Lecture – 38

Statement of the Jacobian Criterion for Regularity

Gyanam Paramam Dhyeyam: Knowledge is Supreme.

Okay, now, I want to recall some facts which we will use in the proofs about field extension. So before that, I will just make a couple of definition, so when I have A local. Actually this definition will not use localness but, so when I take, Sing A, this is by definition, all those prime ideals p, such that A localize at p is not regular. This is not necessary local, so, look at all those prime ideals, so that, A localize at p is not regular. So, this is the subset of the Spectrum of A. This is called singular Locus of A or Spec of A. And we want to prove that the singular locus is open side or close side.

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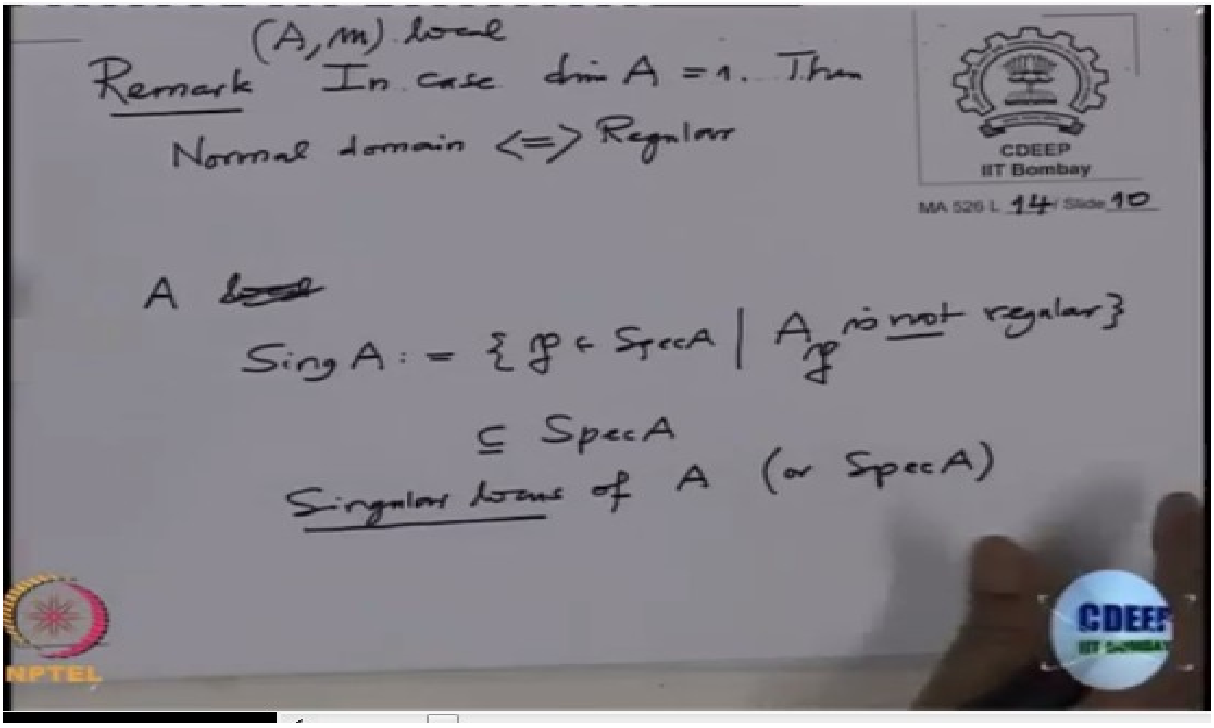

Okay, we want to study these, or we want to give a criterion, how do you decide some prime ideal p belong to the singular locus or not in singular locus? Such a criterion is a Jacobian Criterion. And this is similar to, we will attach some matrix to these and then we will study the rank of the matrix and correct rank will say that whether p is singular or not. First, I want to recall some facts on the field theory. So, everybody know what a perfect field is? Field is called perfect. One definition is every algebraic field extension is separable. This is what we'll use more often. Every algebraic field extension is separable. And when do you say, field extension is separable? So, we have L over K, field extension, algebraic. We will say that an element x in L is separable over K, if the minimal polynomial of x over K, it is the smallest degree polynomial in K x so that, x is 0 of that polynomial. This polynomial is a zero decimal polynomial. This polynomial should not have multiple roots. If this polynomial has no multiple zero's. That's equivalently, so, to saying, if you would have seen the discriminant of this polynomial is non-zero. Do you know what is a discriminant of a polynomial? In general So, discriminant of a polynomial is, the difference of the roots and then squares and so on, so, I will not go much into this. Characteristic 0 fields are perfect. Another thing I will need is, if I have a finitely generated field extension, L over K finitely generated field extension, which is separable, then there exist separating transcendent basis. You know finitely generated field extensions have a transcendent basis. So, transcendent basis means the pure part and algebraic part. Now, when want to find transcendent basis in such a way that the algebraic part is separable part. And such a way transcendent basis is called a separating transcendent basis.

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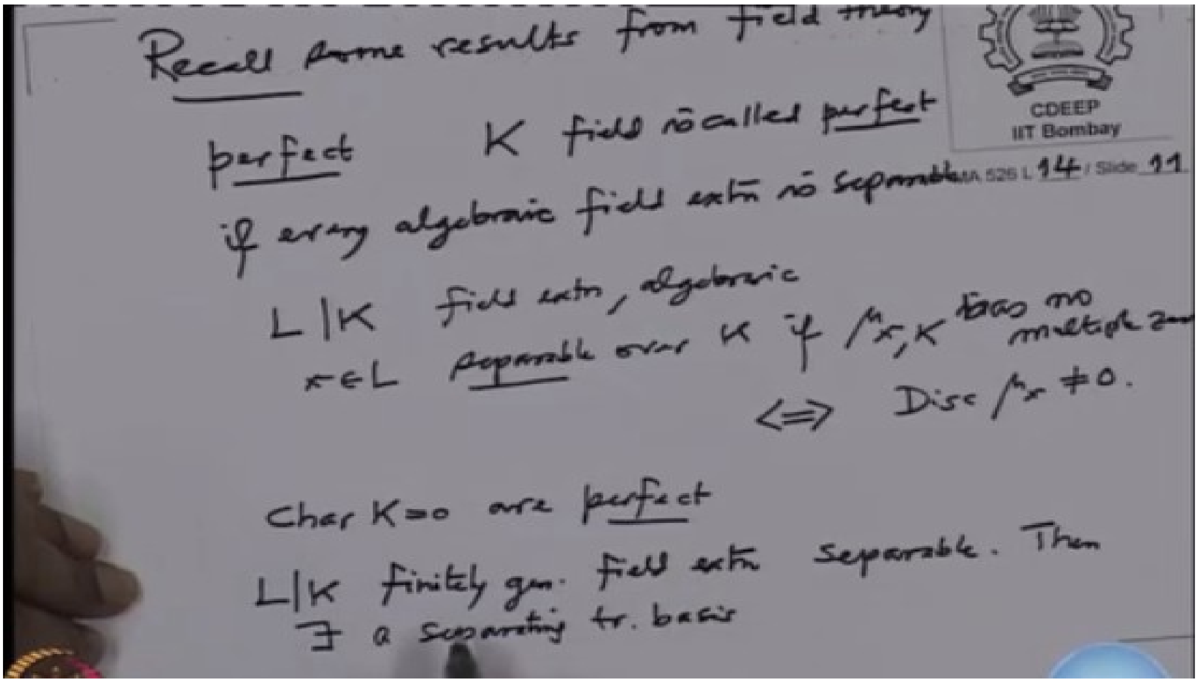

I think better that, I will not state exhaustively but use it when it comes. Now, I want to state the Jacobian criterion at least today. We have an ideal A, let say which is generated by f_1 to f_m, this is in a polynomial ring over a field in n variables. And p is prime ideal in the polynomial ring. First, I will do it for the finite type K algebra, then we will remove that assumption also. $K[X_1,\ldots,X_n]$, which contain A. And I will assume, this p may not be minimal over A, so I will choose the minimal prime each contain p. A containing q, contain in p, this is minimal prime over A which is contain in p, may be p itself is minimal prime. Now, we have these polynomial n in n variables, so as usual the Jacobian matrix, that is partial derivate of f_i, with respect to x_j. So, you get how many rows, how many matrixes, so these are the polynomials now. Now how are you donating this, j in the column index and i is the row index, so this is $m\times n$ matrix. This is actually matrix is M, m, n and over the polynomial ring. And of course, I can read this matrix, mod this p, the given p. So read this matrix mod p that means we are reading these enter is mod p. So, d of f_i by partial derivative so to x_j, these I am reading mod the ideal p. So, this is a way, you'll get a matrix, $m\times n$ again. $M_{m,n}$, $K[X_1,\ldots,X_n]$, mod p. This is the matrix you have like a Jacobian matrix.

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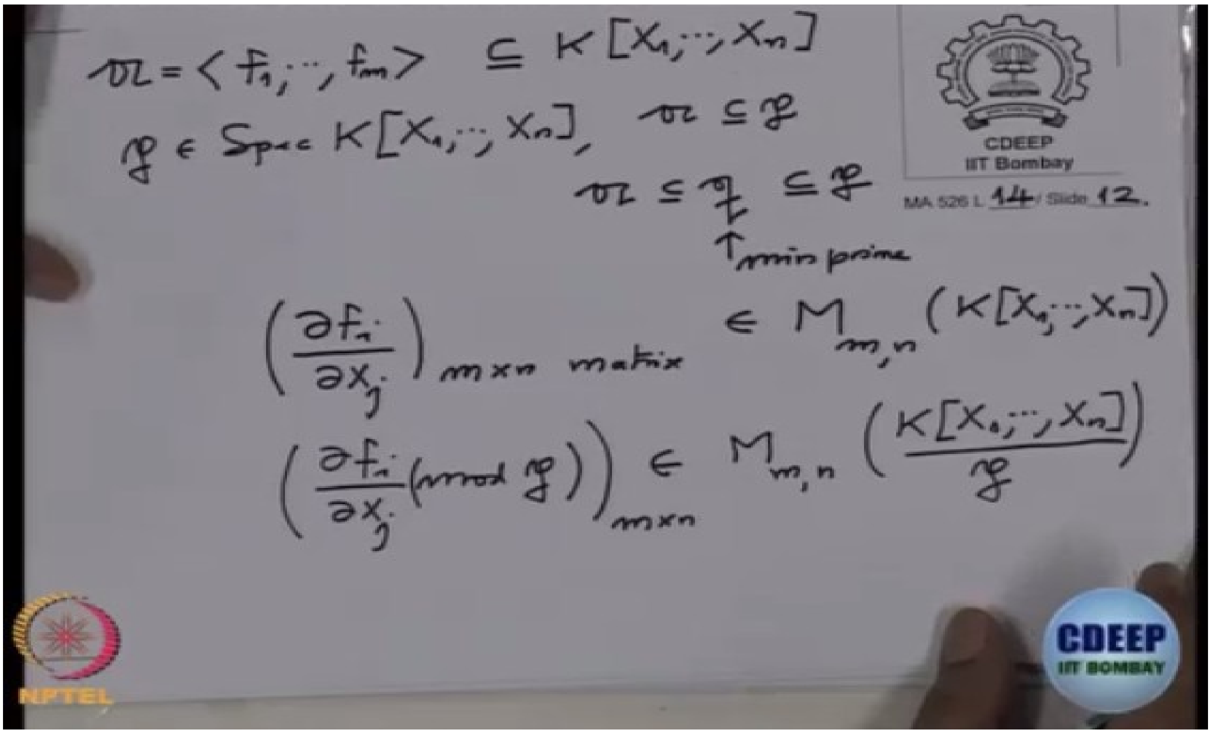

So, the first assertion is the rank of this matrix, df i by dx j mod p is less equal to height of q, where q is a minimal prime over A which is contain in the given p. Second assertion is, equality holds here. If equality holds in 1, then ring is regular. Take $K[X_1,\ldots,X_n]$, mod that ideal a, and of course, this you have to localize at p mod a, this ring is regular. So, that means this two gives you a good criterion because for finite type K algebra at least, given p whether the localization is regular or not you have to compute this matrix and compute its rank and whether equal to the minimal prime which containing p, then it is regular. Third statement is, if I take the quotient field, see the only thing we have given is a and p. So, if you are go mod p, $K[X_1,\ldots,X_n]$, mod p then you get a domain and if you look at the quotient field of this integral domain, I want to write the converse of this, I want to write the if and only if condition. And if you notice in 2, we have only one-way condition, if this then it is regular. So, in 3 we assume that this quotient field is separable over K, K is this even base field. This extension, this field may not be algebraic. May not be finite. So, if it is separable over K then the converse of 2 holds, what does that mean, if this ring is regular, then the quality will hold here. So, three rates, the equality holds here if and only if this extension is separable and the ring is regular. Okay, the separabality hypothesis-- this hypothesis is automatically holds if you're base field you're working with that is perfect, because for a perfect field that is a definition, or also it will hold if this prime ideal p, you started with, actually it a special kind of prime ideal, if it is actually a maximal ideal and if it is actually a point, then it will hold. So, I'll just remark it here, I will write it, so it is, the statement will be clear. So, this separablity assumption in 3 holds if either K is perfect or given p is a maximal ideal. P is a prime ideal in K, not only a maximal ideal it's a point, that means p is generated by elements $X_1-a_1,\ldots,X_n-a_n$. This is also denoted by m suffix a, vary the point. Then it holds. Because in that case, what will it be? The residue class ring will be just K. So, then you are saying K is separable over K that is very rational.

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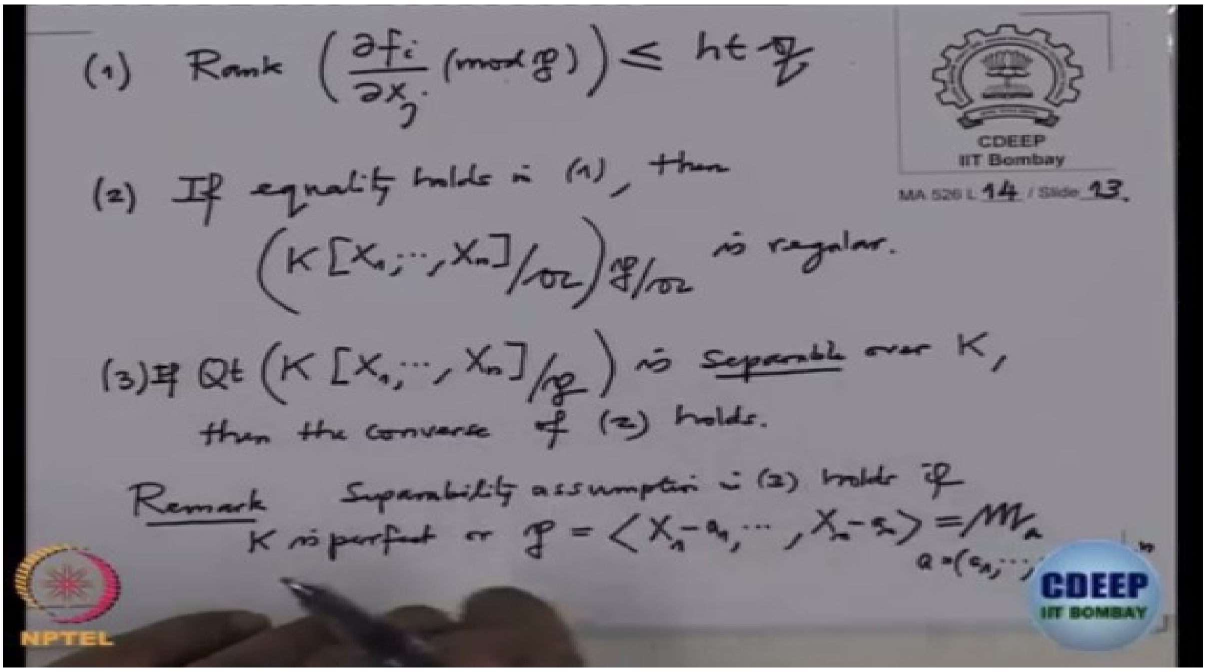

First of all, there are several things should check here. One very important is, suppose somebody takes different generating set for the ideal A, then the matrices will be different and entries will be different and so on. So, one should check that this rank of this matrix doesn't depend on the coordinate that we have chosen. The generating set and then the coordinates, so all these have to be checked. So, this I will do it, so what do I need to do it now? This is precisely, what when means by Jacobian criterion, this theorem, which has this three parts, that is called Jacobian Criterion for regularity. So let us write some Lemma's, which will be are preparing for these. So Lemma, I will need at least, two lemma. So, let us write them, number 1 and number 2. So, p is prime ideal in the polynomial ring in n variables, of height m. So, you remember, one of our main concern is, we have to relate the rank of the matrix with the height of a prime ideal. The part one, there exist, m elements, f_1 to f_m, in p which generates, the localization p $K[X_1,\ldots,X_n]$, localize at p. I can find n polynomials m is height, so that the localization at p, the ideal p will be generated by those m lemmas. So, you have a prime ideal in $K[X_1,\ldots,X_n]$, p the prime ideal here and we are looking at the localize ring. K localize at X_1 to X_n, localize at p. So, I can find elements here, f_1 to f_m. So that if you go, the ideal will push the ideal here, that is generated by those m elements. And 2, if I look at the quotient field of this, mod p. If this is separable over K, then the rank is equal to m. Then the rank of this $\frac{df_i}{dx_j}$, mod p, equal to actually m. Let's understand, this field extension, so I write this, X_1 to X_n, mod p, this is an integral domain and let us call L to be this field extension. This L is the quotient field of this. So, therefore this quotient field is generated over K by the images, this x_1 to x_n or the images of X_1 in this class. Now, what is the dimension of this? We have studied the dimension of this finite type K algebra. So, the dimension of $K[X_1,\ldots,X_n]$, mod p, this dimension will be equal to n minus the height of p. This is if you remember, when we did Normalization Lemma and then several corollaries of that, so this is $n-m$. Lemma means the height of that prime ideal we are taking. But we have normalization lemma, we are proved dimension of these finite type K algebra is equal to transcendent degree. So, this is also equal to the transcendent degree of L over K, where L is the quotient field of this.This is normalization then. This was the consequence of normalization then. So, this L over K as transcendent degree $n-m$, let us call n minus to be K. So, K is the dimension of this ring, which is the transcendent degree of L over K, so that means they either transcendent basis which consisting of K elements. So, that means we will find L here and L_0 here which is generated over K, now what do I call it variables, let us call it y_1 to y_k, this and then this contains K. Now, this part is algebraic part, and this part is the transcendent part. Now, if this extension was separable, then see that was an assumption somewhere and this extension is separable that means, these extension is separable. You can assume, if this extension is separable we can choose y_1 to y_k, so that this is algebraic extension. This is separable. That is called separating transcendent basis. And that when can always do it, that is a theorem in field theory if you have a separable field extension, then you can always choose a separating transcendent basis. Now, we are considering any L_0, take $K[X_1,\ldots,X_k]$ and then polynomial after that, X_{k+1} to X_{k+l}, and see now these are the, I am trying to define a map for each l, l is in between 0 to m. Remember the notation $k+m=n$. m is height of the given prime ideal, n is the number of variables and k is the dimension. So, from here to L, what do we do? For this X_i to X_k you map it to the separating transcendent basis. So, X_i going to y_i, up to l is from 1 to k. So, next time, I will repeat definition of separable extension in general or arbitrary field extension and then we will continue this proof.