Lecture No. 39

Hibert function for Affine Algebra

Gyanam Paramam Dhyeyam: Knowledge is supreme.

Okay, so we will continue studying regular local rings and I will provide the gaps last time you have had. So for example, and also some examples, so to get more into two ideas from geometry especially. So what we want, so we have started with. So let us recall a definition of regular local ring. So (A, m) , noetherian local ring. And of dimension equal to d. So we have seen in earlier like K that this d is less equal to the minimal number of generators for m, which is also dimension of the vector space $\frac{m}{m^2}$ over the residue field $\frac{A}{m}$. So if equality holds here then we call the ring to be regular. So, A is called regular if regular local if equality holds. Okay, so their local ring are usually studied with the help of the associated graded ring, you have seen these also in the proof of dimension theorem et cetera, also when we talked about Hilbert-Samuel polynomial et cetera and so on. So for A we have this object $gr_m(A)$, which is associated graded ring of, this is the direct sum of this successive powers coefficients. This is called associated graded ring of A. And last time we have seen that regularity of the local ring is equivalent to the associated graded ring is a polynomial algebra. This is what do you have seen and obviously in the number of variables as d. Okay, in that proof I've used that the local ring A, and this associated graded ring of the same dimension. This I have not proved, we will prove it, today? Most of the things we are proved, we have to tie-up the thing in a neater way. So that is what do we have used but before I do that, I'll say, I have made a statement that if A is regular then it is normal. And I wanted to prove, if A is regular, then this graded ring is a polynomial ring and polynomial ring if the integrally closed, normal. And from there I said, originally is normal. So we have viewed these two theorems, two things. Namely if A regular, regular local, then the associated-- that is equivalent to saying the associated graded ring is a polynomial algebra or the residual field. This we have use, this we have proved by using dimension of A equal two dimension associated graded. So this you have to prove today, this theorem one. And another thing I've used is if A regular local, then A is normal. And for this I said use this one because if this is normal, graded ring is normal then the original ring is normal. This is what we have to prove.

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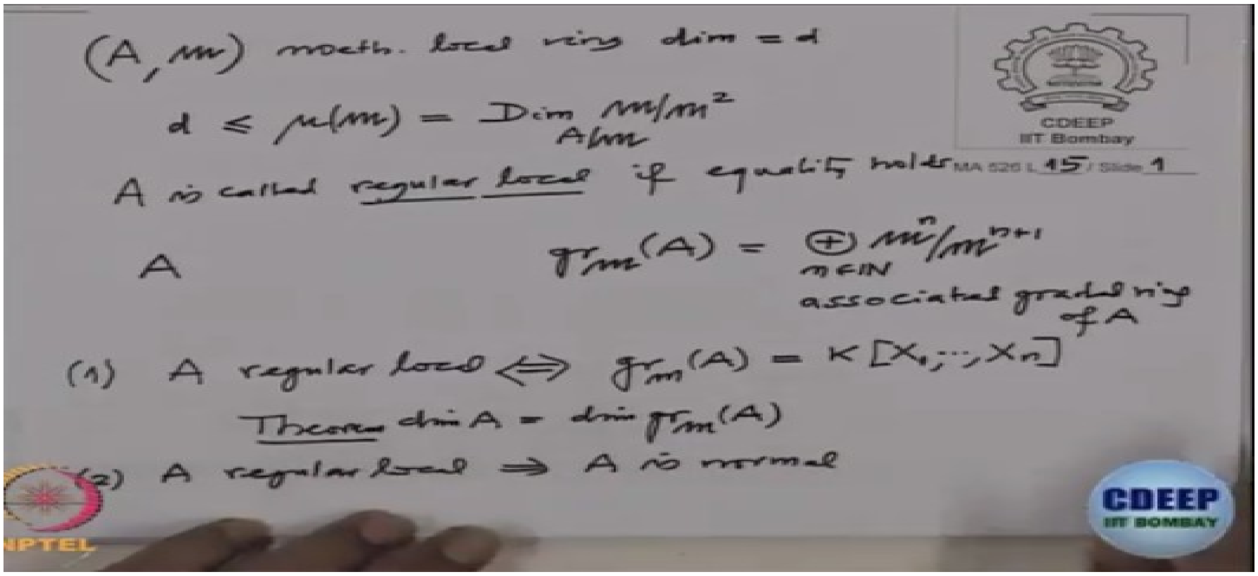

Okay, but before I do that, I want to make some comments about the definition of regular thing. First of all, see you should view these associated graded ring, so from A we have gone to associated graded ring. One advantage is, this is a finite type algebra over a field and for a finite type algebra over a field we have very strong statement normalization lemma and so on. That was not there-- in the case of local ring. Another thing is if you look at this Spec, the prime ideals, it is Zariski topology. And this one, so this one you know better one because this is a cone. See this graded ring is isomorph to polynomial ring in general. I'm talking in general. $K[X_1,\ldots,X_n]$ modulo some ideal J. And this have a property that this is a made up of the lines passing to the origin. So therefore we know approximated this spectrum by a nice object which has this homogeneous property that namely the cone, that is called cone, affine cone. So that is Spec of this. In a example, it will become more and more clear about the geometry, okay. That is one command. And another thing is we want to define in general when the ring not regular, ring is not local, then we want to define it is called a regular, if all localizations are regular. But to do this we have to prove first that, if you have a regular local ring, if you a local ring and if all localizations are regular then the original ring also should be regular. And this is not so easy to prove. For this we need a help of homological algebra. So after this I will develop homological algebra. So, this statement that is local then A is regular if and only if A localized P is regular, for all P. This is not too easy to prove because of this-- to prove this statement homological algebra use, the use of homological methods in commutative algebra became more and more popular. Okay, so, now first I will indicate the some examples first. Of course, some examples we have decided. So for example-- also we have defined for a singular locus. If I have a finite variety V, so V of the ideal, this is subset of the spectrum of the polynomial ring, let us say. And we wanted to define, what are the singular points? So singular points to test whether some point is singular or not. That is what the criteria I stated in last time that the Jacobian criterion. That any way we will prove it soon. But more important is for arbitrary ring for this Spec of A, we want to define, what are the singular point, so Sing of Spec of A. These are the almost prime ideals there in the ring is not regular. A_p is not regular. So those are called singular points. And compliment is called nonsingular points. So we want to check that these singular points is a close subset. And not only close subset, it is thin, that means it has not too many elements.

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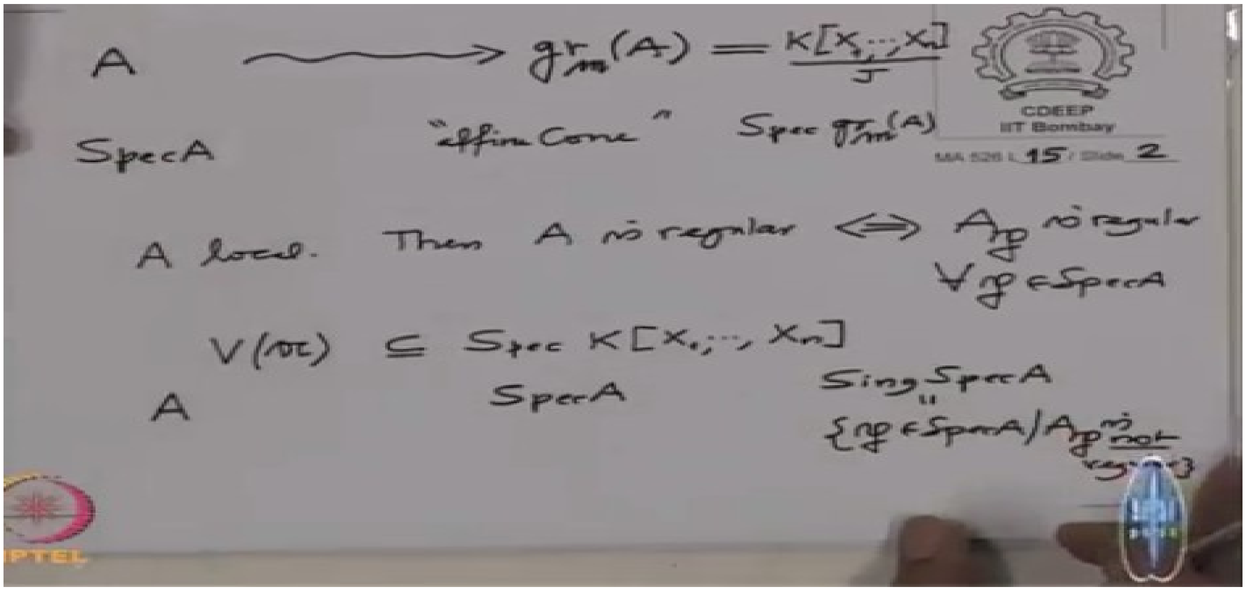

So, the two examples will give more light on that. For example, if you try to draw a graph of some curves let us say, cubic curves. So these are the examples, that gives us some ideal, so this used cubic curves in the plane of course, in the plane. For example, if you take the equation, $Y^3 - X^2$. So the picture will look like this. So at this point will be the singular point. That is-- I will test in Jacobian Criterion also. And this is what do you have learn, from your calculus courses also. This is called cusp. This point is cusp. Or it should take another equation, that'll be $Y^2 - X^2(X+1)$. So, this will look like this. This is called node. This is also singular point. And you see why that is singular point? These or this. Here tangent is-- here there visually two tangents at this point, right? So the tangent did not well defined here also there are two tangents, but did as a double tangent. One is-- so they're hidden. And also they are in the picture it is not seen but the actually there are double tangent. And if you write down, the equation on the tangent by using your calculus courses, then you will have two tangents here namely this X-axis. Here it will be Y equals to X and Y equal to minus X and then another example is, like this. So, here this equation is $Y^2 - X^2(X+1)$ and I am digging the real pictures, so that I'm taking the field integral numbers. So this case it is smooth, the point is smooth. So there is no singular point, no singular point. So this is what we learned from calculus. But as you see, when you're going higher and higher dimension and more and more equations then it becomes more and more difficult to check by using calculus methods. So, we have to have more powerful algebra. And that is provided by counted algebra.

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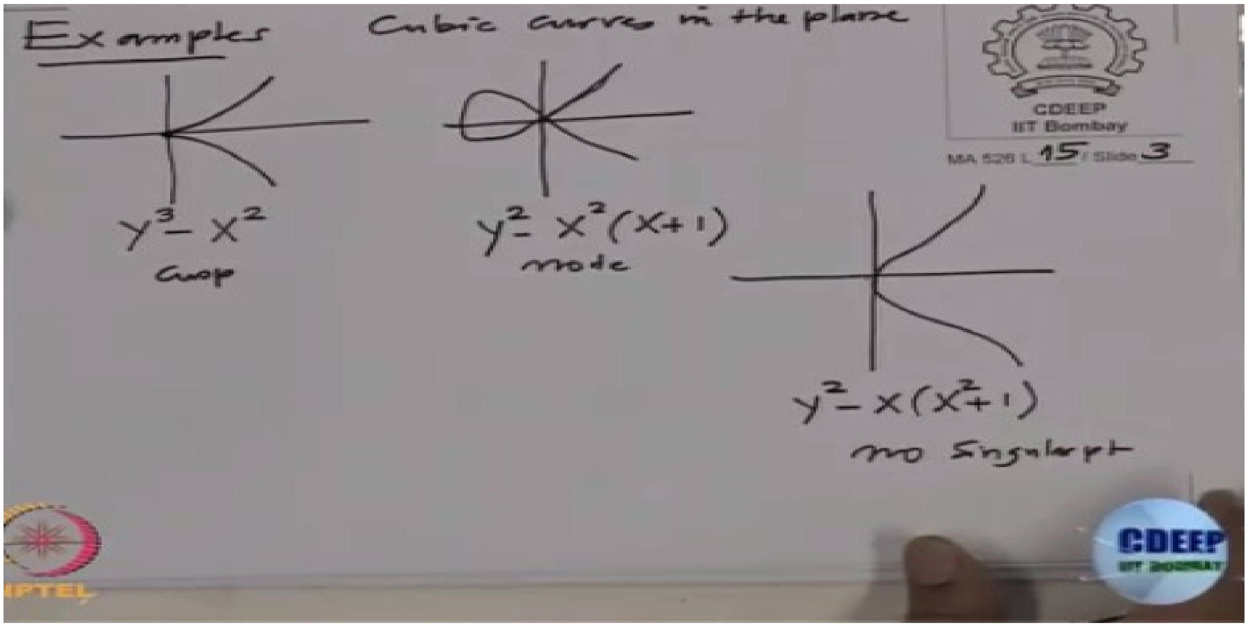

Okay, now, before also I gone. So with the same example, also now let us do it little bit more abstractly. So I will prove now that, so we have, the noetherian ring A, may not be local. And we have the Spectrum here. And we have learned that this topological space has finitely we need is irreducible components. And the irreducible components are provided by the minimal prime ideals. So irreducible components take correspond to minimal primes.

So the first observation is, if some prime ideal P, suppose it belongs to more than one irreducible component. So think of P as a prime ideal, so think of that has a point in a topological space and suppose this point belong to two irreducible components, that can happened, no? So for example, if you have a somebody like this, this is irreducible components, so for example, if you have equation, for example if A equal to $\frac{K[X,Y]}{\langle X,Y\rangle}$ ideal generate by XY. In this case the picture looks like this. So this is one irreducible component, this is another irreducible components and this origin is belong to both of them. So in that situation I want to say that that point cannot be nonsingular. That point has to be singular. That means the local ring at that point has to be non-regular. So if P belongs to, that is a statement more than one irreducible component, then P is singular. Singular means, so that is, P belongs to, that is A localized P is not regular. So let us prove. So P belongs to more than one irreducible component means P will contain, so everything the whole idea is all the geometric statements we will convert to algebra and prove there and go back. So suppose P contains two minimal primes. See P belonged to the, P is on the irreducible component, this statement, when you translate to algebra, it will become P contains that minimal prime. So P contains therefore two minimal prime ideals q and q prime where these are two minimal primes. And what do I want to show you, you want to show these ring A localized P is not regular, right? So that means we have to go to localization at P and because the minimal primes are containing P they will survive here. And they will continue to be minimal primes. Right? But then we have just now seen, this ring cannot be regular, that's what we want to conclude, if it is regular it'll be domain. And domain means exactly one minimal prime ideal, namely the zero, one. But we have two minimal primes surviving there. So therefore A localized P cannot be regular. So minimal primes containing P, therefore q, A localized P and q prime, A localized P, both are minimal primes, in this local ring. But that cannot happen, because this is a domain. If a regular-- it is regular, if A_p is regular then it is a domain, A_p is a domain and therefore there cannot be two minimal primes. So contradiction therefore we have proved that A_p is regular and that means P is singular. So therefore when one wants to draw a picture of a spectrum, of a ring, where P is a point there. If had P it is nonsingular that mean that P should belong to only one irreducible component. Okay, and in this example, so in this example, for example, you did not seen they are two components. See the thing is we have to do locally and localization for some obviously then it is not good enough to see the picture locally. So, localization is not good enough localization. So for that we will study the concept of completion. But I will not come these now. This completion is like a more magnifying glass. So which will see locally and enable you better. And also it gives you a better connection with the analysis. So that corresponds to the powers you think.

So here we are dealing with polynomials et cetera. So completion will deal with the powers. So that will give you more-better pictures. So, this picture, when you would have looked under completion, you will see that two components are going through that origin. This is what we will check when I introduce a completion to you. Okay, fine. Now let us get back to our proof of, so the theorem we want to prove is, actually this is what the dimension theorem should be. If A is noetherian local then the dimension of A equal to dimension of associated graded ring of A. Okay, so proof. I only how to type several steps, so I will be little bit sketchy, but all the proofs we have known earlier in when we did Hilbert functions and Hilbert polynomials and so on. So now this $gr_m(A)$, this is finite type algebra over the residue field. So this is $K[x_1,\dots,x_n]$, where K is the residue field of A, and the maximal ideal is generated by x_1 to x_n . So, m is minimally generated by x_1 to x_n . Strictly speaking it is bar here, right? So this is, this is standard what we called it standard graded K algebra.

So this is K direct sum, $\frac{m}{m^2}\oplus\frac{m^2}{m^3}\oplus\dots$. This is the associated graded ring and this part generates either K algebra, graded ring it generated by this first homogeneous component over K as an algebra. That is what we know. So now I imitate what we did for the local ring the Hilbert-Samuel polynomial. I'm going to imitate that to this graded K algebra. This finite type K algebra, but the definitions are defined by using this x_1 to x_n . And we have to check that it doesn't depend on the generating set. That is the major step. But that is also okay. So let us define, what we have defined, so more generally.

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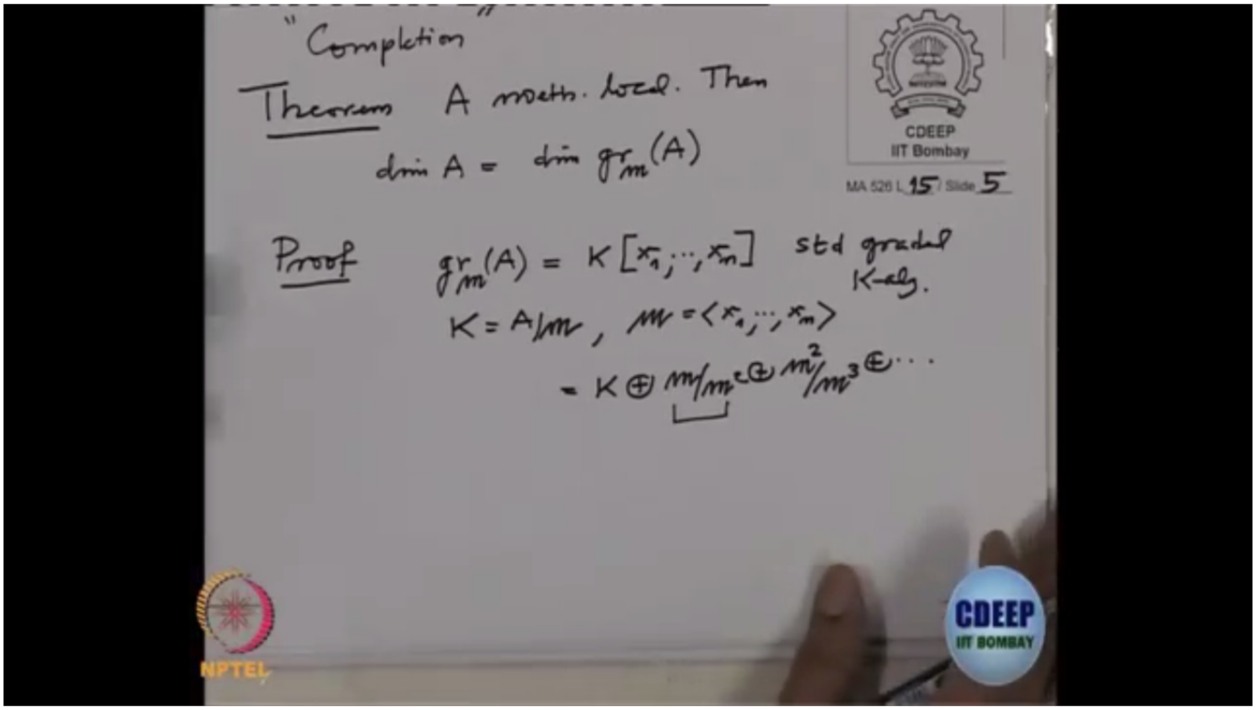

So what we are-- what I'm going to do is, I have a polynomial algebra $K[X_1,\ldots,X_n]$. And I have an ideal in that. A is an ideal in that. And I have the residue class algebra. $\frac{K[X_1,\ldots,X_n]}{A}$. This is $K[x_1,\ldots,x_n]$ and let me call this as A, capital A. This is nothing to do with the original local ring. So this is A, and I'm going to define, I'll define actually earlier lecture. So I just want to recall, for any Am in m, we have defined is A less equal to m, that is by definition. Look at all those polynomials in the x, such that, if degree of f is smaller equal to m. Any element in A is coming from a polynomial and you put instead of X, x. So this is not unique, the expressions are not unique, but, okay. That you just take them and you look at all through degree m polynomial and then take their images in this ring and collect all of them and call that Am. And it is obvious now this is K-vector space and finite dimensional. So it makes ensure define this function that is function from $\mathbb{N}$ to $\mathbb{N}$. Any m we will do K-vector space dimension of this. This is ideal function I denote by h suffix a. and because it depends on this A. This is called Hilbert function of that ideal, of the ideal A. And similarly the Hilbert series that is Ha(T), this is by definition, the series. The coefficients are this value that m. This is Hilbert series of A.

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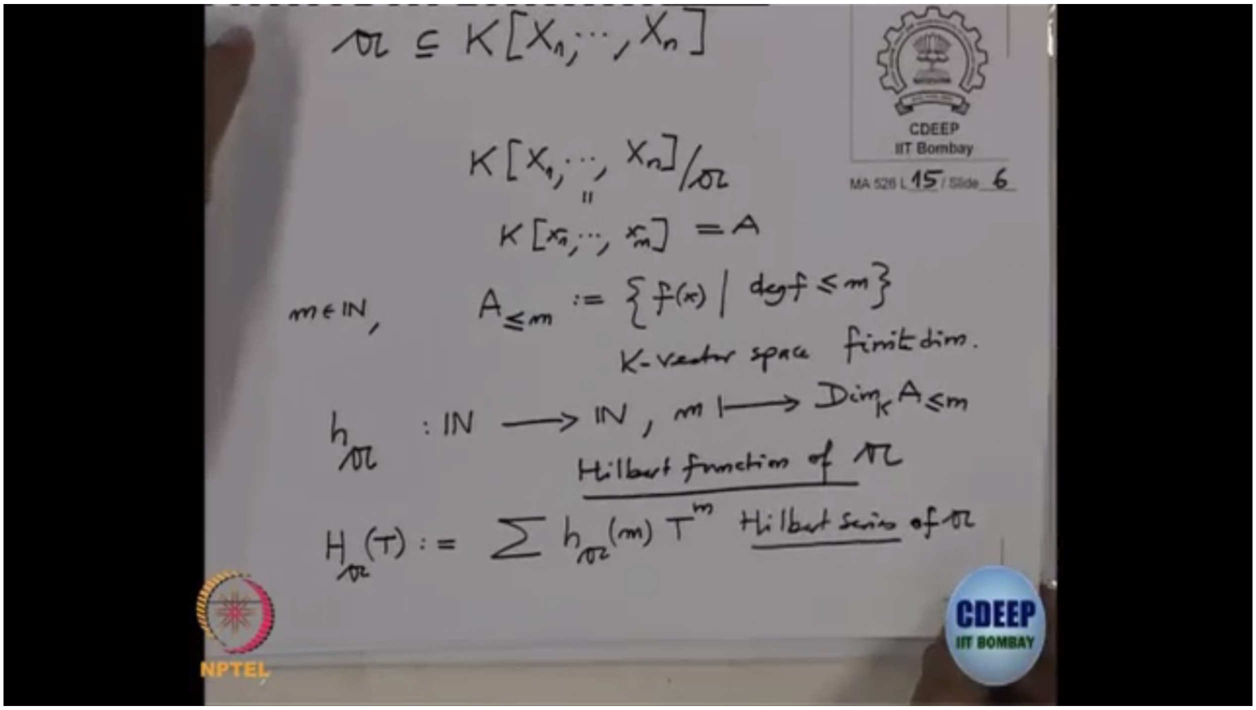

Okay, now let me give one example. So that you get, that this, this Hilbert function and Hilbert series they don't-- they depend on the generating set. This is what I want to show you by examples such. So suppose you have two variables $K[X1, X2]$ and you take the ideal--principle ideal generated by $X_1 - X_2^2$, this is my ideal. Okay, then let us write down the Hilbert function of ideal A at value of m. So I say, that this is $2m+1$, because what do we want to compute, we want this should be the dimension of a vector space A less equal to m., for K. This is what I want to check. So what is this vector space, this is you just go mod that and take all polynomials of degree smaller equal to a m and then compute the dimension. But I use to all those polynomials, see for example, $1, x_1, \ldots, x_1^m$, then $x_2, x_1 x_2, x_1^{m-1} x_2$. These are-- this is in fact the basis of this. Because what do you need. You need polynomial smaller equal to m, right? So, and whenever the where X_2^2 I'll replace it by X_1 , so I don't need extra square. So that's why and up to x. So this is a basis for this. And how many of them are there, there are-- these are m, these are m and these are 2m and this is 1, so these $2m+1$. Right? So this is the Hilbert function. And therefore what is the Hilbert series. You see then, 1 plus T divided by 1 minus T whole square. That is easy computation, I will leave it for you to check. You may just need only this in the calculation. Okay, on the other hand now suppose I-- this is I've taken, this suppose I take, yes, see here, from the x_1 to x_1^m there m in number. And for me x_2 to x_1^{m-1} , okay? But you see now, these we wait on what was A, A was $\frac{K[x_{1,} x_2]}{\langle x_1 - x_2^2 \rangle}$, this was the small x_1 , x_2 is here. Their modular is ideal. But I could have also taken, this is also isomorphic to Just forget X_2 . So this is isomorphic to $K[X]$ either algebra.

So, I could have taken x as a m generating x. And that case what is the Hilbert function then, in that case it's easier. In that case, if I take this as a representation, see, when I say, representation means, we want to write given algebra either coefficient of a polynomial algebra. And there may be many ways to do it. So for example, this was one way. This is another way just you forget X_2 and then you get $K[X_1]$. No, forget X_1, we get-- this is a isomorphic two, polynomial ring in $K[X]$, right? So now if I use X as a generating set for the K algebra. Then the Hilbert functionally be, not, I don't write An suffix here, now. So in that case h (m) will be equal to m plus 1. And the Hilbert series will be whatever. So therefore Hilbert function do depend on the generating state of a finite type algebra. Now therefore the only hope is we prove that this degree of that-- we proved that it is a polynomial function and the degree of that polynomial function doesn't depend on a generating set.

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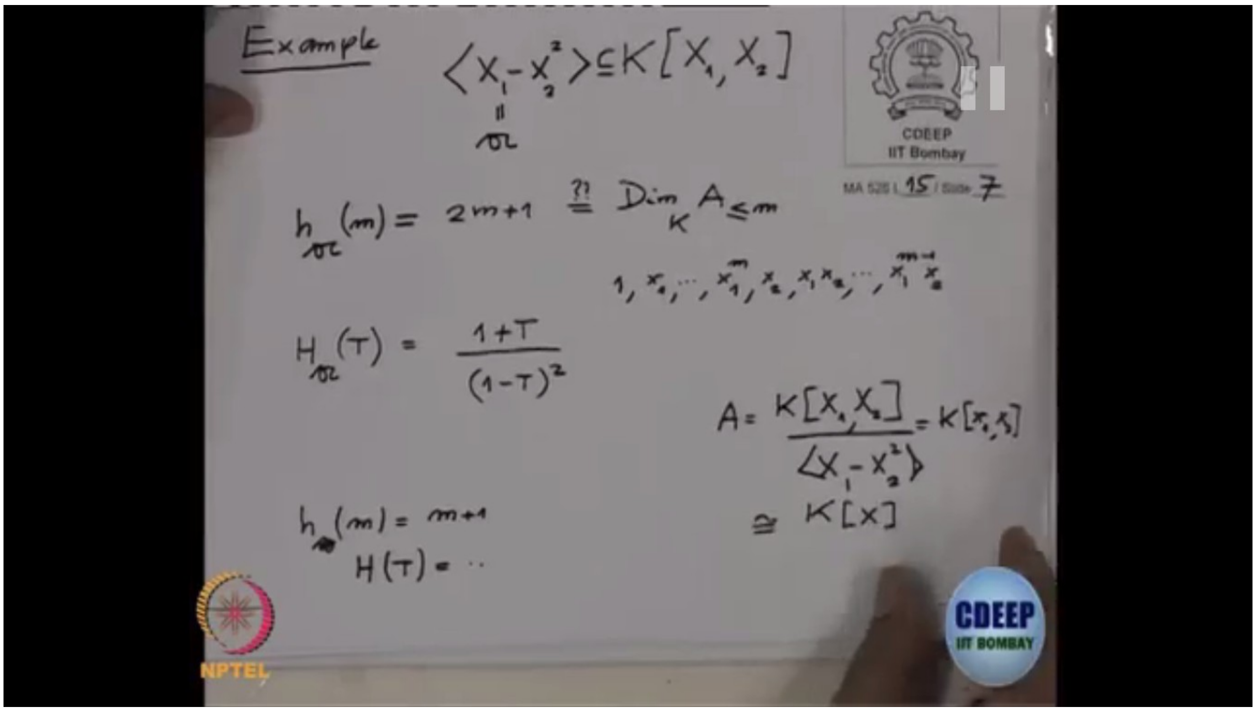

This is what one-- we will prove. Okay, so. Okay, so, we are-- see in another words see this, this A less equal to m. This gives a filtration of the algebra A. And we are use that to define the Hilbert function. Okay, now for a-- for principle ideal you can compute the Hilbert function. I will not compute that, but let us-- I will just at least state. Suppose I have a one polynomial in over a field in several variables. And I want to compute-- I take the principle ideal generated by F, that is A, if the principle ideal is generated by F and then we are taking coefficient algebra A is $K[X_1, \dots, X_n]$ module ideal generated by F. And then what is h suffix F and H suffix F, the series. So I will write the series now, because from each you can read the other one. Because these are the coefficient and these once you learn the coefficients, you know the series. So, for the 0, if everywhere polynomial then it's easy. It is $\frac{1}{(1-T)^{n+1}}$. And if f is non-zero, it is (1 - T) power degree of F divided by $(1-T)^{n+1}$. And this proves by induction on in. By induction on in. For F equal to 0, it is really easy because in that case, what is our A is a polynomial algebra. So, if F equal to 0, let us see, it's easy. This case A less equal to m is nothing but K polynomial ring in anywhere if else less equal to m. This is a vector space of polynomials in n variables of degree up to m. But this is same as direct sum $K[X_1, \ldots, X_{n-1}]_{\leqslant i}$ and X_n^j . The direct sum is running over i plus j equal to m. And i and j are a natural numbers. We are in this computing the dimension of this, but we can easily compute the dimension of this by induction. And just tie it up, so that I will not carry out that calculus. Similarly same thing you can do it for nonzero F. And for nonzero F the usual trick we have to do is, we have done it earlier also that you have to consider multiplication by a f on the polynomial ring and then we get exact sequence.

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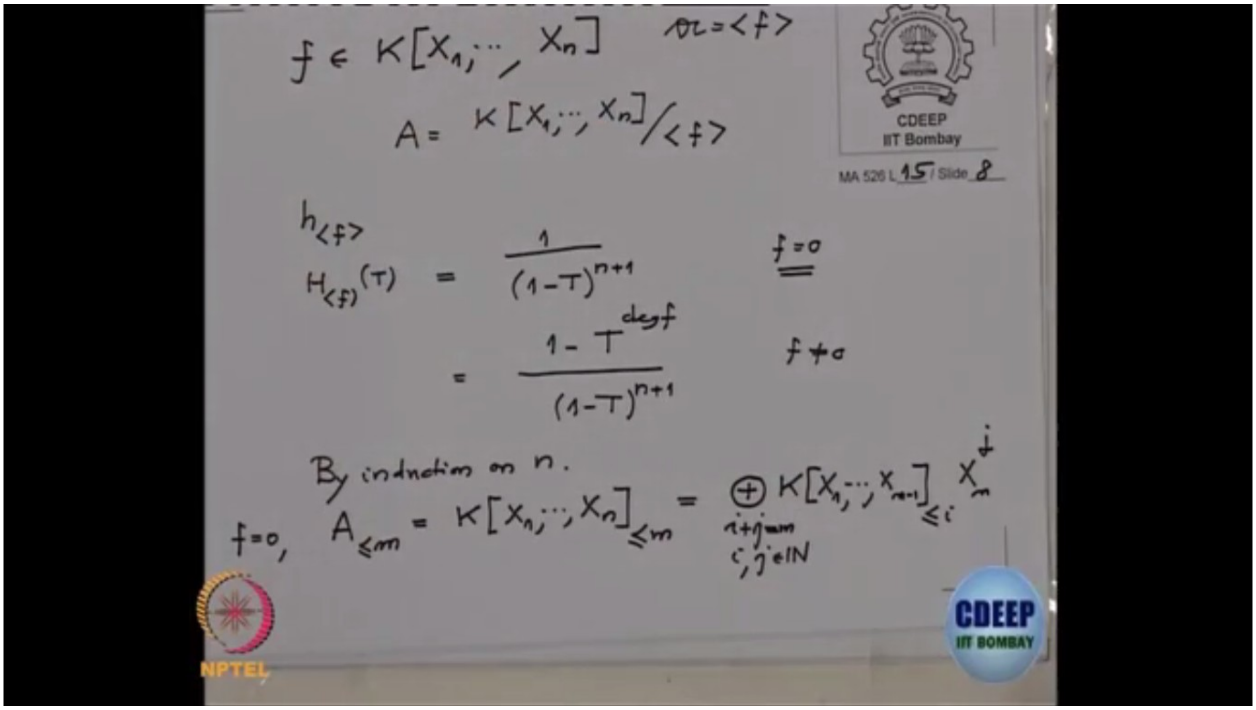

And then you read. So that is-- this we have done, but in this format. But the idea is clear that you have to consider multiplication map by F on the polynomial ring and then write a short exact sequence then from there you write the dimensions, okay. Okay. Now, again and the node that if you have two ideals A and B in the polynomial ring. And let us say homogeneous ideals. Then I want to check that the Hilbert series for the some ideal plus Hilbert series for the intersection is same as Hilbert series for the ideal A plus Hilbert series for the ideal B. So you should write T here actually . This is like this-- we prove this kind of formula in almost every subject. Now, linear algebra you prove this, even said, can you prove this. Its inclusion, exclusion and et cetera. So the-- this is we can say inclusion, exclusion for the Hilbert series. This is not so difficult to prove again. I will indicate the step. You see, you consider the map from $a_{\leqslant m}+b_{\leqslant m}$, mod $b_{\leqslant m}$, this is clear the notation. This means the polynomials here of degree of 2m and so on, right? And there is a natural map here. Which is sudjective, just going bar, right? So F going F bar, that is a map. What is a kernel? Kernel of this map is precisely $a \cap b$ less equal to m. Once, you have that then you'll get an exact sequence. And then you go mod, go mod these guys so that means you work in this bigger vector space $K[X_1, \ldots, X_n]_{\leqslant m}$, this vector space. These are all their subspaces, right? You work in this vector space and then when you go mod this then you get a formula. You know, from the short exact sequence no plus, minus, plus, minus, that at formula, we surely will give this formula.

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