INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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CDEEP IIT BOMBAY

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Lecture No. – 04
Chevalley's dimension

Alright, so let us continue about the dimension, so first of all note that as I said while we will use the fact that this polynomial ring in several variables over a field this is Noetherian and therefore any algebraic set V, $V_K(A)$ this is intersection of finitely many hyper surfaces, $V_K(f_1)$ intersection-intersection, intersection $V_K(f_m)$, finitely many hyper surfaces. Not only that this ideal A has a primary decomposition, because in Noetherian ring every ideal is a primary decomposition that means this ideal A is written as intersection of finitely many primary ideals, Q, R, where Q_1 to Q_r these are primary ideals, this is why in fact Noether have approved primary decomposition for ideals in Noetherian ring, and their radical they are, when you say primary ideals corresponding to the P_1 to P_r or their radicals, these are the, their one calls this Q_i to be P_i primary.

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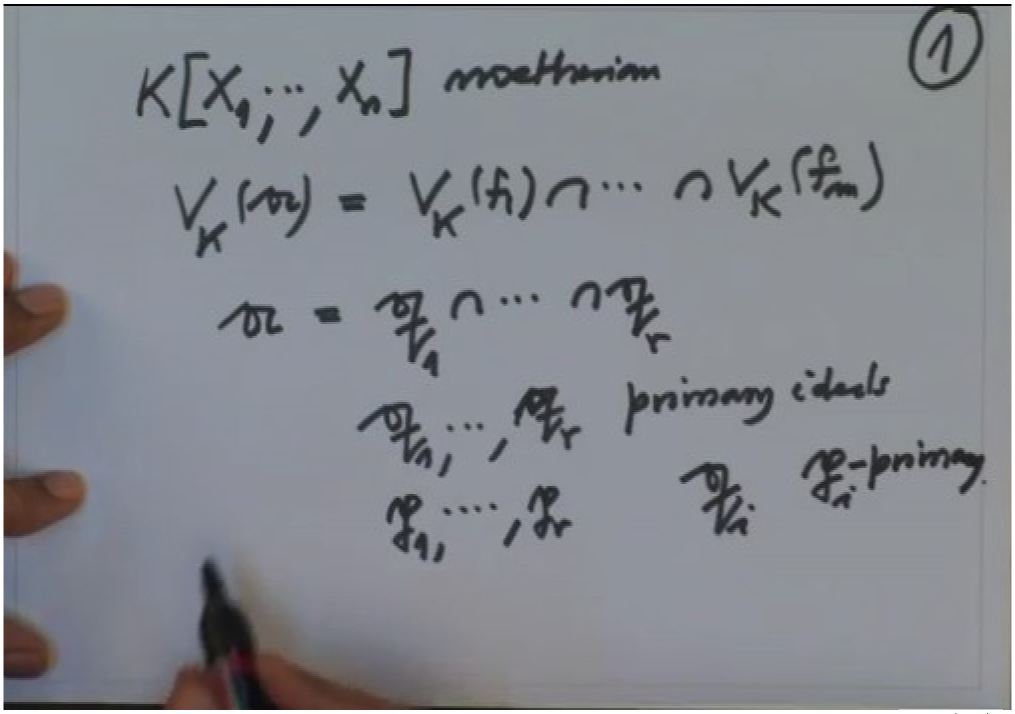

And therefore when you apply V and look at the properties of V then therefore $V_K(A)$ will become $V(Q_1 \cap Q_i ... Q_r)$ but then this will become union, so this will be $V(Q_1) \cup V(Q_2) ... V(Q_r)$ but this Q_i 's are the radicals are, radical of Q_i is P_i and there V doesn't depend on the radical therefore this will be $V(P_1) \cup V(P_2) ... V(P_r)$ therefore if one want to study this,

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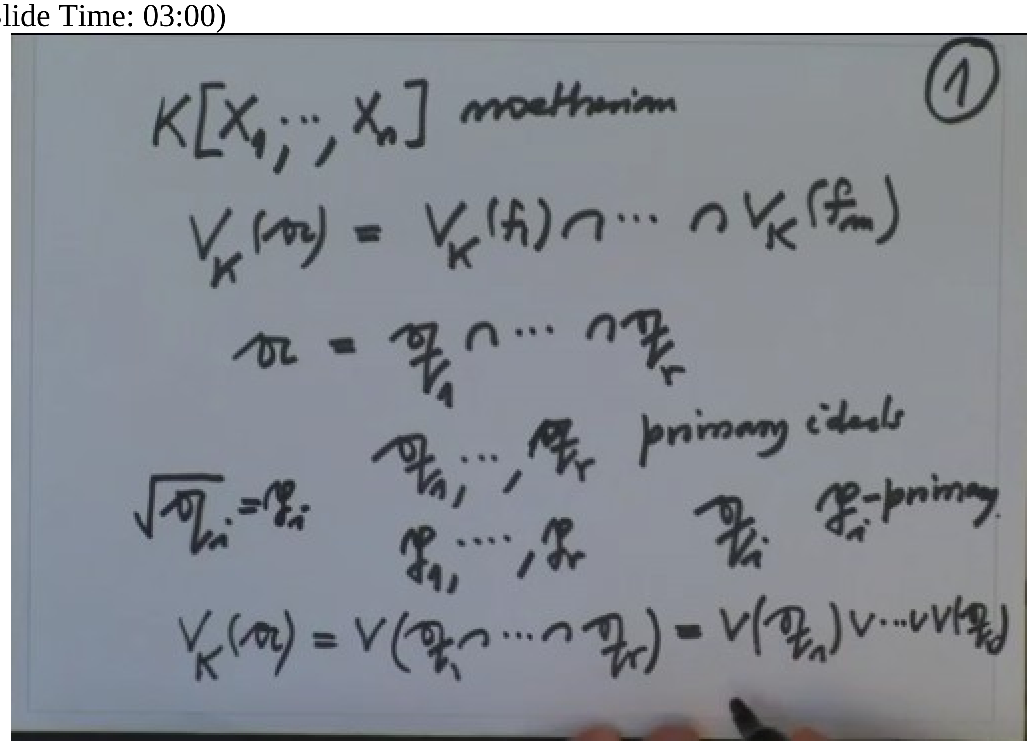

I wrote V so this is V_K everywhere, therefore if one want to study this V, I have to study this union, and therefore the dimension of this will be the sup, because that was our property 2, so therefore and also know this is

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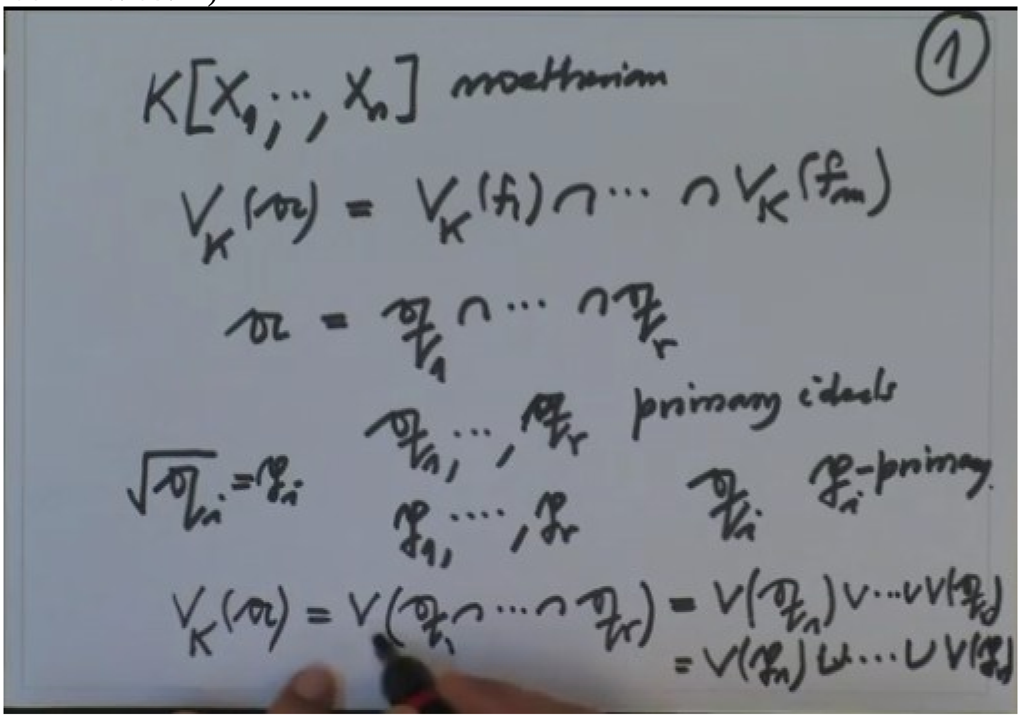

therefore what we proved is every algebraic set in K^n is a finite union of algebraic sets of type $V_K(P)$, where P is a prime ideal in the ring $K[X_1,\dots,X_n]$, but these are precisely the irreducible algebraic sets, this needs little proof but I'm not going to prove it, you take it as a exercise therefore what we have proved is every algebraic set is a finite union of irreducible subsets, alright.

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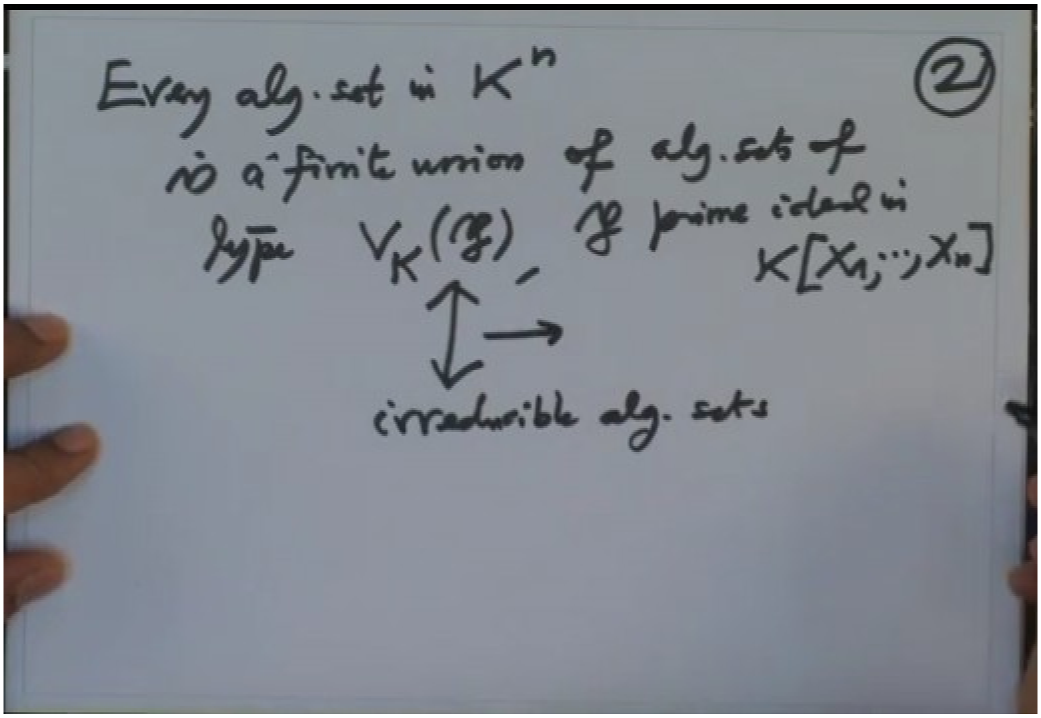

Therefore 2 says, so we have proved therefore V algebraic set is a union of let me call it $V_1 \cup V_2 \ldots V_r$ where this V going to V_r are irreducible algebraic sets, and therefore our property 2 says that at least one of them will have the same dimension as V,

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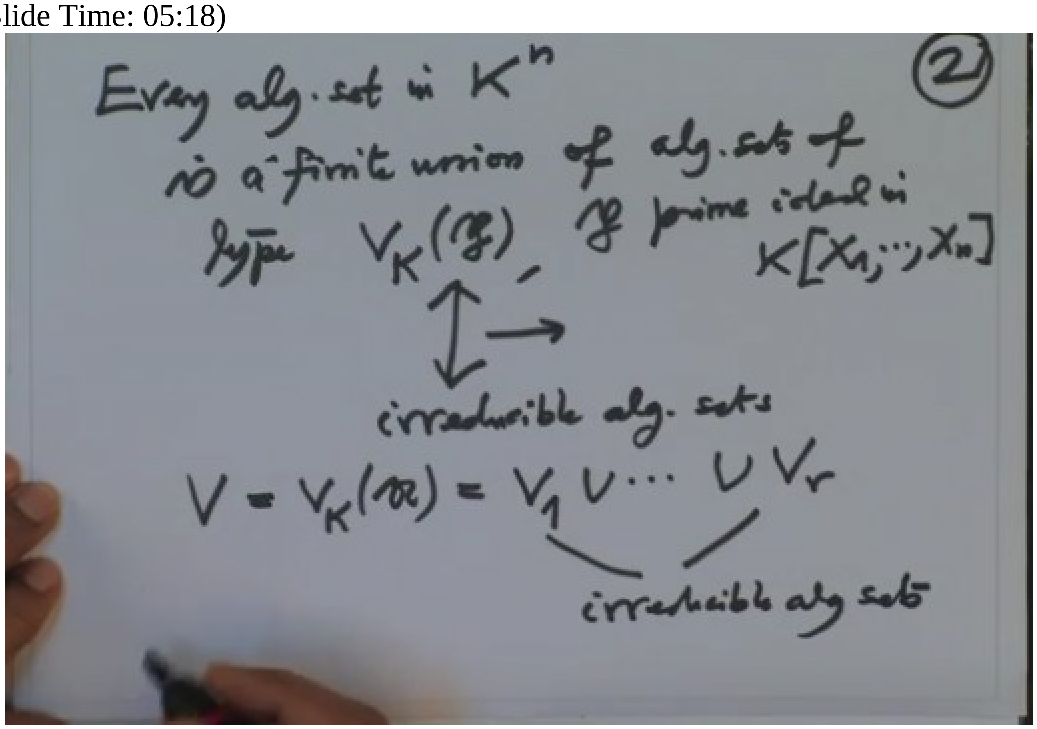

so dimension of, property says that dimension V should be equal to the supremum of dimensions of V_i's and therefore this is a supremum over finitely many V_1 to V_r and therefore supremum has to attain, therefore dimension of V

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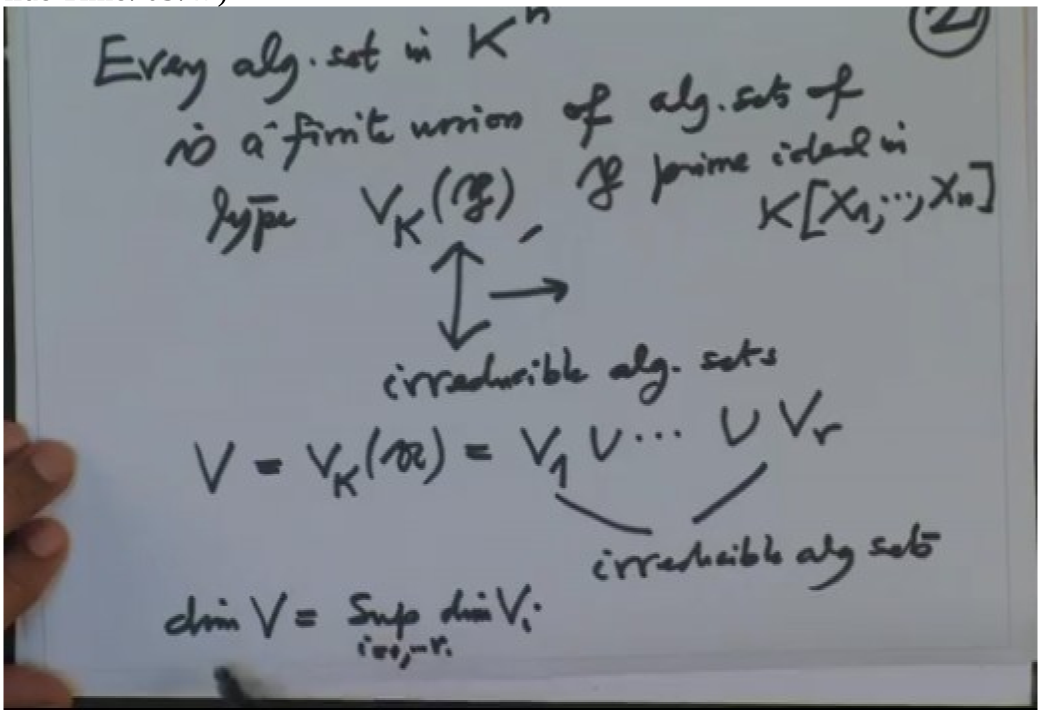

will be equal to dimension of one of them, let's say V, let me call it V naught, and this V naught for some irreducible subset of V.

If dimension V naught is positive, then the property 4 says, then by property 4 there exists an irreducible algebraic set V_1 of V_0 proper, so that the dimension drop by 1 such that dimension of V_1 = dimension of V_0 – 1 that was a property 4, alright.

So there if d was the dimension of V so therefore if d is the dimension of V, we continue, we continue this argument so that means we can find a chain, than there exists a chain V_0 properly contained in V_1, properly contained in V_d of irreducible algebraic sets of V, algebraic subsets of V such that at each stage dimensional drop by 1 such that dimension of $V_i = d - i$, for each i from 0 to d,

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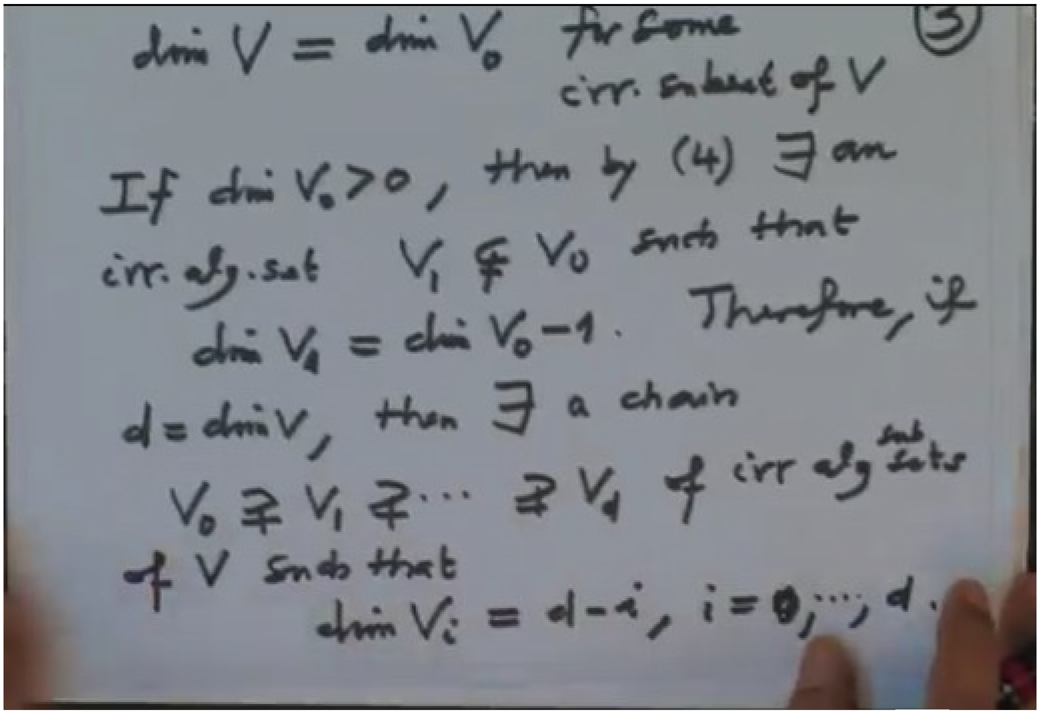

conversely if I have a sequence on the other hand given a sequence, sequence V_0 contained in V_1, contained in V_r of irreducible closed subsets of V with V_i, of course irreducible closed and part of irreducible closed is V_i 's are not equal to empty sets, then dimension of V should be bigger equal to dimension of V naught, because all these are contained in V, and at each stage the dimension is dropping by one, so dimension of V naught will be at least r, see this will be dropped by 1 so this at least dimension of V naught be at least r, this is by the property 2 and 3, so it is therefore we can make the following definition, so this allows us to make the following definitions, and what is the definition?

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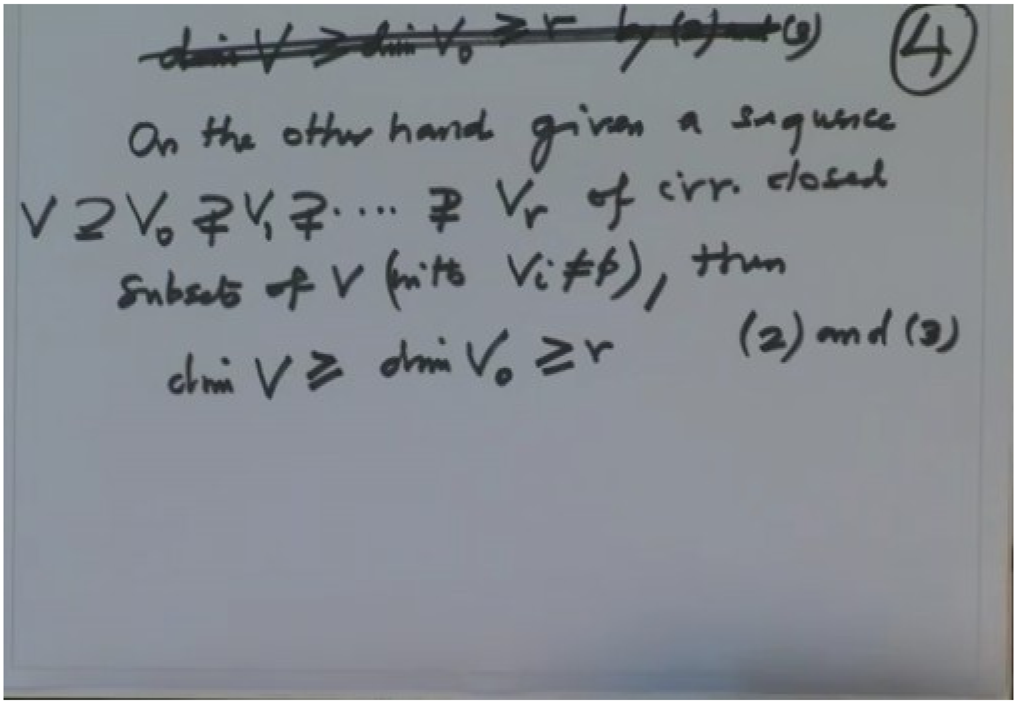

So definition, the dimension of an algebraic set V contained in K^n is dim V, this is a definition I'm making sup of r, r is in $\mathbb{N}$ such that there exists a sequence V_0 contained in V_1 contained in ... V_r of irreducible algebraic sets, algebraic of sets of V, that's it, so dimension is supremum of the chain of the length of the chains of irreducible subsets contained in V,

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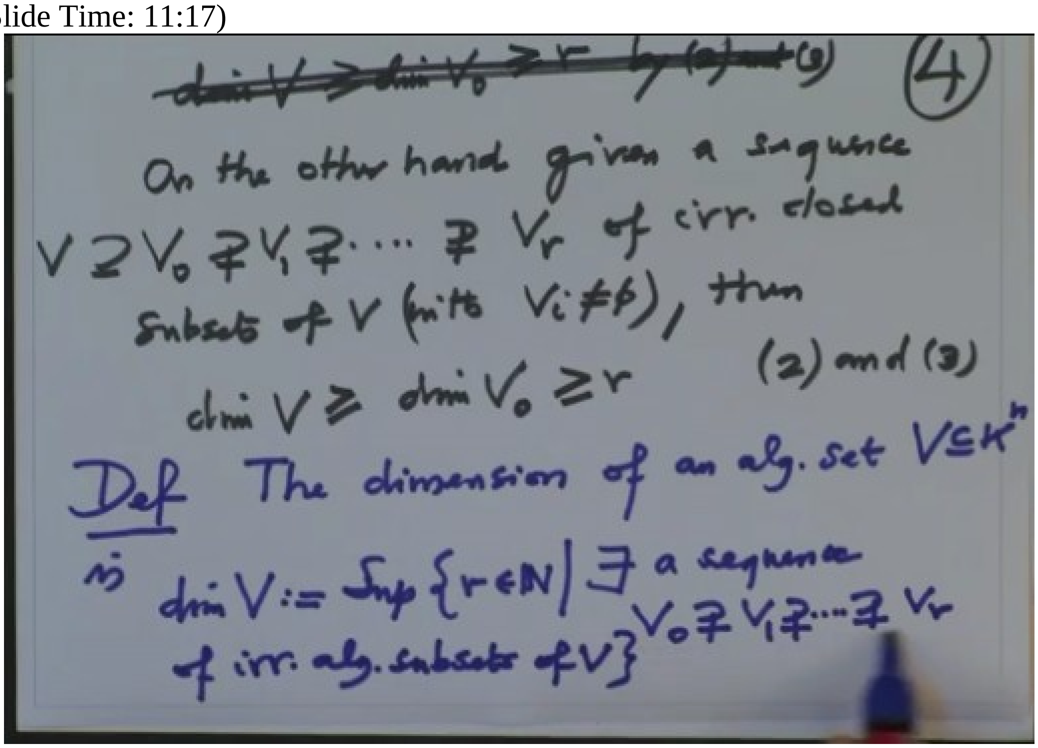

that is what we come to the conclusion that dimension should be this, and if we define this it satisfy the desired properties.

Once you have this, and here also note that when, we should make this convention, supremum of empty set is -1, this is actually not a convention but it is a force convention because if you take an empty set in natural numbers, what is the supremum? We should put that to be -1, so once you have defined this, it is easy to check that it satisfies the properties 2 and 3, 2, 3, 4 so now easy to check that this definition has properties 2, 3 and 4, we have only assume 2, 3 and 4 and come to this conclusion that the dimension should be this one that is a supremum of the length of the chains of the irreducible closed subsets contained in V.

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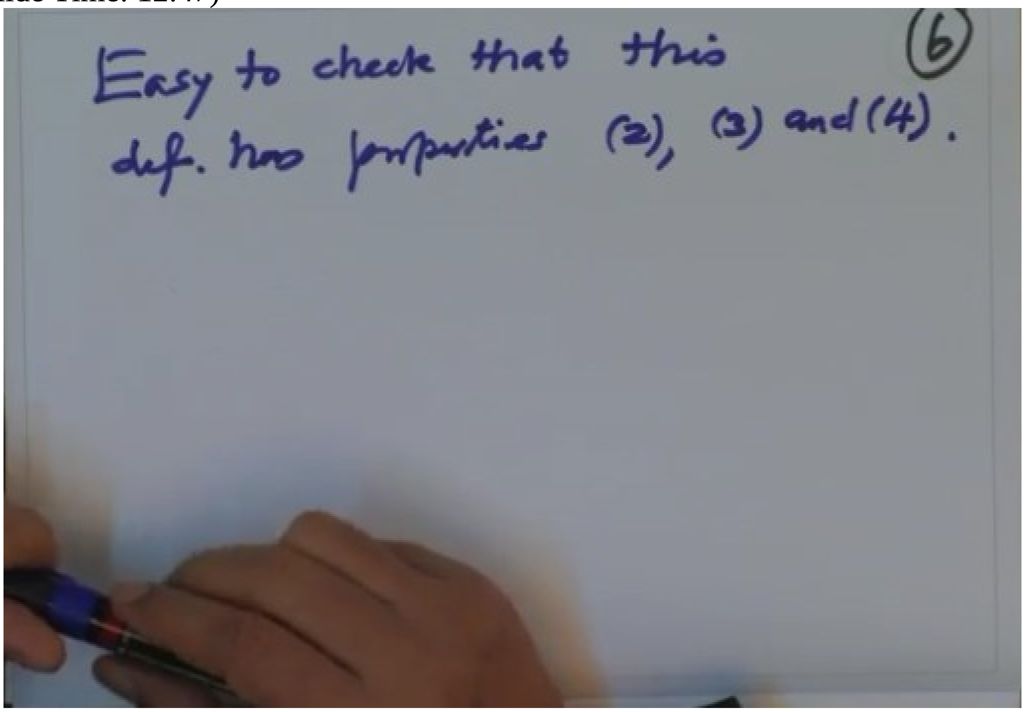

And now when I want to translate this to algebraic definition, now how do I translate to algebra? So translate this into algebraic definition, how do we translate? So that means I have to translate a chain so V_0 which is irreducible therefore it is V_K of some prime ideal P_0, and then contained in $V_{1,} V_1$ is irreducible therefore it is V_K of some prime ideal P_1 and so on, and then this V_K of prime ideal P_r, this chain I have given they are irreducible subsets of the given V, V was $V_K(A)$, alright.

Now how do I come to, how do I recover this prime ideals? That's very easy that is Nullstellensatz Hilbert is early proved that if I take ideal of this apply I_K to this chain, we get, what do we get? I_K of this will be radical of A so that is A, I'm assuming A is radical ideal, so when I apply I_K I get back my A, now this inclusion will get reverse, so this will be contained in I_K of this that is $P_{0,} P_0$ contained in this, P_1 contained in this, contained in P_r, and what is P_i 's? P_i 's are prime ideals, where P_i 's are prime ideals in the polynomial ring, so this will give you, what is called a Krull dimension of $\frac{K[X_1, \dots, X_n]}{A}$, because if I call this ring as R, the Krull dimension in the lectures how would I define? I'll look at the chains of the prime ideals here and take their lengths and take the supremum,

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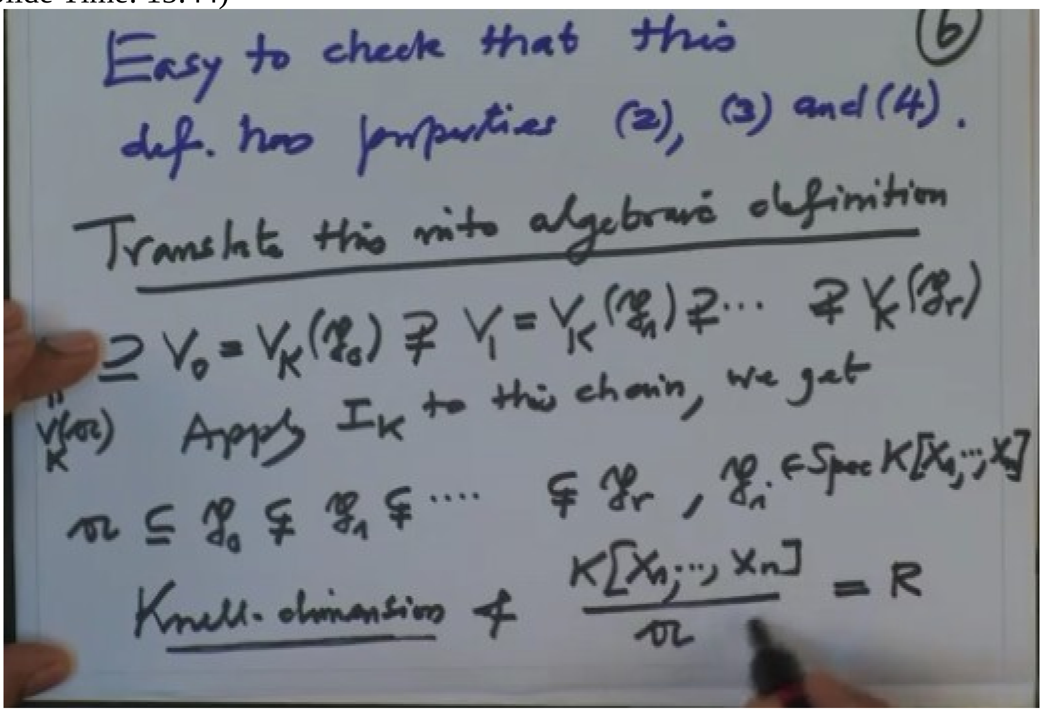

but now the chain has become the reverse way, because our this operators I and V they are inclusion reversing operations, and therefore I got the definitions of a Krull dimension, so therefore Krull motivated by this, in general define what is a Krull dimension of a ring? So let me recall that, so using this Krull defined more generally Krull dimension of an arbitrary commutative ring.

So what is the definition? So definition, let A be a commutative ring, then dimension of A is by definition sup r in $\mathbb{N}$ such that there exists a chain of length r, so P_0 contained in … P_r of prime ideals in A, this supremum is called the Krull dimension of the ring, because the Krull was the first to defend this,

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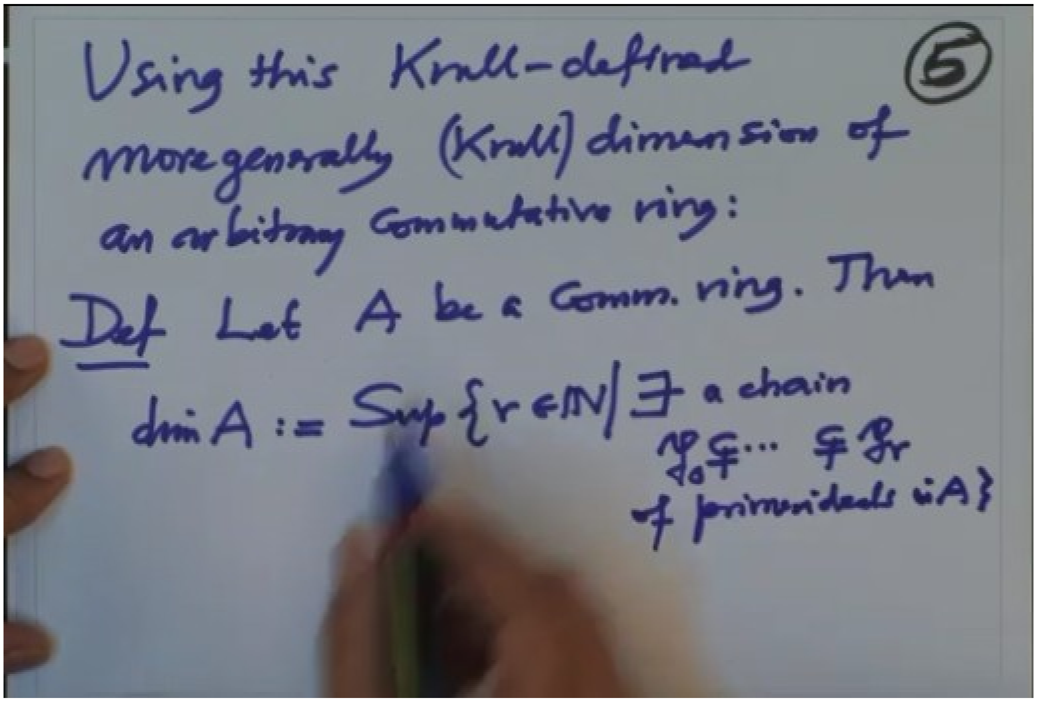

and also note that if A is 0 ring that is equivalent to saying there is no prime ideal there, therefore the supremum is taken over empty set so therefore in this case one puts dimension of A = -1, this is a definition of this ones.

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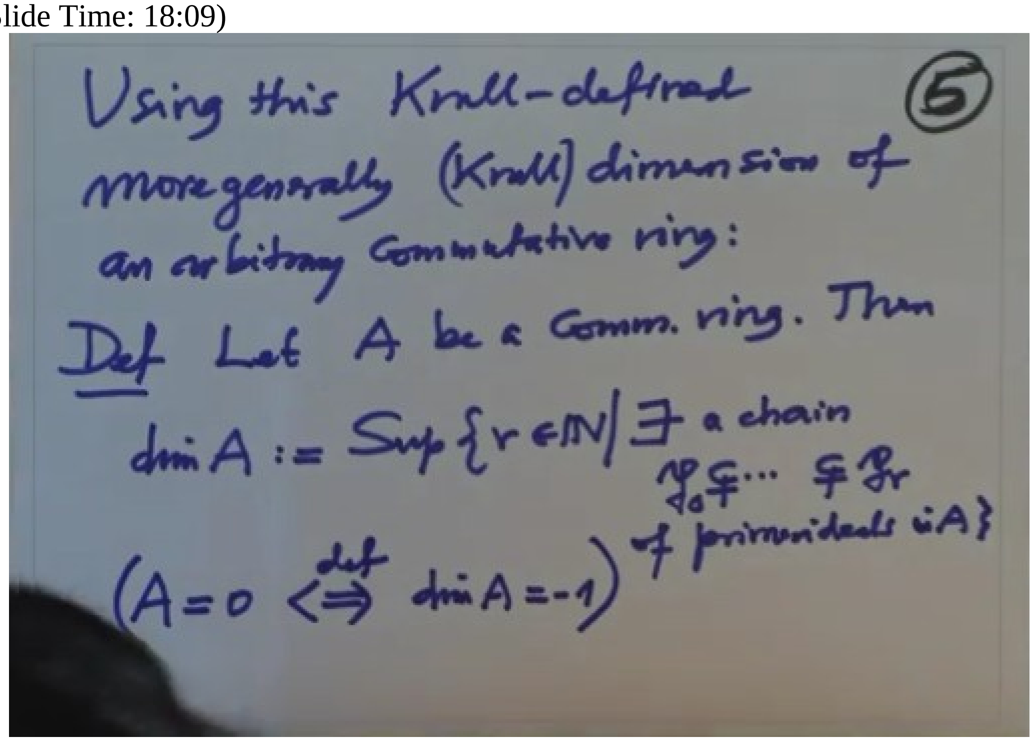

So all this was, all this used the motivation which came from Hilbert Nullstellensatz, so therefore Hilbert Nullstellensatz HNS, this is a real corner stone, this is a corner stone of algebraic geometry, it's the starting point, alright, so this, so dimension with this therefore dimension of variety, dimension of algebraic set, V algebraic set in K^n this is nothing but dimension of the coordinate ring of V that is you take $K[X_1,\ldots,X_n]$ and go modulo ideal of V, $I_K(V)$, that is the definition, alright.

Now having defined this the question is how do we compute this?
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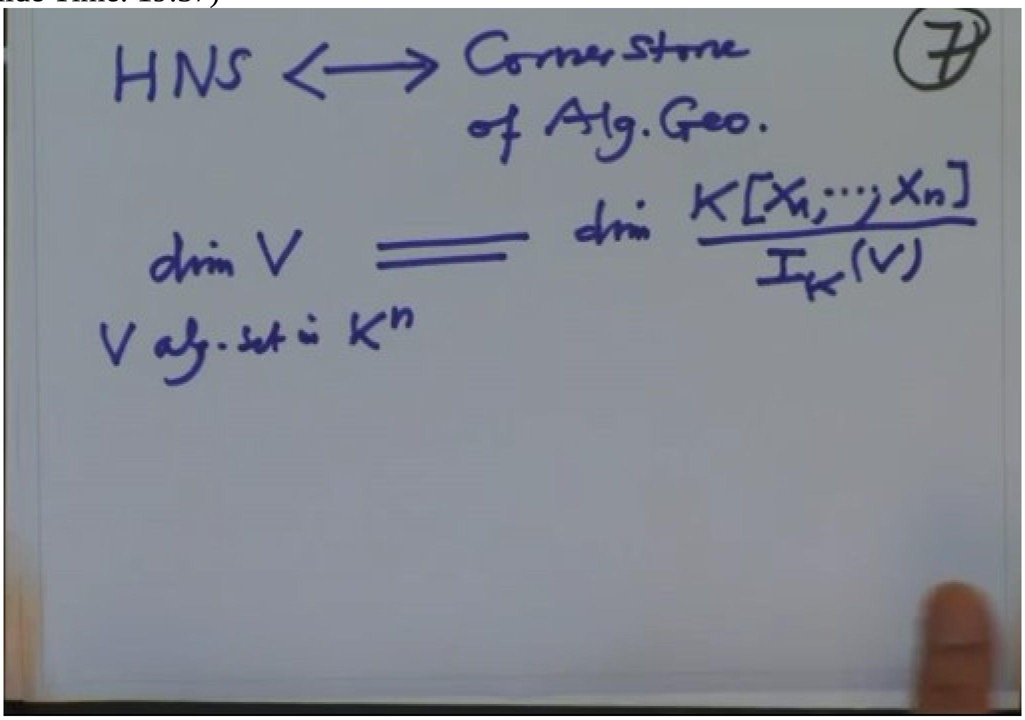

This definition of supremum of lengths of the prime ideals, this is looks quite, quite, quite complicated, so therefore can we say that now the definition is, how do we compute this? So later in the course, so from this definition it is not even clear that if I take V algebraic set in K^n, algebraic set whether the dimension of V is finite or bounded by n, this is not at all clear, okay, so to do this what one does is the following, there are two more concepts of dimension, two more numbers, so one is to V, so the property 5, so note that property 5 says that if I have V given then I can find a hyper plane, hyper surface H so that we intersection H has, if d is positive, if the dimension of V is positive then I can find the hyper surface so that its dimension drops by 1, and keep doing this so that means after finitely many steps I will come so that the dimension becomes 0, that means I can also, H will be, so keep doing this, so H will be given by single polynomial, so this is V(f), $V(f_1)$ and so on, so after d stages it will come, the dimension will become 0, so that means what? This means algebraically the following, so V has the coordinate ring that is $\frac{K[X_1,\ldots,X_n]}{A}$ and this intersection operation means when I go to algebra it will become ideal generated by the sum ideal, so this will become, so V intersection H1 will the coordinate ring will be this ring $\frac{K[X_1,\ldots,X_n]}{\langle A,f_1\rangle}$,
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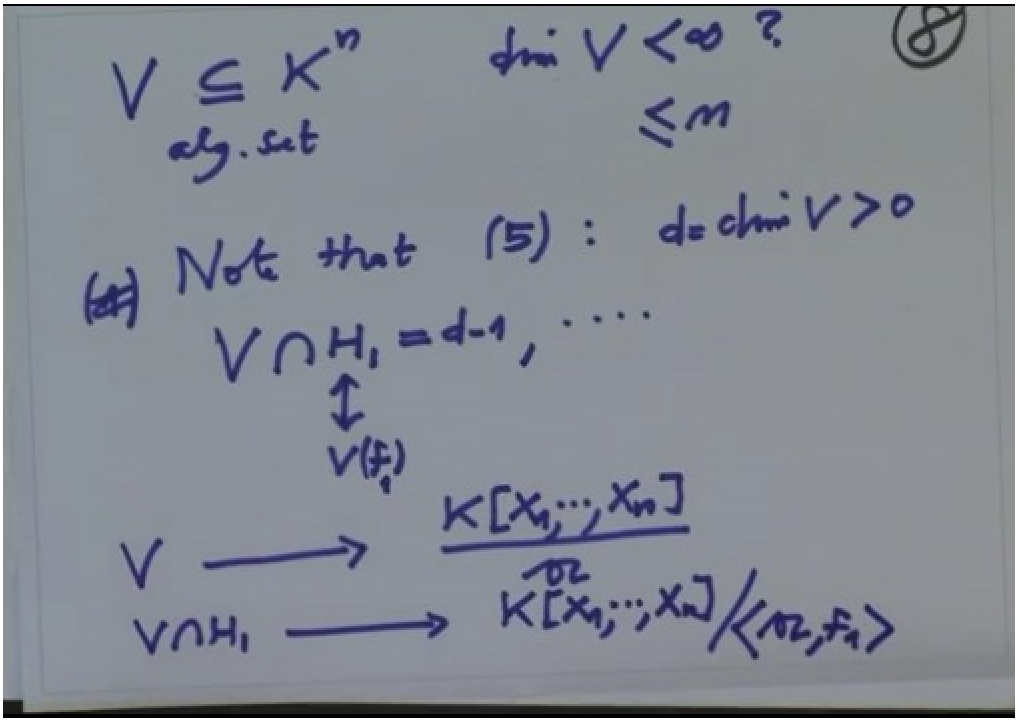

and this ring has dimension d, this ring will have dimension d-1, and we keep doing this d stages that means what? That means we can find the polynomials f_1 to f_d definitely, because it might happen that the dimension become 0 so utmost it can go to d stages, so we can find the polynomials in d polynomials in $K[X_1,\ldots,X_n]$ such that when I go $K[X_1,\ldots,X_n]$ mod ideal generated by A and this polynomials f_1 to f_d this dimension of this ring should become 0.

So note that when the dimension of the ring is 0 means what? So analyze this, dimension of A is 0 means, that means the only chain should have length 0, that means sup of the chains, so this sup means this will be equal to 0, that means utmost there is only one prime ideal there is no other prime ideal contained in that, so this means the first consequence is every nonzero, every prime ideal in A is maximal, because we have a prime ideal P itself should be maximal, if it is not maximal it is contained in the maximal ideal, but then the supremum will increase, so therefore every prime ideal is maximal, so such rings are called Artinian, so dimension 0, dimension A = 0 and A Noetherian these rings are precisely called Artinian, Artinian rings.

So this gives us a dimension, this gives us a definition of Chevalley's dimension, this is Chevalley's dimension,
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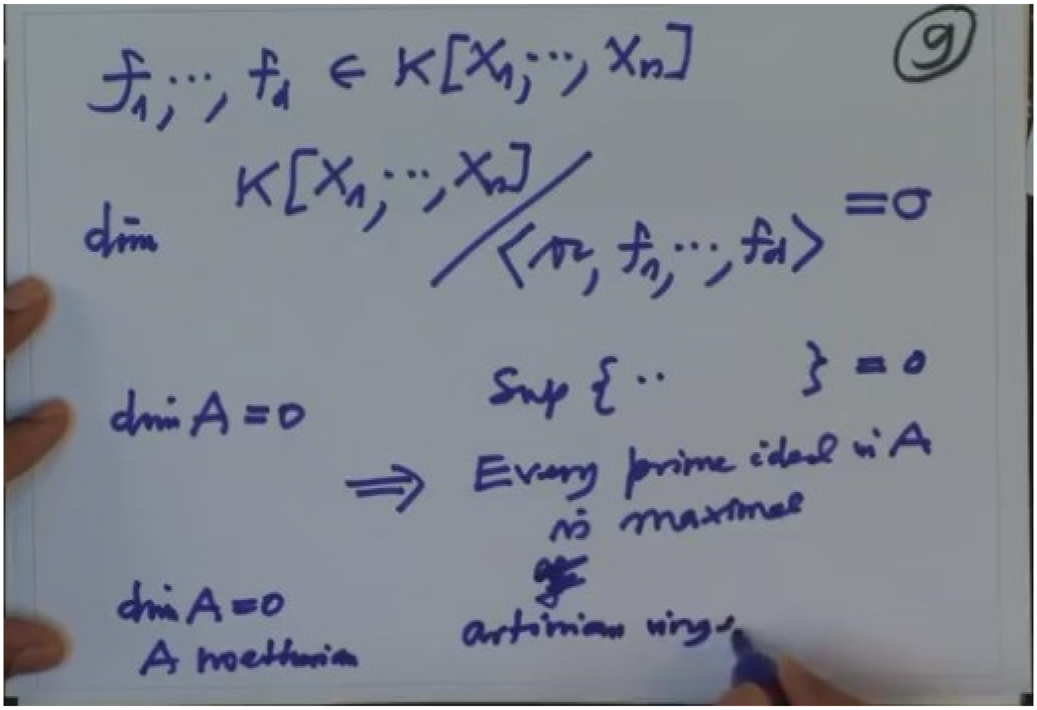

so what is it for arbitrary local ring? So also in between one should have remark that if I want to study dimension of arbitrary algebraic sets I could study dimension at a point, and this means you take the coordinate ring that is $\frac{K[X_1,...,X_n]}{I_K(V)}$ and this point will corresponds to the maximal ideal, so localize this at that maximal ideal, because maximal ideals are the last in that chain always, so therefore if you want to study this it's enough to study this and therefore that will reduce study of dimension to the dimension of the local rings, so reduce the study to local rings, so that means what? Dimension of A this is same thing as dimension, sup dimension of A localize at M where M is maximal ideal in A, once you have done this then you can define Chevalley dimension, then Chevalley dimension of a local ring if A is local, Chevalley dimension is the infimum, so that is also denoted by SA, this is by definition infimum of r such that there exists A_1 to A_r elements in the maximal ideal M(A) because A is local with maximal ideal M such that A mod, if I go mod A_1 to ideal generated by A_1 to A_r this ring is Artinian.

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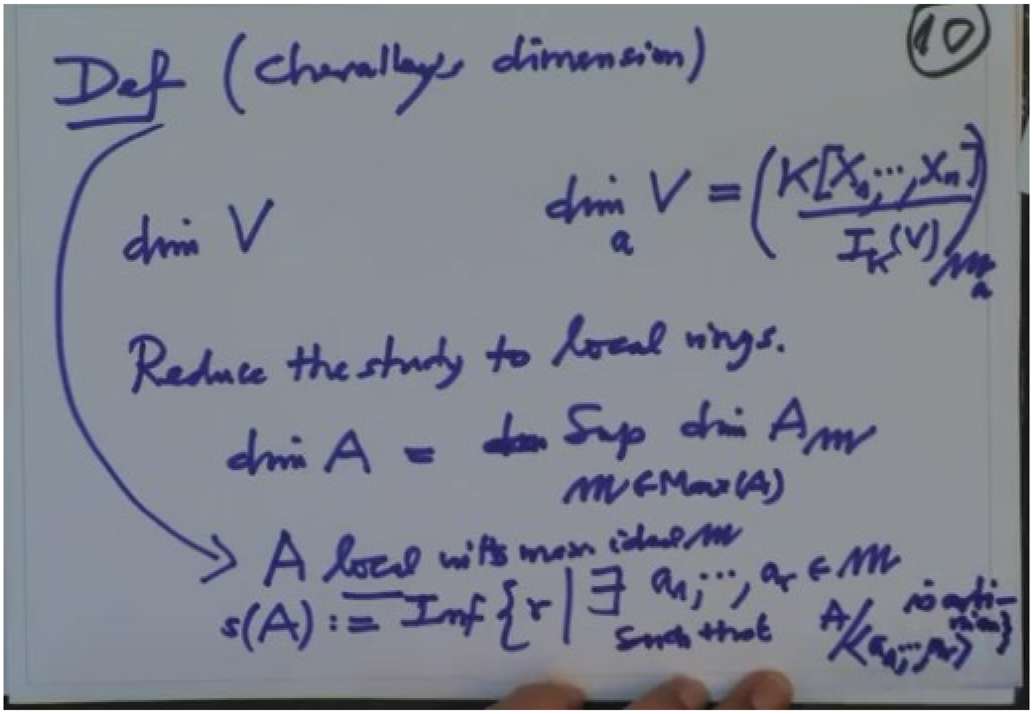

We know this exists because we have proved it for algebraic sets that we have such cut successfully by the hyper surfaces by using the property 5, and therefore the Chevalley dimension exists, and then one proves that dimension theorem says, dimension theorem says that the Krull dimension = Chevalley dimension = degree of the Hilbert Samuel polynomial, so this is what the precisely the dimension theorem, this is proved in the lectures, so I will not repeat what is the definition of Hilbert Samuel polynomial etcetera, but this gives a fairly good ideas of dimension concept, so couple of remarks here that we know dimension of a field so some examples, dimension of a field, field K this should be 0 because the only prime ideal is 0 and nobody else, so therefore the sup is 0, dimension of PID, dimension of $\mathbb{Z}$ is 1 because the chain is 0 in the prime ideal and then ideal generated by the prime numbers that is prime ideal and that's all no more because nonzero prime ideals here are maximal, so the dimension of K[X] is also 1, also if I take the power ring K[|X|] over a field in one variable that dimension is also 1, because it's a PID, in fact this is a better PID than this, so similarly dimension of the polynomial ring this will correspond to dimension of K^n which is M and so on.

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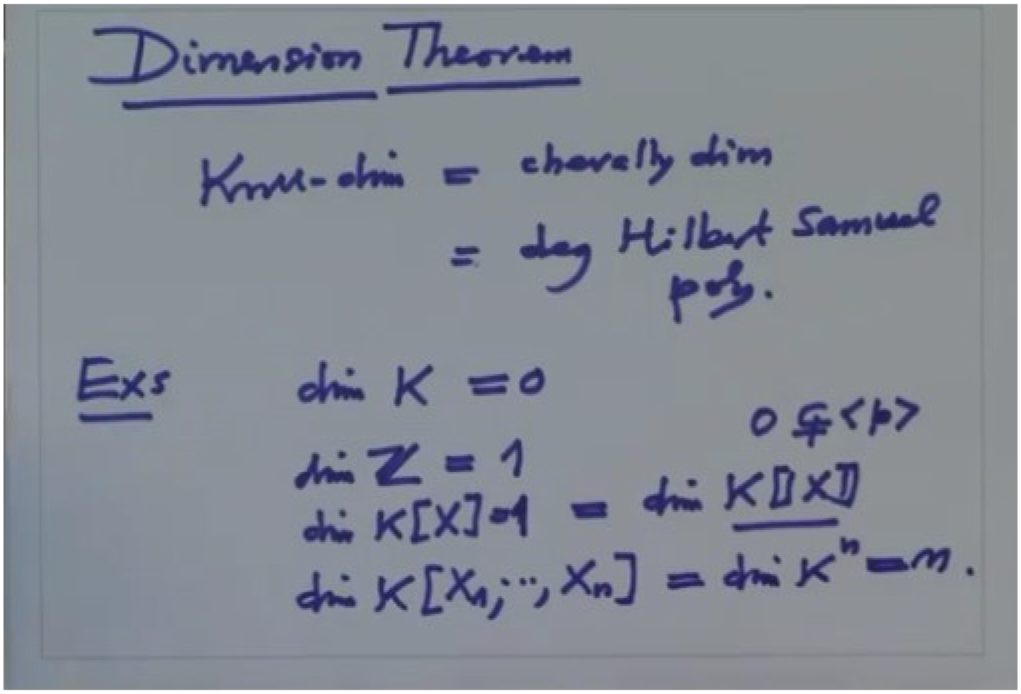

Now this dimension theorem will allow us to compute dimension of more rings and also it shows that dimension of a local ring is always finite because so this is for a local ring, Noetherian local, because it's a degree of some polynomial and therefore it is finite, and normally in all this subjects, especially commutative algebra one usually assumes the rings are Noetherian because it is not worth studying so much non Noetherian ring because they don't corresponds to any geometric objects, therefore one usually assumes the rings are Noetherian, and with this I will stop this lecture. Thank you very much.

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