Lecture - 37

Connection between Regular local rings and associated graded rings

Gyanam Paramam Dhyeyam: Knowledge is supreme.

Today, I will do what are called Regular local rings. Before I formally start it, I want to give couple of motivations for studying this. So for example, we have seen if K is a field and you have finite type algebra or K. Then this corresponds to so called algebraic varieties in K power n, so we want to study this and when talks about variety when it's equipped with the topology from the risk topology. So when want to study points on this and especially the local behavior at the points. So this local behavior at the points, if you, this V the variety and a is a point there, then there any corresponding and if you call this as a, this a will corresponds to the maximal ideal m and, local behavior of V at a, that will be determined by this localization. This local ring. This is called a local ring of V at a. So especially when would like to know when is this point non-singular. Non-singular, actually this concept more coming from the analysis. For example if you remember what is inverse function theorem or implicit function theorem. So they corresponds to these points the differentials, the study of differential maps and the functional matrix or Jacobian matrix and the rank of that matrix and also similar thing we want to do it for this. And as you can see now this depends on the coordinates here. So one would like to have coordinate free approach. So that is where I want to start more generally. So now, instead of varieties and finite type algebras and so on. We start with a noetherian ring A. A is noetherian ring and then you have this Spec A and here is also there is a rescue topology, now the points are not maximal ideals. The points are actually the prime ideals. So and given p prime ideal, we have this concept a localize at p. This is a local ring. And again the idea is by studying this local ring and its various properties, one would like to get information about this p in a neighborhood. So that is what it is. And I want to prove at the end Jacobian criterion. So we will prove Jacobian criterion for regularity. This, not today maybe next time. So this will decide this is the criterion which will decide when this local ring at p is regular or not.

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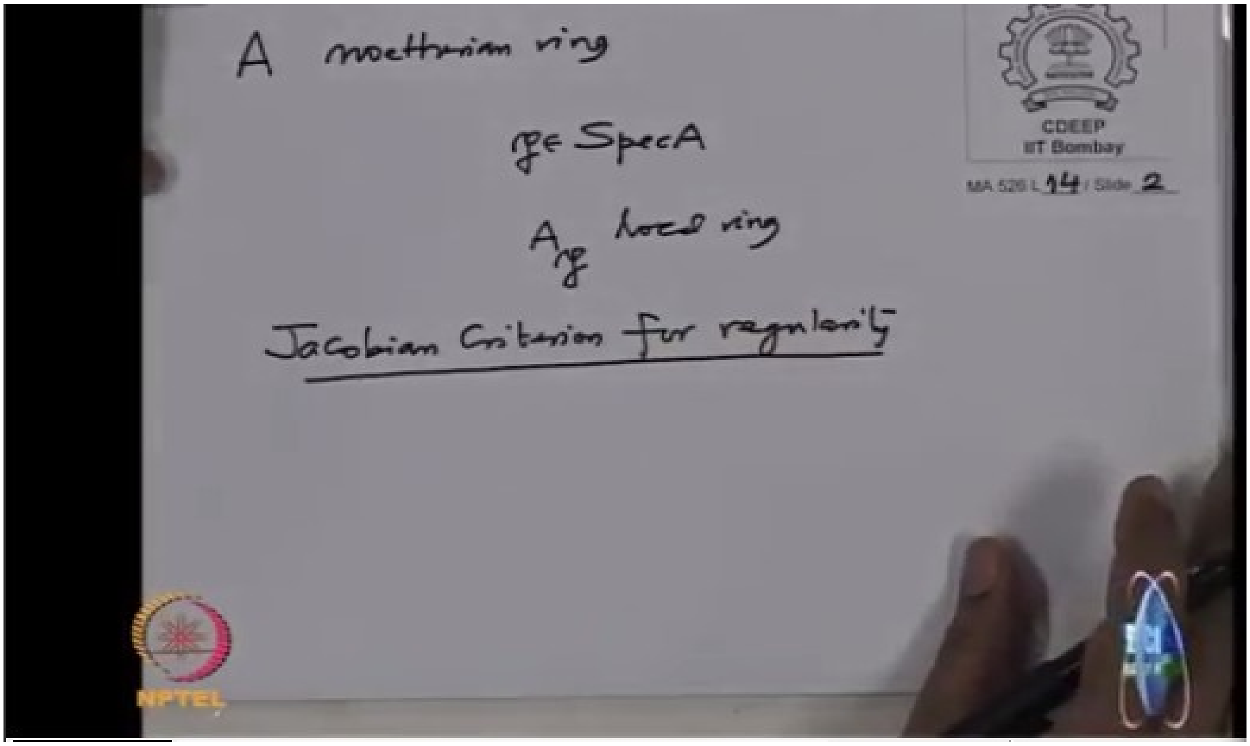

So let us recall what we have proved so far. So if you have a local ring A. So I will assume now onwards the ring is local. So this is local. So there's only one maximal ideal. And sometimes we will say with residue field $\frac{A}{M}$. So it can denoted by triplets. A, M and $\frac{A}{M}$. So let's have local ring and when I say local we will part, as a part we will assume that is it noetherian else. And even more, sometimes when you want to assume more then we will assume that this is a local ring of a finite type algebra over a field. Just to get the feeling from for the geometry feeling. Okay, so what we have proved is, this is dimension of A, we have defined this dimension is a cool dimension and we have proved it is finite and the three number namely the degree of Hilbert Samuel polynomial, it's a valid dimension and the Krull dimension all this, even numbers are equal. We know it is finite. And not only that we have also proved on the way that this dimension is bounded by the minimal number of generators for the maximal ideal. This is the minimal number of generators for m. And this is well-defined that any two sets, any to minimal sets of generators are m, they have the same cardinality. They are the same cardinality and the cardinality is nothing but the vector speed dimension of $\frac{m}{m^2}$, as a vector space over $\frac{A}{M}$. This is Nakayama's lemma. So it can be strict in equality. But when it is equality, those rings are very important and they are called regular local rings. So that is the definition of regular local rings. So A is called regular if Krull dimension of A, equal to minimal number of generators for M, which is the vector space dimension of $\frac{m}{m^2}$. Now, also you can define, if you have prime ideal in general, we call the ring is not local. In non-local case, a ring is called regular if every localization. A localize at p, p is a regular. For every p in, the Spectra. Now it's a big task how do we test some given local ring whether is it regular or not regular.

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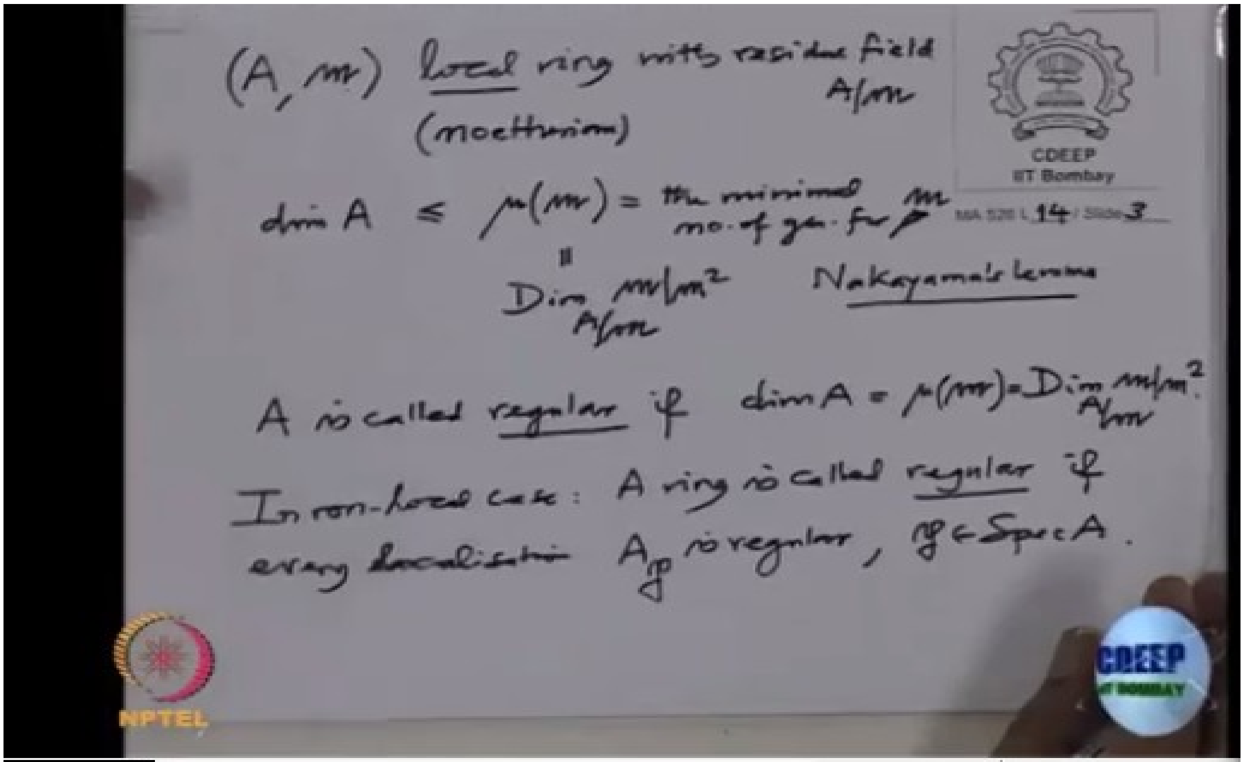

For example, if you want to test, if you take the circle coordinator become circle or real $\frac{\mathbb{R}[X,Y]}{X^2+Y^2-1}$. Is it regular? Okay, such questions we will address after acquiring some kind of basic properties of this rings. So getting back to the local situation, when we have a local ring Am . To study this, especially the dimension, if you notice we have attached. This is associated graded ring, $gr_m(A)$. This is the direct sum, $\frac{m^n}{m^{n+1}}$ n is 0,1,.... So we know, using this ring we have defined Hilbert function and degree and so on. So this is called associated graded ring. Okay. Now, the first important observation is, let's us write it as a theorem (A,m) local then A is regular. If only if the associated ring is a polynomial algebra, plus d where K is a residue field. While strictly speaking I should say isomorphic. As K-algebras. And where this d is obviously d has to be the dimension of the ring.

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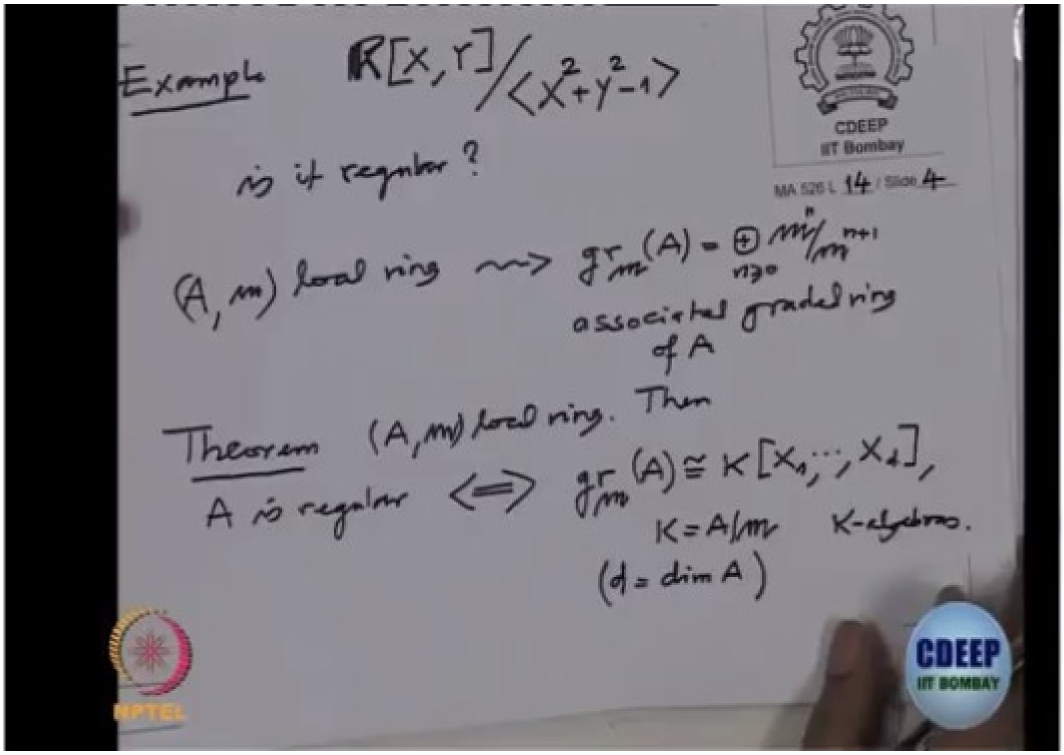

Okay. So, proof. One thing I note here first. Note that from the dimension theory, we have proved one easy consequence is dimension of the ring. Local case, this is the dimension of the $gr_m(A)$. This, I don't remember whether we have, I have formally proved it. But in any case, I want to discuss more about this in the tutorial sessions. So this. Okay, so once you note this, then this is easier because we know that if the ring is regular, if A is regular then we know the dimension of A is the minimal number of generators for m. That means as a this $gr_m(A)$, as a algebra over $\frac{A}{M}$ generated by homogeneous elements of degree 1 and precisely d in number. So this is, finite type, K algebra generated by d elements of degree 1. Because maximal ideal is generated by d elements. So that means, we have a natural surjective map from polynomial algebra in d variables to the associated graded ring. So this X_i s' will go to the generators of m_i s'. Mod, of course, they're mod $\frac{m}{m^2}$. And this is surjective is clear because this is generated by this elements. So this is surjective K algebra on morphism. Now I want to claim it is isomorphism. If it is kernel adjective also. To see that if kernel is non-zero, then it will have at least one on the relevant and this polynomials algebra over a field which is an integral domain. So if kernel is non-zero and if I go mod that this will be isomorphic to the, let's call kernel to be A. So then we will get this Isomorphism as a K-algebras, but then, the dimensional, this will be equal to dimension of this. And the dimension of this would have dropped at least by one. Because A is non-zero ideal and we have seen A, when an integral domain, when you go mod non-zero element dimension will drop at least by 1, because the zero ideal at the end will disappear from the chain. That will be contradiction. So, this will mean that dimension of d equal to which is dimension of graded ring which will be smaller, strictly smaller than d which is absurd. This d is, well, this is d. Is d is the dimension of the polynomial ring, if this a is a non-zero ideal and this dimension would have dropped at least by one. So this is another than dimension d, therefore is not possible therefore kernel is zero, therefore it's an Automorphism, so that proves the associated graded ring is a polynomial algebra.

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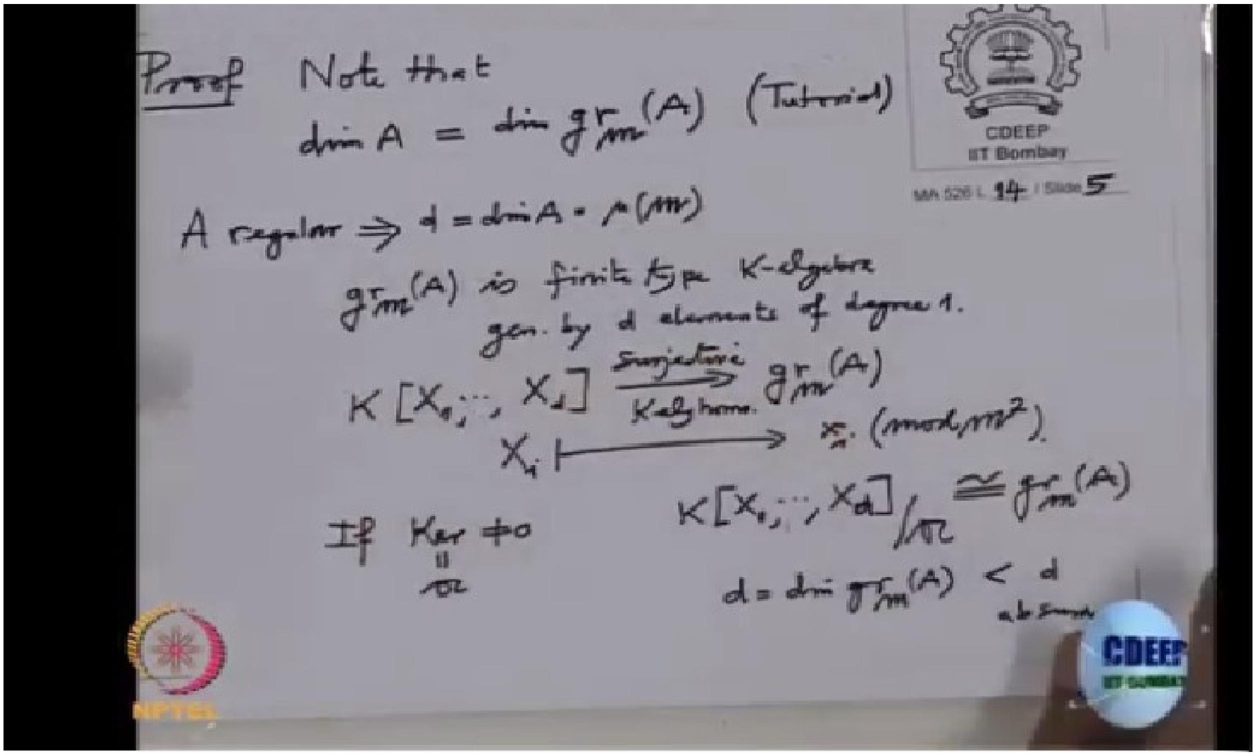

Now we have to see the converse, converse A. So conversely assume that, the A is graded, associated graded ring is isomorphic to the polynomial ring. So we noted the number of variable should be equal to the dimension d, if this, then this Isomorphism is graded Isomorphism. So this is a K-algebra graded Isomorphism. Graded means homogeneous that means homogeneous elements of some degree will go to precisely homogeneous elements of the same degree. So in particular, the homogeneous component of degree 1 which is $\frac{m}{m^2}$, and here homogeneous degree component of the homogeneous component of degree 1 which is precisely the vector space generated by capital X_1 to X_d . This is isomorphic two, this is the k-vector space, generated by these variables X_1 to X_d . So basically these vectors and dimension-d. So the dimension of these vector space, let's call it v, this dimension is d, therefore this dimension is d, as a k is $\frac{A}{M}$. But Nakayama's lemma, this is precisely m. And this d was a dimension. So we prove d equal to $\mu(m)$, so therefore by definition A is regular. Yes, because you see this isomorphism is precisely x_i going to the, where this algebra, graded algebra as a algebra over zero with component generated by degree 1 elements. And we're mapping the variables to those generators.

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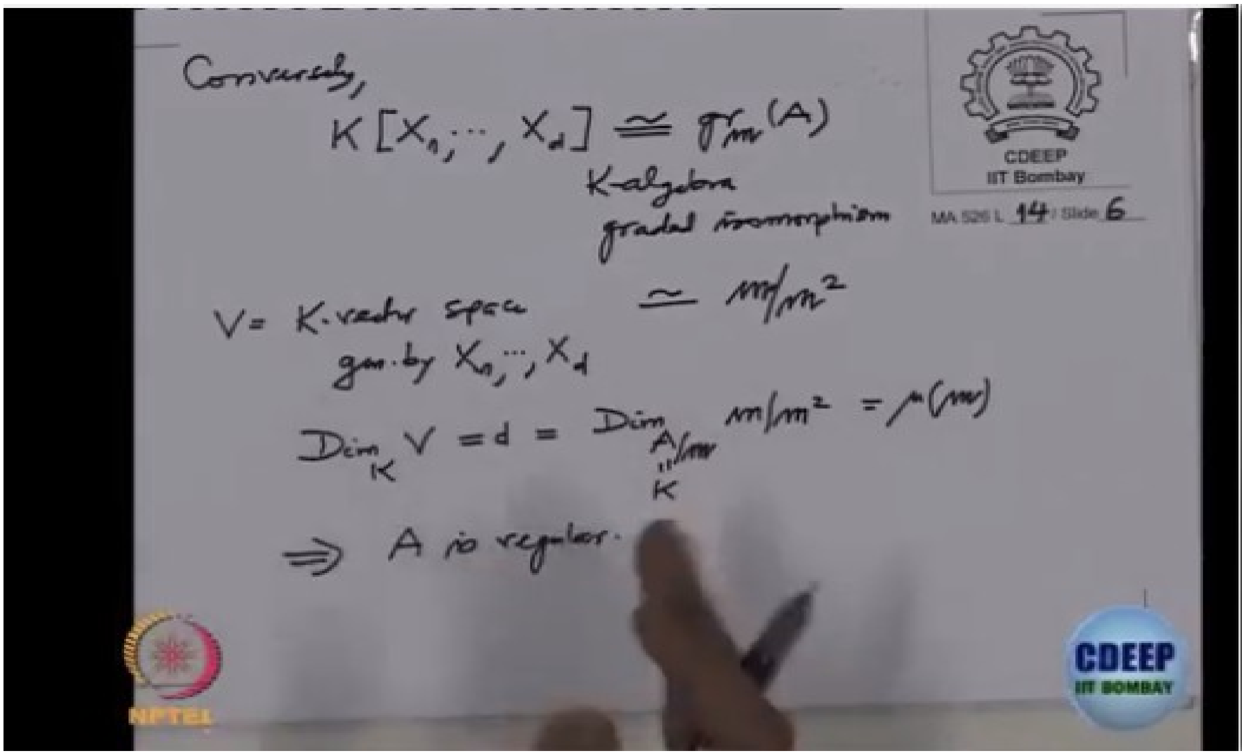

So it is and then extended. So it is homogeneous isomorphism. Okay. So, if you have a finite type K-algebra, that is I will denote it by K, small x_1 to small x_n . So this is coefficient of the polynomial algebra modulus of ideal. And now if you take prime ideal in this finite type K-algebra p, that prime ideal first of all will corresponds to a unique prime ideal in the polynomial ring which contains this ideal a. So therefore I will write, instead of writing prime ideal in this that is a prime ideal in the polynomial algebra. It's in the Spec of $K[x_1, \ldots, x_n]$ and which contain, a is containing p. For such set also there is notation $V(a)$. $V(a)$ is a subset of all those prime ideals in the polynomial ring which contains cf. So if you take any such p, so p here, then, and if, let's call this ring as a or let me call it R. Then r localize at p, is our A. A is R localize at p. so in any case, in this case, the graded ring we have a the gram will look like polynomial algebra $K[x_1, \ldots, x_n]$ modulus of ideal B. And this is a graded algebra. This is K-graded algebra. So before I go on, I want to deduce. So we will come to this case soon. So I want to write one or two corollaries. Corollary to above theorem that every regular local ring is integral domain. That follows from the fact that because if A is regular local then we have seen this associated graded ring in the polynomial adjective. So, and, associated graded ring is therefore integral domain. And if associated graded ring is integral domain then the original ring is integral domain. That is easy to see. So I will leave it for you to check this easy fact, $gr_m(A)$ is a domain then A is a domain. I will just say it orally, what we need to check. So, if we have given this is an integral domain, this is a graded integral domain.

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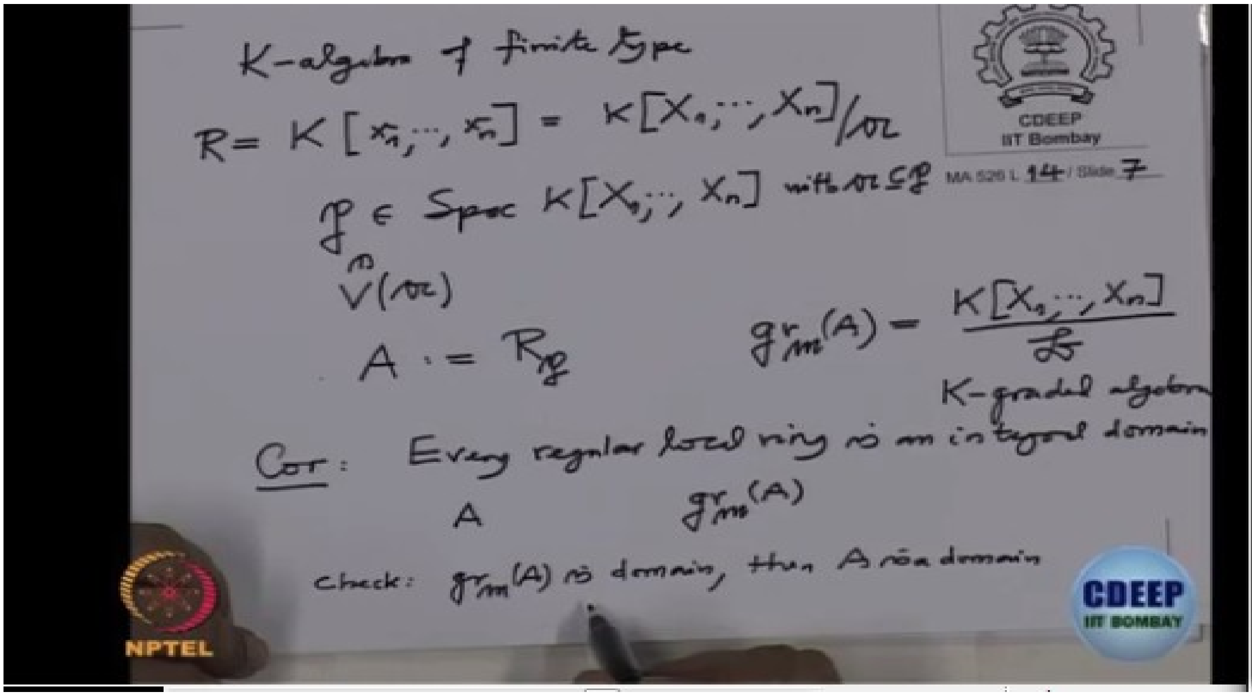

So graded ring algebra is domain if only if there is no homogeneous zero divisors. If you use this observation, then you can easily check A is a domain, because if A is a domain then A times B zero for some A and B in the ring. And there in the maximal ideal, we go in local ring. So you choose power of maximal ideal where A will belong, but not in the next power and similarly for B, then when you read the images of A and B, associated graded ring, think of them as homogeneous elements of degree, whatever the powers you've chosen, and it was if so and complete the proof by using that. Okay, the second. So this is the part one of this corollary. Second part is regular local ring is normal. A is normal. So we've seen in part one it is integral domain. So when you say ring is normal, so it's domain. So normal means, so recall that normal means integrally close in its quotient field. Total quotient ring is precisely. This is who take the minimal primes and take the union of that and then invert all of them. So it becomes the, or then we had to look at this, take the non-zero divisors and invert all of them. So past from A to $S^{-1}A$, where S is set of non-zero divisors in A. So this is the, this map will the injective and that is the maximum possibility. Because we cannot afford to invert anybody other than non-zero divisors, because if you do then this will become zero. So, this is injective, therefore it makes M should talk integral closeness of A in this way. And if that happen, then call the ring to be normal. But in this particular case, because it's a domain this inverting all non-zero divisors mean, inverting all non-zero element that means this actually is the quotient field in this case. So, normal means A is integrally close here, every element of this as inverse A which is integral over A, it must be in A. Typical example of normal domain is integers. The quotient field is $\mathbb{Q}$ here, and $\mathbb{Z}$ is integral close in q, that is a famous observation by Gauss. And more generally don't need $\mathbb{Z}$, but you have UFD, A is a UFD. The same proof we'll tell you UFDs are normal in their quotient fields. So, K is quotient field of A, and this is normal, the same, same idea of this, the idea of being used in the concept of GCD. How can set of GCD and then that was used in this proof by Gauss. So UFDs are normal. Obviously A. So it's a big theorem that regular local ring is a UFD. And that is what we've given to prove but these proofs involves homological algebra. And that is I will prepare in the beginning and the theorem at the end will be regular local at UFD. Of course, but I'm not using this corollary. This corollary is simpler than this. So how do you prove it? Again, the idea is the same, if you remember the assignment when that integral extensions the supplements which I've written. There it was, you will see that, if you want to prove A is normal, to prove A normal, enough to prove that the associated graded ring is normal. But in, if A is regular local ring then the associated graded ring is actually the polynomial algebra over a residue field. And again by theorem of Gauss that polynomial algebra in seven new variables is a UFD. So therefore it is normal. And therefore by that observation, A is normal.

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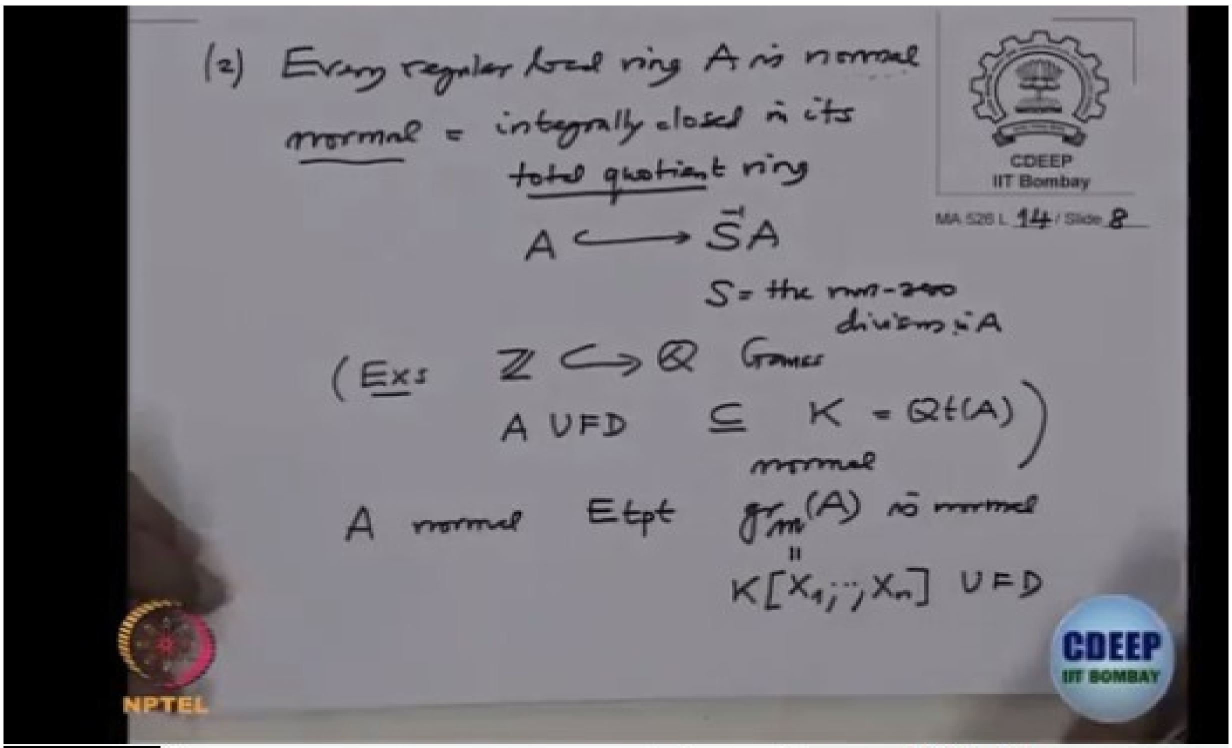

So one or two examples which you studied in earlier courses, let me mention few of them, so for example if you would have studied let's say A equal to this number rings which arise from number theory. So, $\mathbb{Z}$ adjoin the route - 3 for example. This is of imaginary quadratic, a ring of quadratic integers. So this is, in this case, the maximal ideal is, and I'm talking the maximal ideal. So first, this is not a local ring. So, let's use take the ideal M which is generated by 2 and $1+\sqrt{3}$. $1\pm\sqrt{3}$ this ideal. So, when I localize now, A localize at M. This is not normal. Is it not integrally close. Because just look at, i will give an element which is integral over these but it's not there. So, first of all, let us take $\mathbb{Z}$ equal to one plus, root minus three divided by two. This element will satisfy polynomial like this. Z^2-Z+1 is 0. That you can check very easily because whenever square this, this is. So when you square this, this is 1 square that is 1, -3 which is -2, and then, -2 + 2 $\sqrt{-3}$ and divided by this 4. This is Z^2-Z , that is -1 $\pm\sqrt{-3}$ by 2. So these two gets cancelled, this is 2 here. And then + 1. So this is the + sign, this is the - sign. This time we will get cancelled. And this is with minus, minus half. This is half year, this is half of that so that we'll get zero. So therefore $\mathbb{Z}$ satisfy integral equation over this, but that is not there.

Because there two, so it's not normal. So therefore in particular this ring is not regular. In particular, Am is not regular.

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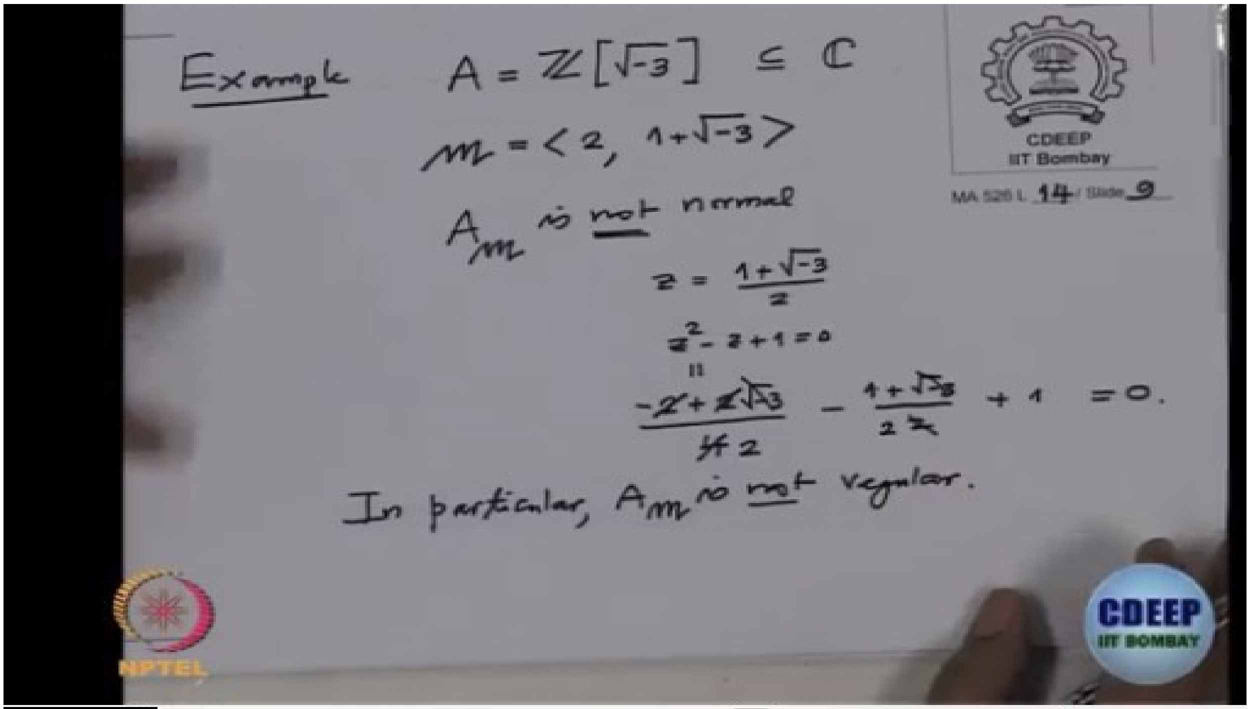

One more observation that in case of dimension one, so remark in case of dimension of A is one. We have to get back to the case very local now. A local and somehow dimension of A is one, then normality, normal domain and regular, these are the same.

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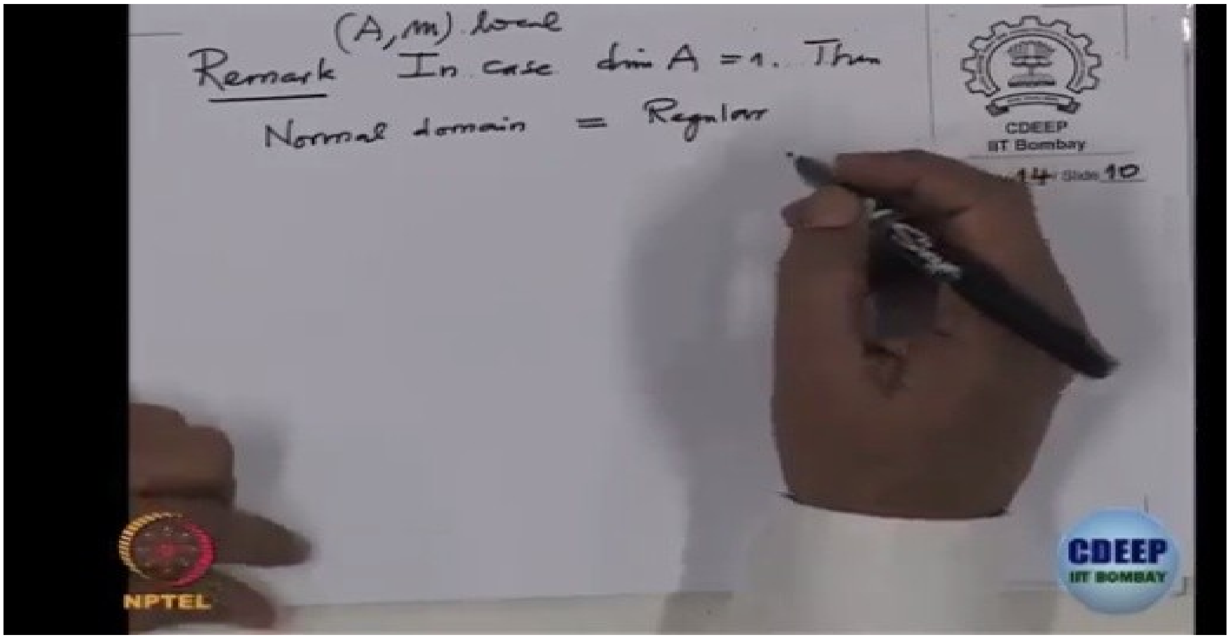

They are equivalent concepts.