose polynomials $f_1, \ldots, f_l$, so that ideal generated by that is the kernel that was the claim, the first claim. Here this kernel of, f_l equal to this, with f_i in this. So fis are already we have tested and we want to prove the equality here but equality also, we have already proved this inclusion by induction all these guys are in the early on, so later, so this is already clear only we have to prove the other way. So to prove the other way I will take f here and prove that f is a combination on this. All right. So f here means what? F here means, if I put variable to the small x_k it become zero. So that means we have only up to $k+l$, so I put the last earlier variable, small x k and the last one which is small x a plus l, that means this polynomial $x_1, \ldots, x_{k+l}$, this is 0 means, if I just look at this that means g will divide this polynomial. So that will mean that g divides f, first $k+l-1$ were able to do a small exercise and the last one capital. This divides where the division is in L_{l-1} , X_{k+l} . So that means, so this divides this, so just write down what does it mean, so that means there exist polynomials are r is in $K[X_1, \ldots, X_{k+l}]$, s should come in the denominator. So it should be not in p so it is in $K[X_1, \ldots, X_k] \setminus p$ such that if I plug it in the first $k+l-1$ are do the small x, I will get equation like this, $f(x_1, \ldots, x_{k+l-1}, x_{k+s})$. Times $h(x_1, \ldots, x_k)$. Remember this h is coming from that g when we wrote g as a rational function so this divided by $f(x_1, \ldots, x_{k+l-1}, X_{k+l})$ this is equal to $r(x_1, \ldots, x_{k+l-1}, X_{k+l})$ divided by $s(x_1, \ldots, x_k)$. This divide this, so write this f is the multiple of this g. And multiple means, so this g was written as s. I will show it, g was written as this. So I just cross multiplied it. So just cross multiplying it will get this. Now, this equality means, what? You cross multiply it so cross multiplying what do you get? You will get like this. So now cross multiplying it this s times, f times, h so I will write h s f. This is s times this f, this h equal to r time this fl. So this is equal to r times f. And we are interested in the coefficient of as a polynomial in the variable x_{k+l} . so first of all, these belongs to the Kernel of f_l . So the kernel of $l-q$ because think of this is a polynomials $k+l$ and compare the coefficients. The coefficients when you do this equality coefficients are equality, coefficients are equal so the other side is so that show that these polynomial is actually in the kernel of $l-1$. And kernel of a $l-1$ is generated by polynomial f_1 to earlier polynomial, $f_1, \ldots, f_{l-1}$. And now look at this h and s, they are not in p. So therefore, when I localize it, it will go down. There I can go down in the denominator, so therefore this f will belongs to the ideal generated by $f_1, \ldots, f_{l-1}$ and along with this f_l locally that is clear because h and s they are not in p. So that implies f belongs to ideal generated $f_1, \ldots, f_{l-1}$ and this ideal is now not in the a but a localized p. In the ring a localized p. So that proves that the, you wanted to prove the kernel, so the prove equality is here. You see, the proved equality here. Because we took arbitrary f here and we proved this f is in ideal generated by this locally. Okay, so that proves this claim. Now in the claim I'm going to use it for the last guy that is l equal to m. So that means, what we have proved is kernel of m is generated by $f_1, \ldots, f_m$ and this was in actually the field $k[x_1, \ldots, x_k]$ and then the polynomial after that. This was it. But this is contained in a localized p. So therefore when I localize it you get so locally they are correct. So now what is the statement we want to prove? The next statement that the rank. The rank of the matrix is m.

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$$\langle f_1, \cdots, f_\ell \rangle \subseteq \operatorname{Ker} \varphi_\ell$$

$$\ni f(x_1, \cdots, x_{k+\ell-1}, x_{k+\ell}) = 0$$

$$g \mid f(x_1, \cdots, x_{k+\ell-1}, X_{k+\ell}) \qquad L_{\ell-1}[X_{k+\ell}]$$

$$\Longrightarrow r \in K[X_1, \cdots, X_{k+\ell}]$$

$$s \in K[X_1, \cdots, X_k] \setminus \mathfrak{p} \quad \text{such that}$$

$$\frac{f(x_1, \cdots, x_{k+\ell-1}, X_{k+\ell})\, h(x_1, \cdots, x_k)}{g(x_1, \cdots, x_{k+\ell-1}, X_{k+\ell})} = \frac{r(x_1, \cdots, x_{k+\ell-1}, X_{k+\ell})}{s(x_1, \cdots, x_k)}$$