Lecture – 36

Characterization of Euqidimensional Affine Algebra

Gyanam Paramam Dhyeyam: Knowledge is Supreme.

When you say hyper surfaces that they should be equi-dimensional of dimension 1 less than the ambient space. So, this is what I want to characterize and they should hyper surface it should mean that they are define by one equation. So this is what the theorem we will prove in the following. So theorem: A is the finite type algebra over K. A is quotient of a polynomial algebra, X_1 to X_n , module over ideal A. Then the following are equivalent. 1, A is equi-dimensional of dimension, exactly one less than the ambient space that is n-1. And 2, now, before I write the statements to make them clearer, I am going to identify the spectrum of A, prime ideals of A to $V(a)$. $V(a)$ simply means, all those prime ideals in the polynomial ring which containing and then they will correspond to prime ideal theory and conversely. So, this is a closed subset in Spec of $K[X_1,\ldots,X_n]$. And this what do I mean by over ambient space. Because with this identification, we are thinking this embedded in Spec of $K[X_1,\ldots,X_n]$. In fact, as a close subset and in fact not arbitrary embedding the closed subset. And when I say embedding, that means it's a homomorphism from this to this. So, this is risk topology and this is risk topology coming from this. This is actually a homomorphism, closed homomorphism. Closed embedding one calls it. And the minimal prime ideals here will correspond to the minimal prime ideals here. So, they'll be precisely the minimal prime ideals over this ideal a in the polynomial ring. So, the second statement is every prime ideal which is minimal over a, and when you say minimal prime ideal that means I am now thinking here in the Spec of $K[X_1,\ldots,X_n]$. So, every prime ideal which is minimal over a is minimal among non-zero prime ideals in $K[X_1,\ldots,X_n]$. Third one, the radical of the ideal a is generated by a single element for some g in polynomial over K. So that means, you see here $V(a)$ will not change if I go to the radical ideal. And if you would have proved this, that will mean that this is defined by one equation, so that is the hyper surface. Such a thing is called hyper surface. This equivalently justify, this equivalence in fact analog of what we did we can linear in hyper surface in the vector space is defined by one equation. That is one less dimensional of space in a finite dimensional vector space is actually it's define by one equation.

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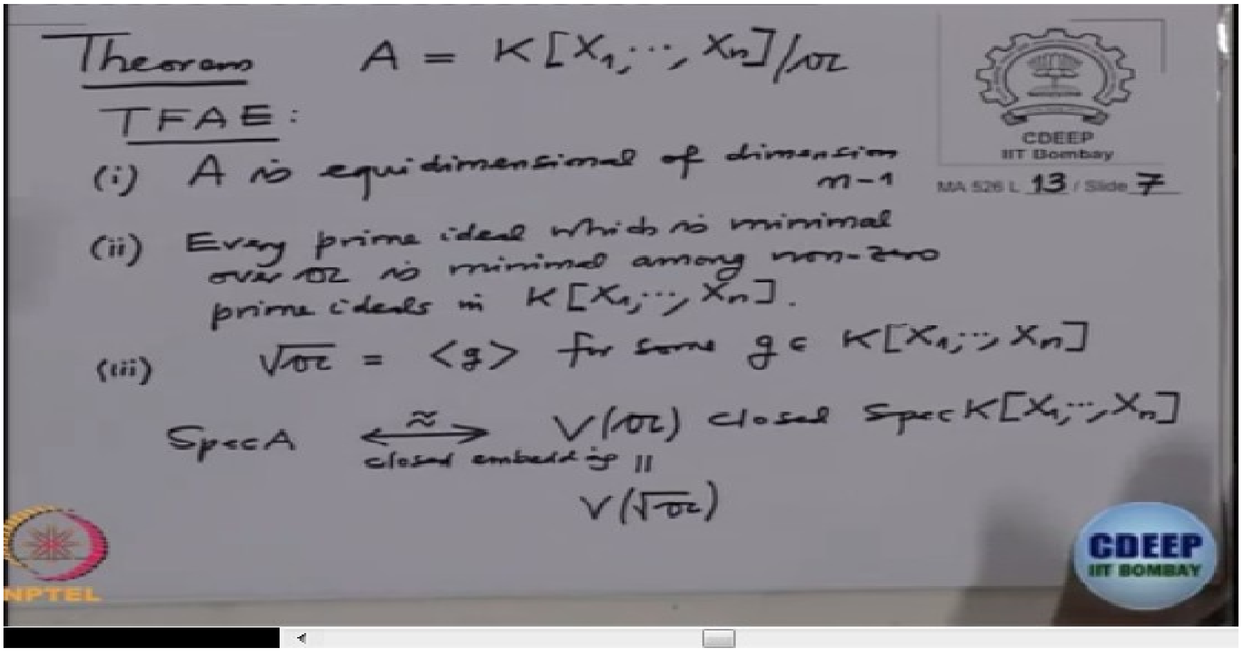

Okay, so let us prove this. So, proof. As I said, I will always make use of the identification that above I mentioned. So let me write M equal to, these are the all elements, all prime ideals in $K[X_1,\ldots,X_n]$ such that p is minimal over a. This is the finite set. In fact these are precisely the minimal elements in the Associated primes of the ideal a, $K[X_1,\ldots,X_n]$ mod. This is same as min Associated primes of $K[X_1,\ldots,X_n]$ module a. And what do I have to prove, if I want to prove 1 implies 2, what do you want to prove? I want to prove that any element here is in fact minimal among all non-zero prime ideals in $K[X_1,\ldots,X_n]$. Okay, so first of all note that if I take a dimension of $K[X_1,\ldots,X_n]$, take any element here in M, so if I take any p in M, then this dimension is dimension of A by the image of p mod a. But this is n-1, because of our assumption, because we are saying A equi-dimensional of dimension n-1, that means all minimal prime ideals, all equi-dimensional components will give the same dimension and that is nothing but dimension of this. So, we know that for each n in M, dimension of $K[X_1,\ldots,X_n]$ mod p is n-1. So first of all, we know because now the dimension of $K[X_1,\ldots,X_n]$, this dimension is n and going mod p dimension drops exactly by 1, so in particular p cannot be zero. If p were zero, this is not correct. So p has to be non-zero, first. Not only that, p has to be minimal among non-zero prime ideal because if not, what will happen? If p is not minimal among non-zero prime ideals, that means p contains some non-zero prime ideas. Then 0 is a prime ideal in $K[X_1,\ldots,X_n]$ and another q will be there, and this p will contain that q. If p is not minimal among non-zero prime ideals then there will be q in between and now this link I know, n-1, because this dimension is n-1. So therefore this chain will increase at least by 2, so that will show that dimension of the polynomial ring will be at least n plus 1, but that is not correct. Therefore, that proves 2. So, that implies p is minimal element in Spec of $K[X_1,\ldots,X_n]$ minor 0. That is the meaning of second statement that it's minimal element here, minimal among non-zero prime ideal of this. So that is 2. So that proves 2. So one implies 2 we have proved.

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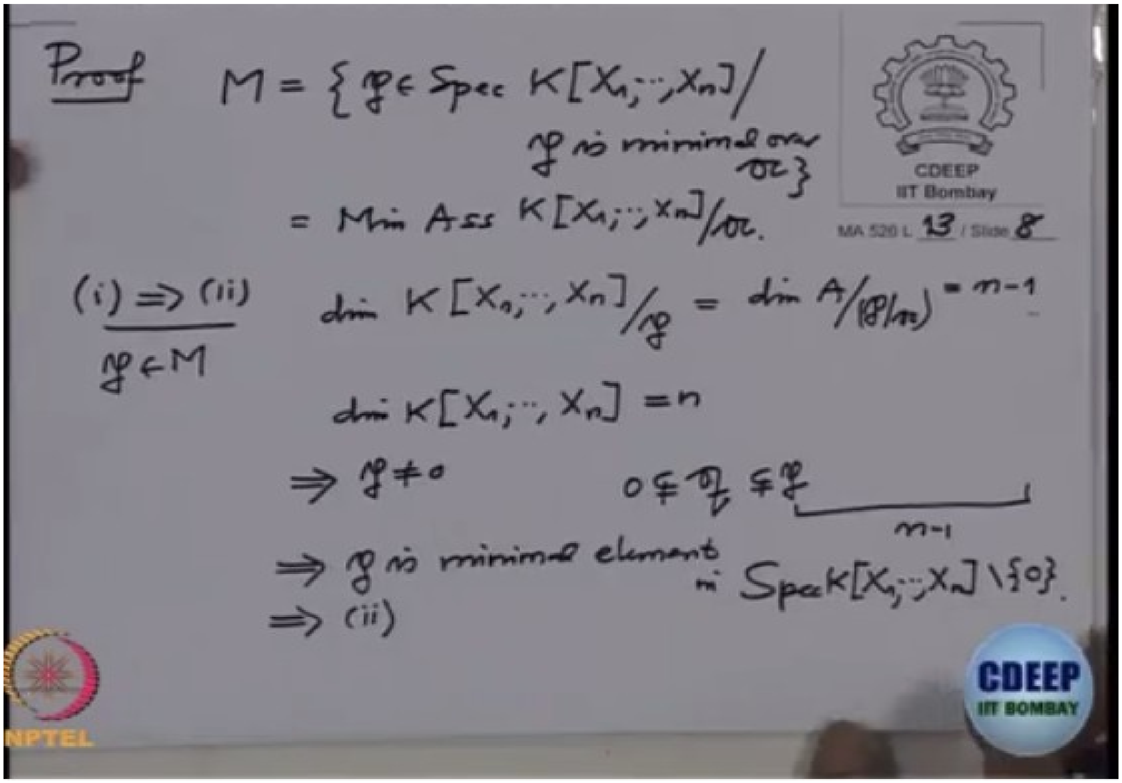

Now, let us prove, 2 implies 3. So, 2 says every prime ideal p in the set M is minimal among non-zero prime ideals in the polynomial ring and from here we want to conclude that the radical of the ideal A is principal ideal. This is what we want to conclude. So what is given is p is in M, then we know that height of p has to be 1 because p is minimal among the non-zero prime ideals. So there can't be any prime ideal in between zero and this p. Therefore height of p cannot be more than 1 and on the other hand it is less equal to 1, so it is exactly 1. Now, where p the prime ideal in the polynomial ring over a field, $K[X_1,\ldots,X_n]$. And this polynomial ring is actually factorial. When I say factorial that means, unique factorization domain. Another reusable word is used for unique factorization domains in algebraic geometry which l is the factorial. So, this means this every non-zero element, non-unit has a unit decimal decomposition and you need to unit decimal decompositions are essentially unique that means, the number of factors and they are unique up to order and unique up to units. So, I am going to use that fact. So this is simple Lemma but very useful all the time. So, in general if you have a factorial domain, assume noetherian also, then every prime ideal of height 1 is principle. This is very simple. Let us finish of the proof. Proof is, so we are given a prime ideal of height 1, let p be prime ideal and height p is one, so therefore p cannot be 0, 0 is prime ideal. So it definitely has one element say f in p, f is non-zero but f is element in A. So therefore it has a factorization. So this f, I can write it as $\pi_1 \ldots \pi_r$, where this are irreducible elements in A.

Actually they're prime elements because you know in a factorial domain, 0 divisibility and prime is same. So they will generate a prime ideal. So for example, if I look at π_1 , if it's a non element definitely. So it has at least one of the prime factorial there. So, π_1 for example, this will generate prime ideal and it is definitely containing p and this is non-zero. And because P is height 1, there is no option but to add equality here. So that will imply p equal to generated by π_1 . So what did we prove? We proved that every prime ideal in a factorial domain of height 1 is principle. Converse is, because we put this Krull's principle ideal theorem if you have a height1 prime ideal. If you have a principle prime ideal, then height of that will be exactly 1. Okay, so now that is precisely the situation we have, when P is in M, height p has to be one and therefore p is principle.

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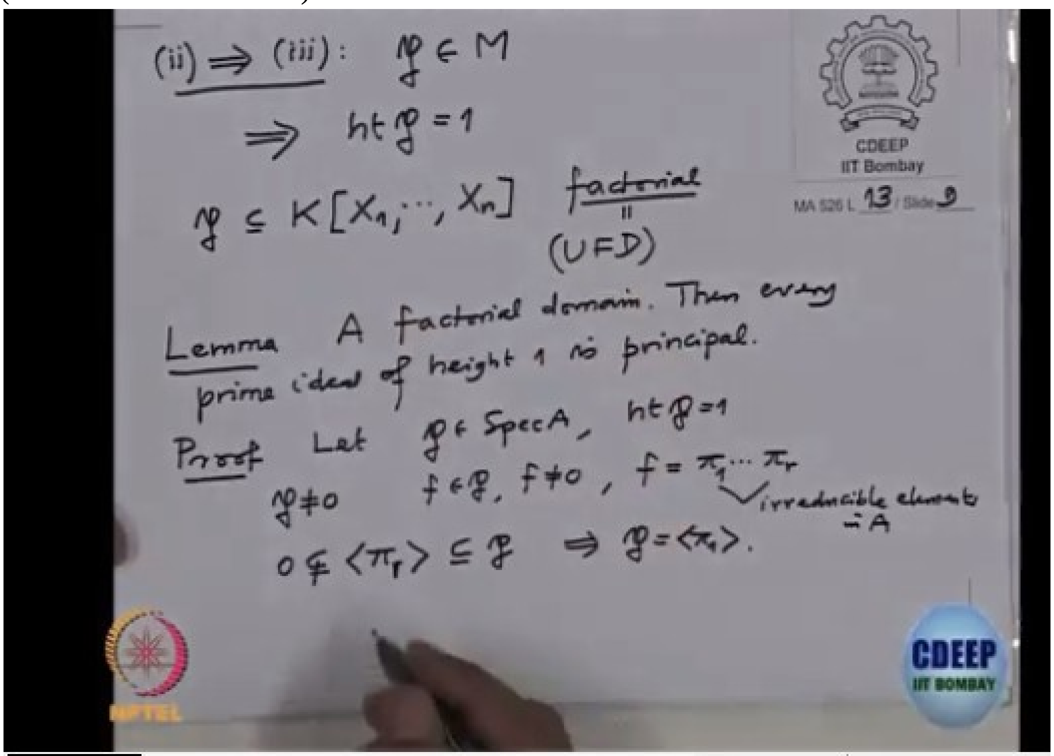

So chose a generator for p. So p is therefore generated by some element. Let me call it, this generator will depend on this P, so I will write g_p . For some prime element g_p in the polynomial ring. Okay and we're interested in now proving the radical ideally generated by 1 element, right. So which 1 element is. So look at the radical of A, radical of A is intersection of the minimal primes. So this is intersection P, p where is in M. M is precisely minimal prime ideals or A. So this is radical and p generated by g_p . So this is intersection of ideal generated by g_p . Where p is a variable number. This is a finite intersection and I claim that this is not generated by g, where g is the product of g_p . This is the final product. So, that is clear, because to prove that you prove that g is here and each one of them is nearly so. This, just check this. I mean, just write here, check this. And you may, you will have to use the fact that you are really unique factorization domain. Okay, so that proves 3. Now 3 implies 1. So we have given that radical of ideal, but even ideal A is principle ideal. And then from that we want to conclude that A is equi-dimensional of dimensional and minus 1. Okay. So suppose, radical of A is generated by g for some g, in $K[X_1,\ldots,X_n]$ and now what can you do? We have an element in factorial domain. Therefore we can consider which is irreducible factorization. So let g equal to $g_1 \ldots g_r$ be a prime factorization of g. Actually I should say d and up to order and up to a units. So now g_i , and g_j are different prime factors here that means the g_i cannot divide these and the other way. Okay, so g_i s are of primes. g_i s are prime elements in $K[X_1,\ldots,X_n]$.

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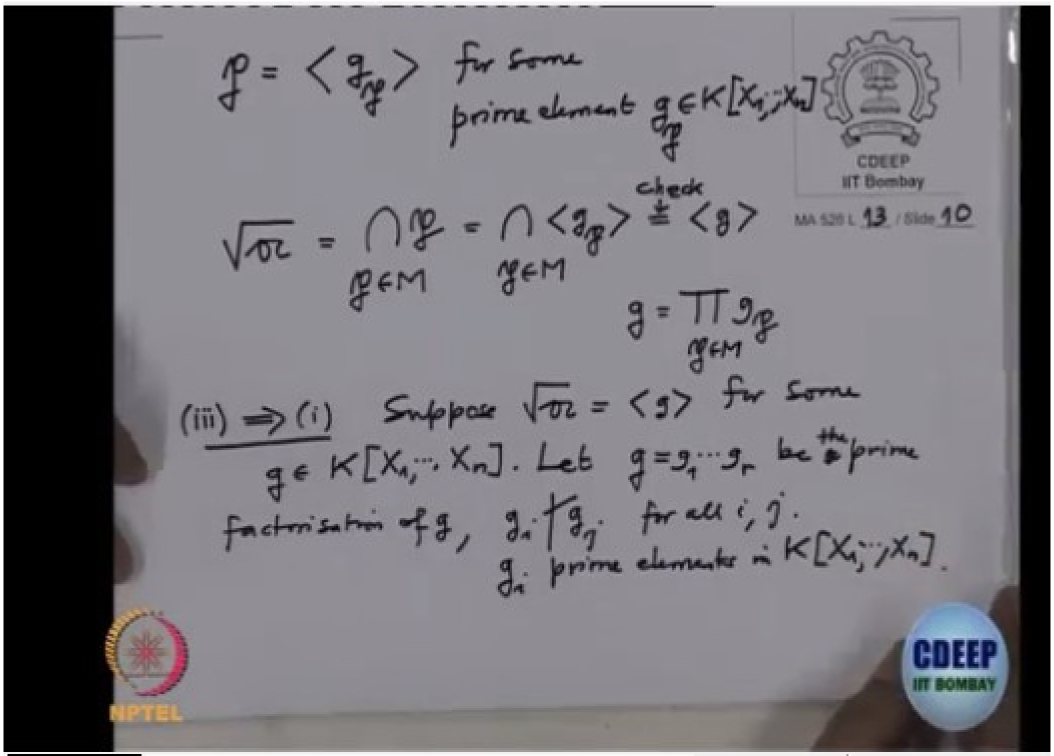

So therefore they will generate a prime ideal. So let P_i ideal generated by g_i . So in Spec of $K[X_1,\ldots,X_n]$ and it is clear also that which I take with the, we know that ideal generated by g is the radical ideal of a and which is intersection of p_i , i is from 1 to r. Because this equality is because, so the g_i s, g is the product. So this ideal generated by g and this intersection is same. Therefore, in this case, m is inside set, p_1 to p_r . And now I want to check that this a is equi-dimensional of dimensional $n-1$ that means I should check that, so I know the associate minimal elements in the associated in the primes. They're precisely p_1 to p_r . Each one of them is a principle ideal and we want to check that the dimension drops exactly by 1. So we need to check only that dimension of A module of this p_i , so that is strictly speaking I should write $\frac{p_i}{A}$ is this ring is precisely dimension of $K[X_1,\ldots,X_n]$, module over p_i , but that is same a s dimension of $K[X_1,\ldots,X_n]$, module of ideal generated by the prime element g_i . And what do we want to check, we want to check this is $n-1$. But that is clear because we see this is we are proved that, if you go module over an element which is a non-zero divisor than the dimension will exactly drop by 1. So, therefore, because g i are prime elements, non-zero, so non-zero divisors are domain, therefore dimension will exactly drop by 1 form this. So this dimension of $K[X_1,\ldots,X_n]$ -1 but this dimension we know, that is n, so it's $n-1$.

This is true for all i. So, therefore A is equi-dimensional of dimension $n-1$. Actually, the dimensional theorem I have use here, to say that the dimension drops exactly by 1, but this also you can avoid by using dimension, dimension theorem. See this is milder use of dimension theorem. When can actually prove it by hand, this dimension is $n-1$. Similarly also, we know the proof $K[X_1,\ldots,X_n]$ equal to dimension equal to n, here also we don't really have to use the dimension theorem you would follow from normalization level. So, some of the things can be avoided but anyway since we have the machinery we will use it. So that prove this theorem.

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