INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

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Lecture No. – 34

The Spec Functor

So now here also immediate corollary to this is, if I have a prime ideal P in any ring A Noetherian, then height of P is finite,

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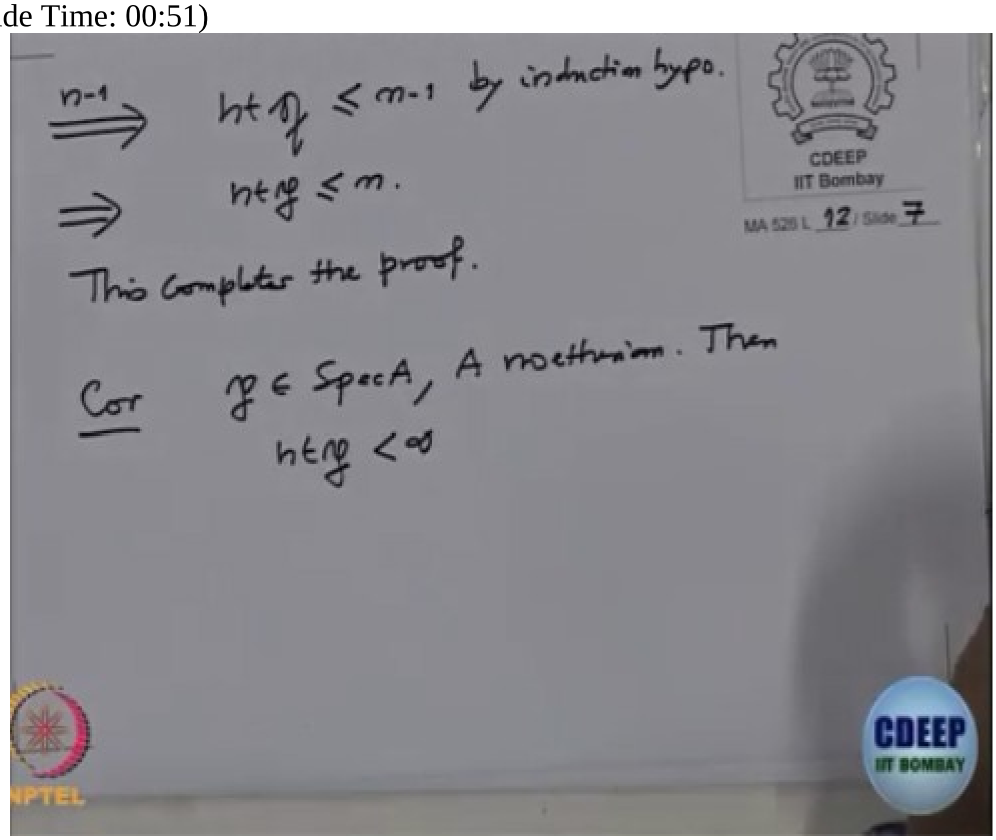

because this P will be generated by some element and then we apply the Krull's generalize theorem to be this piece of P, height P will be less equal to the number of generators so actually height of P will be less equal to mu of P, that's we have noted last time no, so therefore in particular dimension of a local ring is finite, so in particular dimension of a local ring which is height of P, in general actually dimension of local ring, A local, this is finite.

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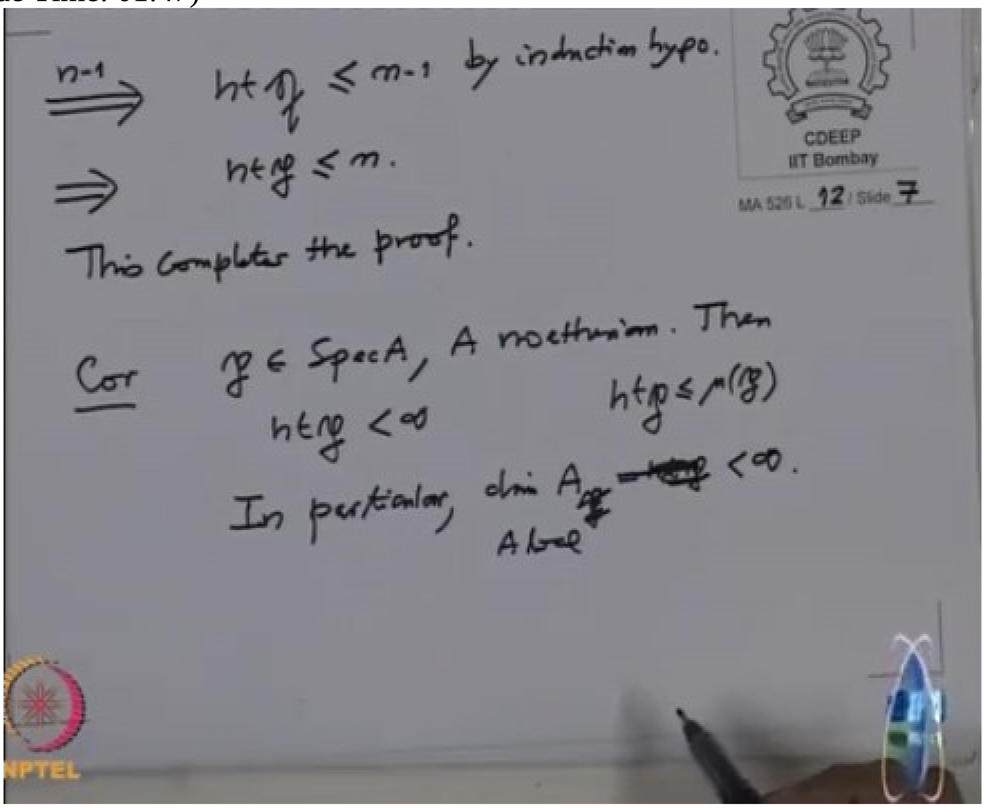

So dimension theorem corrected this and because it's a degree of Hilbert Samuel polynomial etcetera, so okay, if you notice I wanted to write a proof in the exercises that remember the, in the normalization lemma we proved that finite type algebra over a field is integral over polynomial algebra, and the number of variables if therefore the dimension, right, so for a finite type algebra I wanted to prove that the polynomial, the sum I wanted to prove that the, I wanted to have analog of the Hilbert Samuel polynomial for finite type algebra, so A is finite type over a field K, and we somehow know not by using dimension theorem we somehow know that the dimension of this finite type algebra A equal to the number of variables which appear in the Noether's normalization lemma, so this d which appears in NNL,

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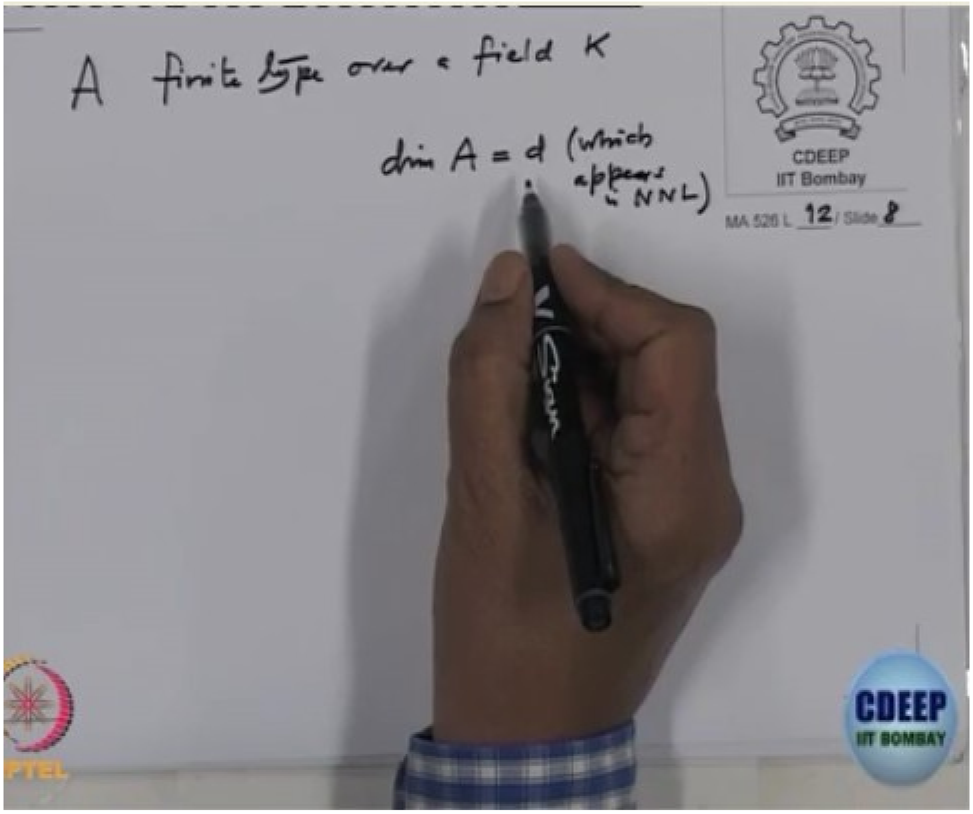

but this I wanted to do also show it by using the proof similar to Hilbert Samuel polynomial and I wanted to cook up some polynomial, polynomial function with rational coefficients whose degree is related to this d, d or d-1, so that will match it the dimension theorem, right.

So but remember that to define that polynomial I had to have, I have to have a graded module, and graded rings and so on, so finite type algebra is just A just $K[X_1, \ldots, X_n]$ modulo of some ideal, and this ideal may not be homogenous ideal or anything, so what I wanted was suppose this small x_i 's are the images of capital X_i , this residue class ring then from such a, I want to define a polynomial function,
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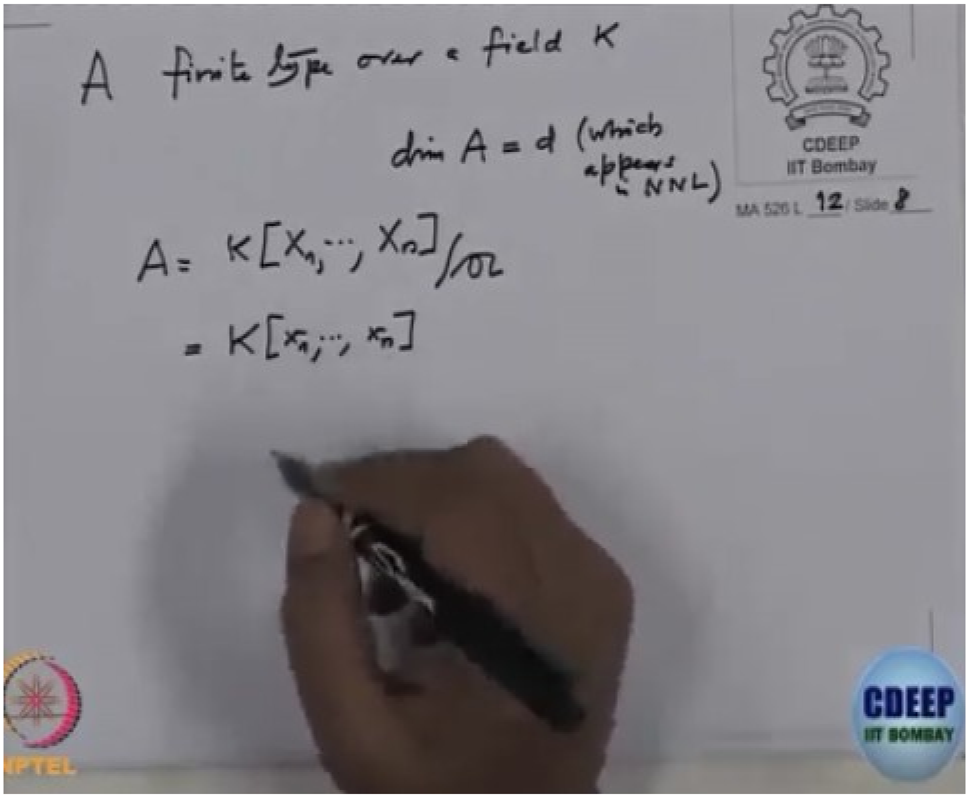

I want to define some numerical function and check that numerical function is a polynomial function, and that d, I wanted to check this d, right, so first of all that too many hurdles, number 1 hurdle is some this generators may, it depend that polynomial function which I will define will depend on this generating set, so somebody may have different algebra generators of these, and then the polynomial corresponding to that as a, if you have read that it's not unique, the polynomial is not unique but what is unique is the degrees unique, so that is what I wanted to, you do it more carefully and maybe on the way your difficulty will disappear, okay.

So the polynomial what I wanted to define here was, so this also I will do it precisely today in the exercises, so that will get tied up with many other things, okay.

So another thing now today I just wanted to mention that if I have ring homomorphism from A to B, then we have a map from spec B to spec A namely any prime ideal Q in B you contact it to A, right, and so contraction of a prime ideal is a prime ideal,

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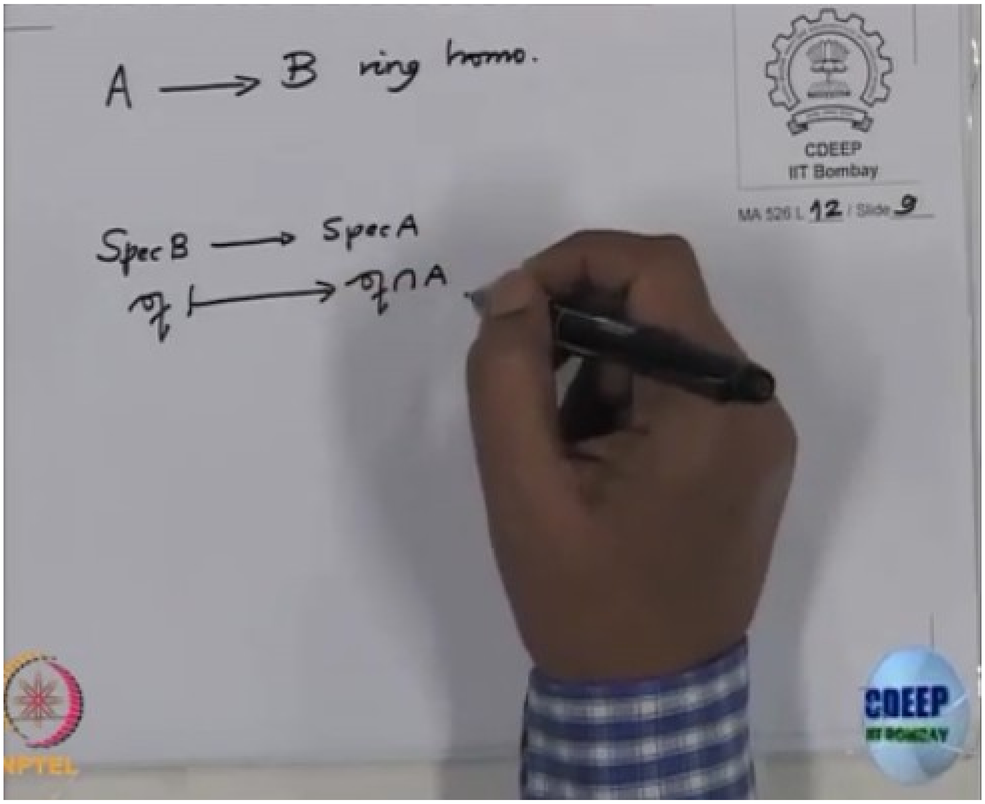

but this will not map in general maximal ideals, 2 maximal ideals because contraction of a maximal ideal need not be maximal ideal but that we know it for if you assume further A and B where finite type algebras, but in general we may not have that, so and also I guess you know what is a topology on spectra, right, so consider this now as a topological spaces with Zariski topologies, so this is a topological space, this is a topological space then I want to check that this map is actually continuous, so then this map is continuous in Zariski topology.

Do you already know that or shall I check it? Okay, so how do we check some map is continuous? Inverse image of an open set is open or inverse image of a closed set is closed and so on, so that means we want to check, so let us give some name actually so this is ϕ, then usually this is called ϕ^*, so and a closed set we know,

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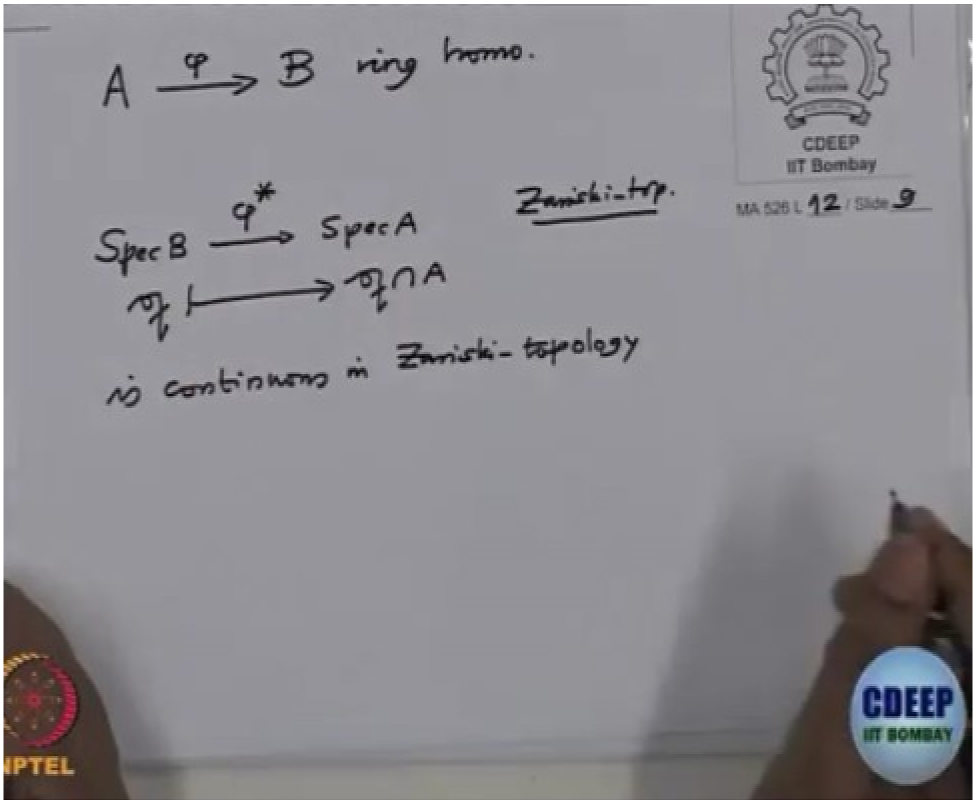

closed set is typically of this form V(a), where A is an ideal in A, so when I want to pull it back under this, so phi star inverse of this and what will it be? See you need to prove it is closed set, so V of somebody but that should be an ideal, so who is a likely candidate? A, you take this ideal A and extend it to B, so A times B. Either check this, this is easy to check because you can take one element here and check it here and other also because if somebody contains here that means somebody contains Q, Q ideal, prime ideal Q if it contains A times B, then contraction of Q will contain definitely A, so that means contraction will belong to this image, so that means phi star of Q will belong to $V(a)$, and conversely also similar or I wanted to also remind you that to check some map is continuous you can do little bit economically better, because you only have to check that the basic open sets are the inverse image of the basic open sets are again open,

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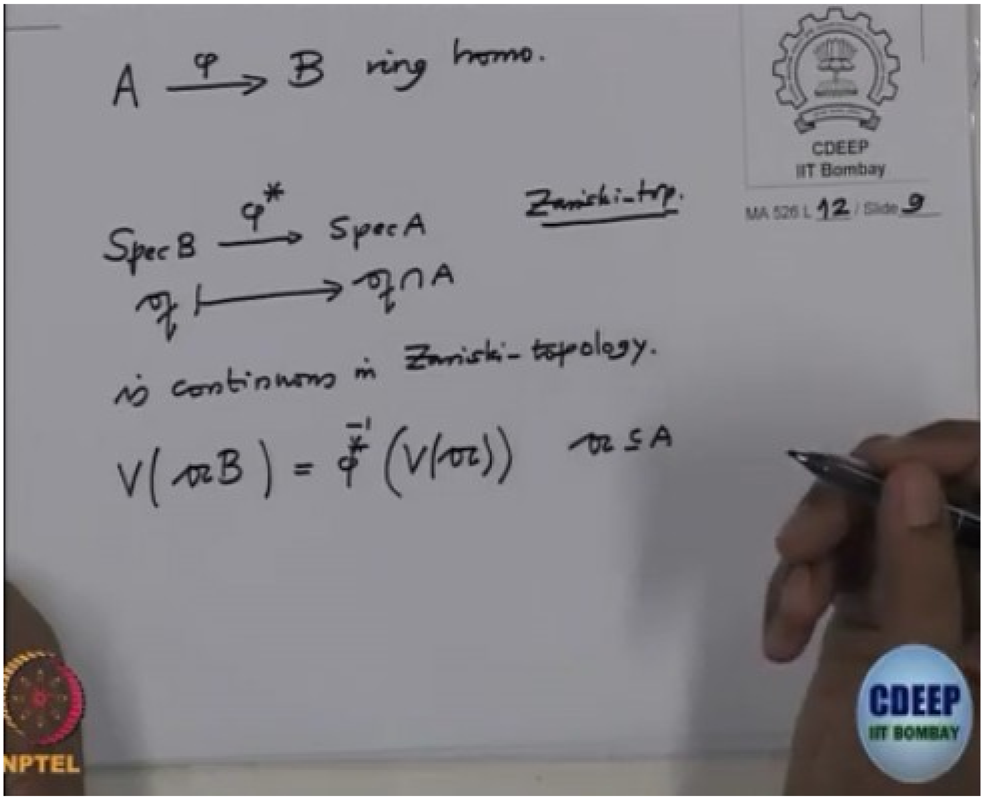

and basic open set you know, what is a basis of the topology? So this word is not like what we use in linear algebra or algebra, but this basis means similar to that, that means any open set or you think of it is a collection which is collection of open set, so that intersection of any two members of that collection will contain somebody like that, then it is called a basis or equivalently any open set will be union of this collection, so there is a nice basis from Zariski topology.

So what is the basis do you know? So basis for a Zariski topology on spectrum of A, so that is you take any F in A and look at $D(F)$, this is by definition, this unit open so take the complement of $V(F)$, $V(F)$ is same as V of ideal generated by F, so this is definitely open,

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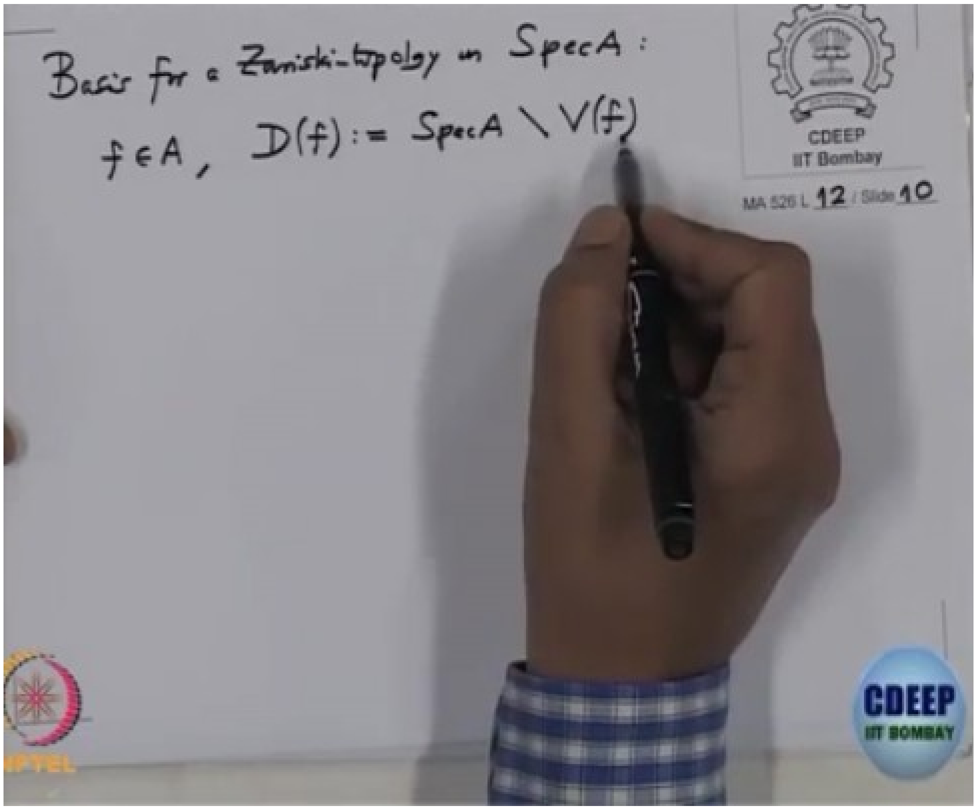

and any open set will definitely contain this because any open set is of the form complement of $V(a)$, but A will contain some element and that, so it's clear that this is a basis and so it is, see this is also directly you can define, this is all those prime ideals P in a such that if it is not in P,

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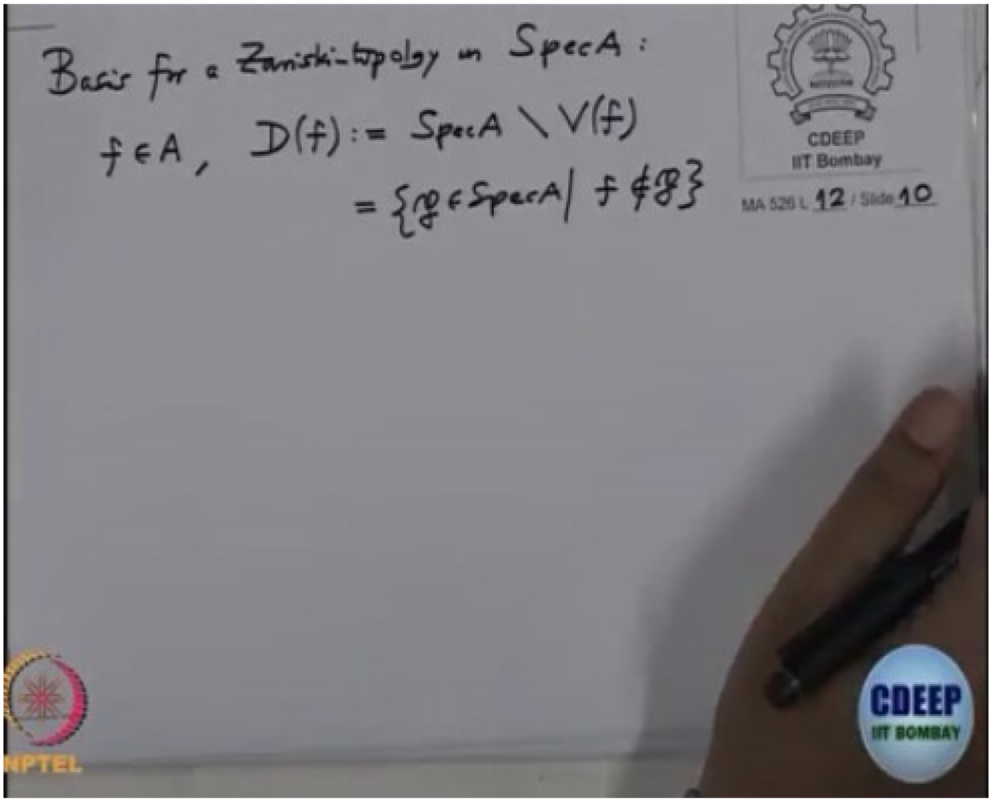

so one problem normally at the beginners level one normally come across is the notation, and one remedy for that notation is so when you want to do algebra think of prime ideals, so think of spec, this is the same thing algebra, I'll give the notation P, right, because that will also, and when you want to do topology geometry etcetera, instead of this you write x, topological space and instead of this you write small x a point, and then this P and this x there is one to one correspond, right, the same makes we are denoting in this notation by P and the same P in this notation we are denoting by X, so this you should also keep in mind this is P_x, right.

Now what happens? Now we have this ring A, we have $\frac{A}{P_x}$, no this is a integral domain, so it has a quotient field, so that quotient field in algebra notation is this, right, that I want to denote by there is no field anywhere, this is a field so that field I want to denote by κ_x, it only depends on x, and so when you think of topological space and X this is a field that edge to that point it's called the residue field at that point, and there is a natural map here, there is natural inclusion here, right.

Now let's check where do F go here, if I have an element F here, where does it go? It goes to $\overline{F}$, the residue class of F in $\frac{A}{P_x}$, so that is $\overline{F}$, and this $\overline{F}$ goes, this is natural inclusion so it will keep going to $\overline{F}$, this $\overline{F}$ I want to denote by $F(x)$, this is just a notation, but now see the magic, so when will this F belong to P_x?

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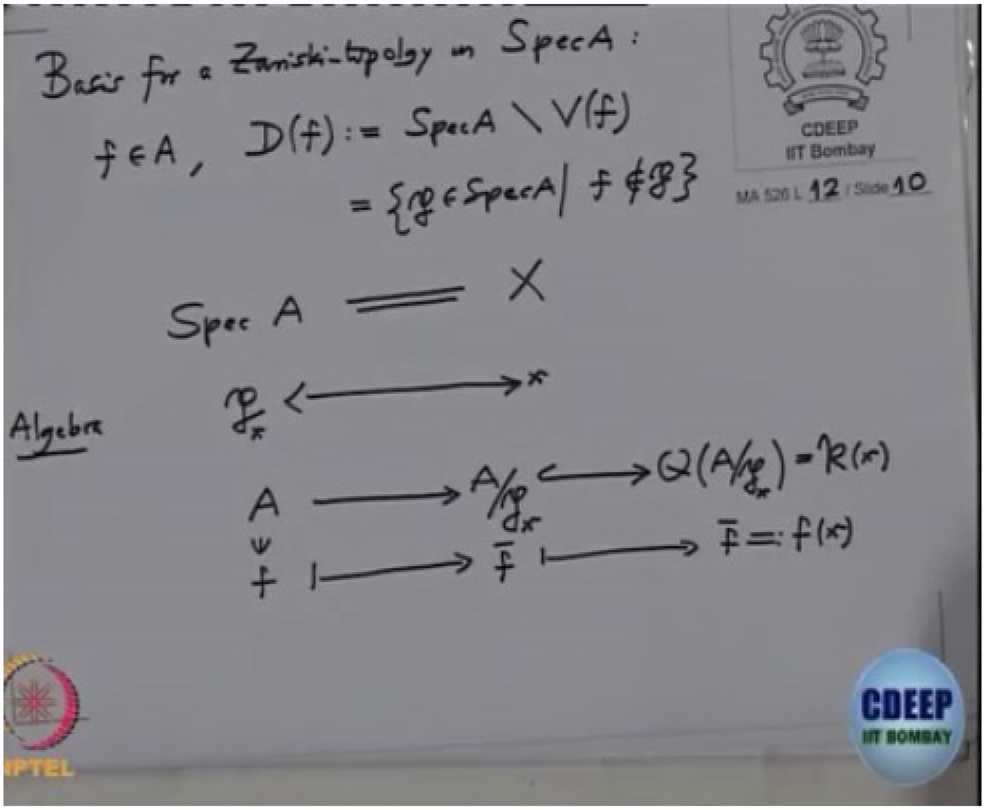

So let us write down that, when F belong to P, P means P_x , if and only if this is 0, so that means $F(x)$ is 0.

So this is, you get a feeling like evaluation and a point and zero of F and so on, so this is very, very handy especially when you want to check this is a topology, right, suppose you want to check,

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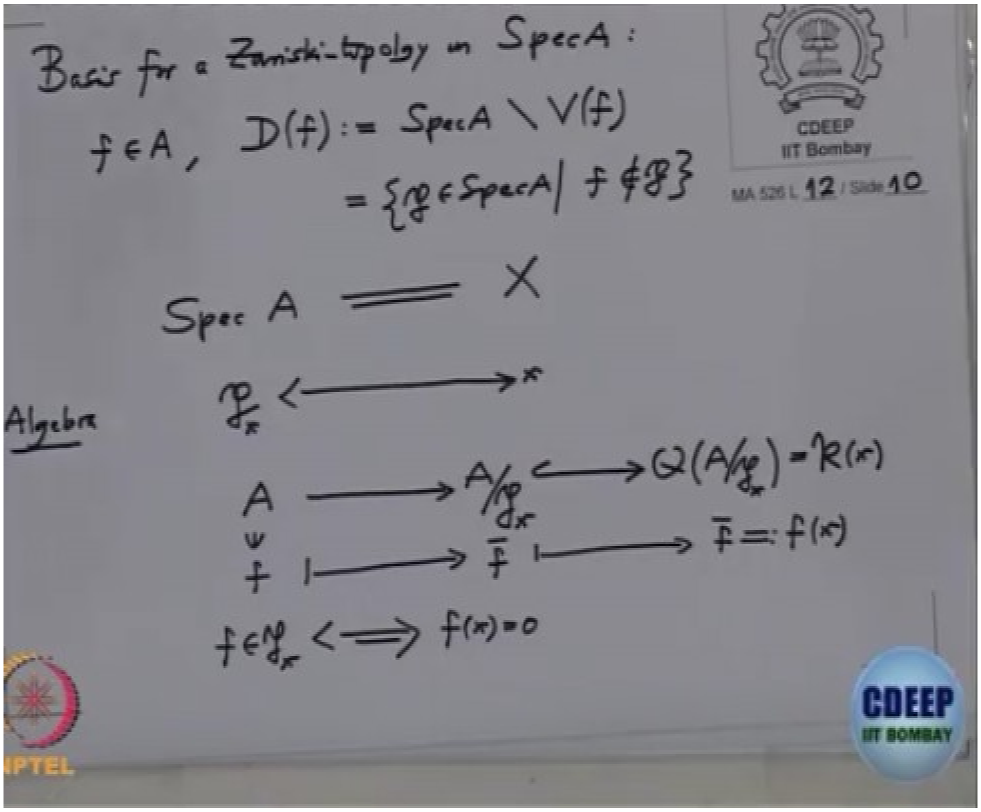

let us check one equality already so that will feel is at this, so tell me one property that $V(a)\cap V(b)=V(a+b)$, and $V(a,b)=V(a)\cup V(b)$, so let us check this first, second first for example, so that means I want to check that let us take $x\in X$, now I want to check that x belong here if and only belongs here, so we will prove that x belong to LHS, if and only if x belong to RHS,

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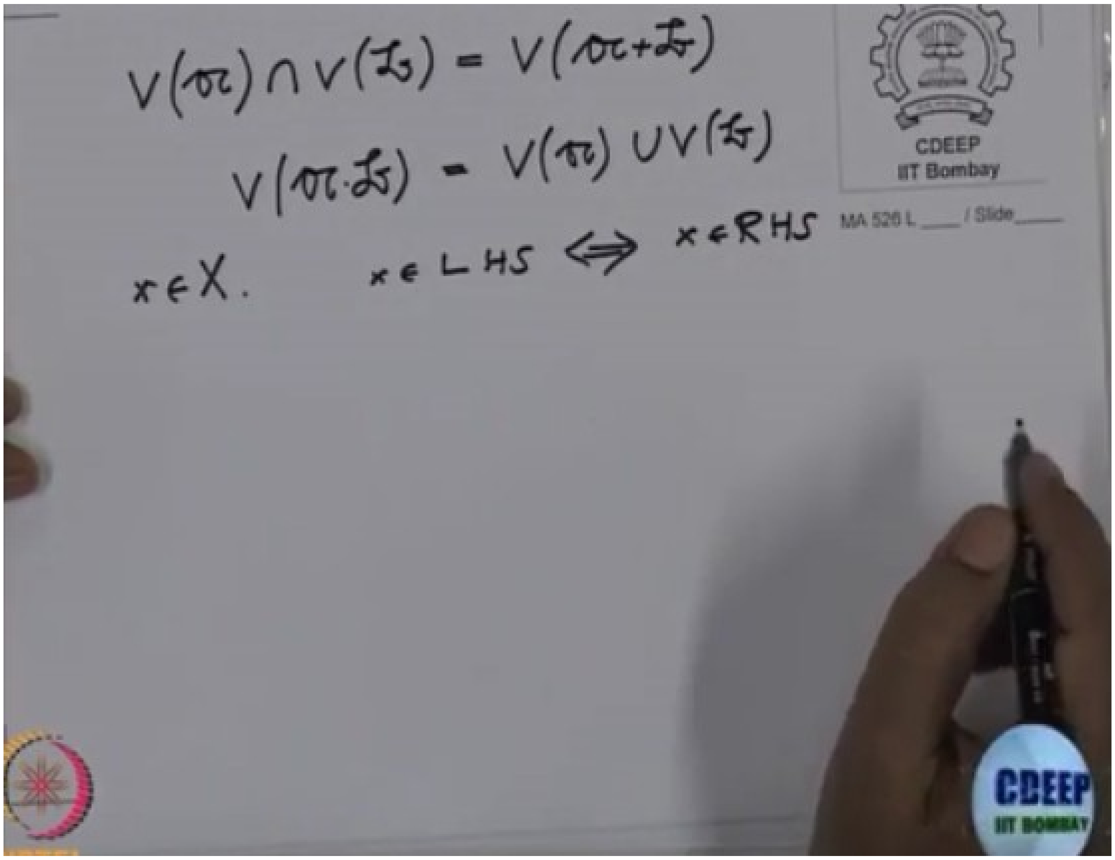

so x belong here means what? That means the prime ideal contains A times B, that means if you take all products that will vanish at that x, so this is F times G at x is 0 for all F and G in, for all F in A and G in B, but you see this, this evaluation map, this is a ring homomorphism, so whether I have product and evaluate or evaluate and product that is a same operation,

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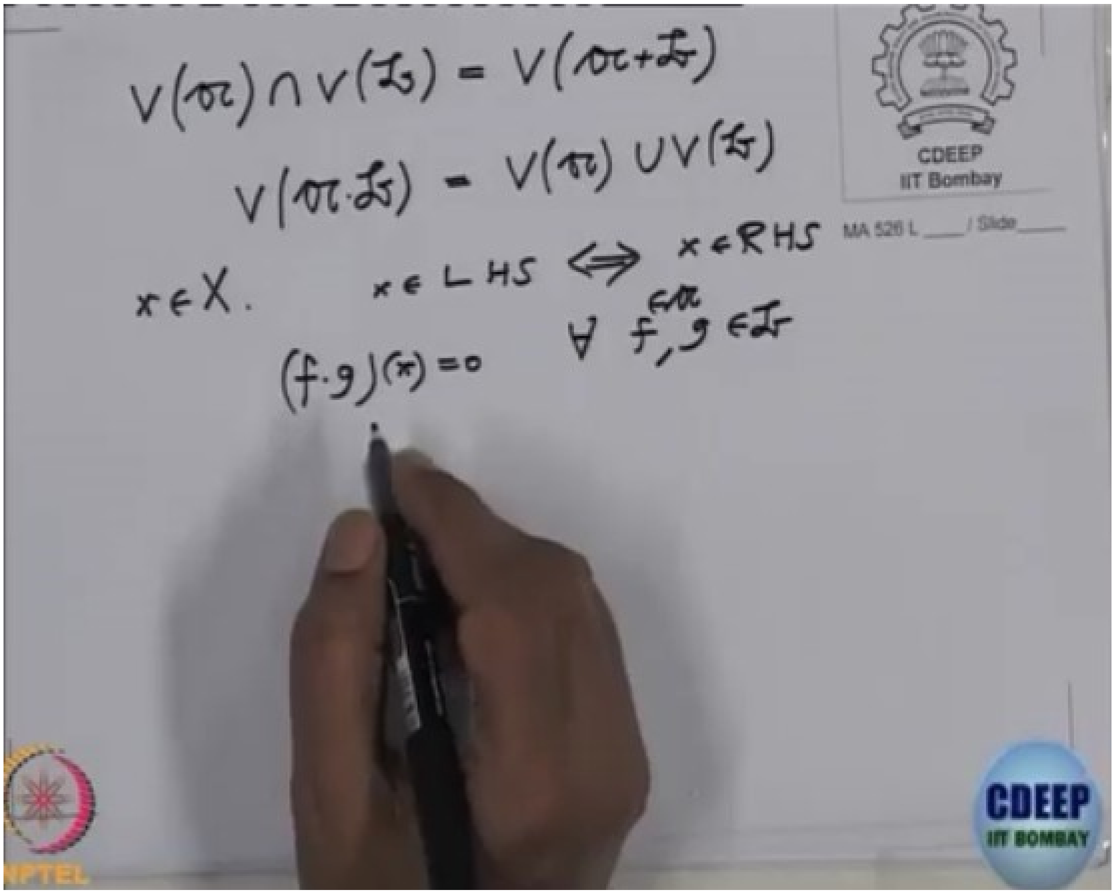

so this is same as F(x) times G(x), but F(x) and G(x) are element in the field, so that will imply F(x) is 0 or G(x) is 0, and then therefore, so this is usually how this is a quickest way to prove it, okay,

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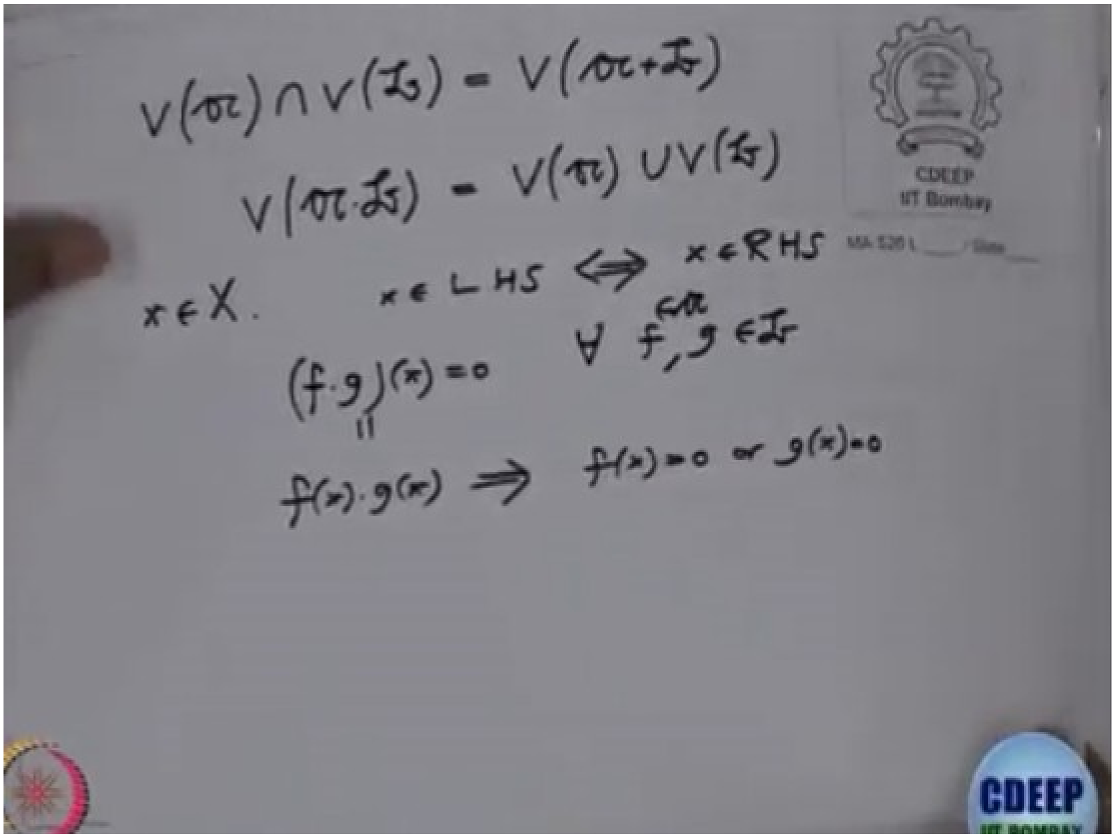

and this notation is due to Grothendieck.

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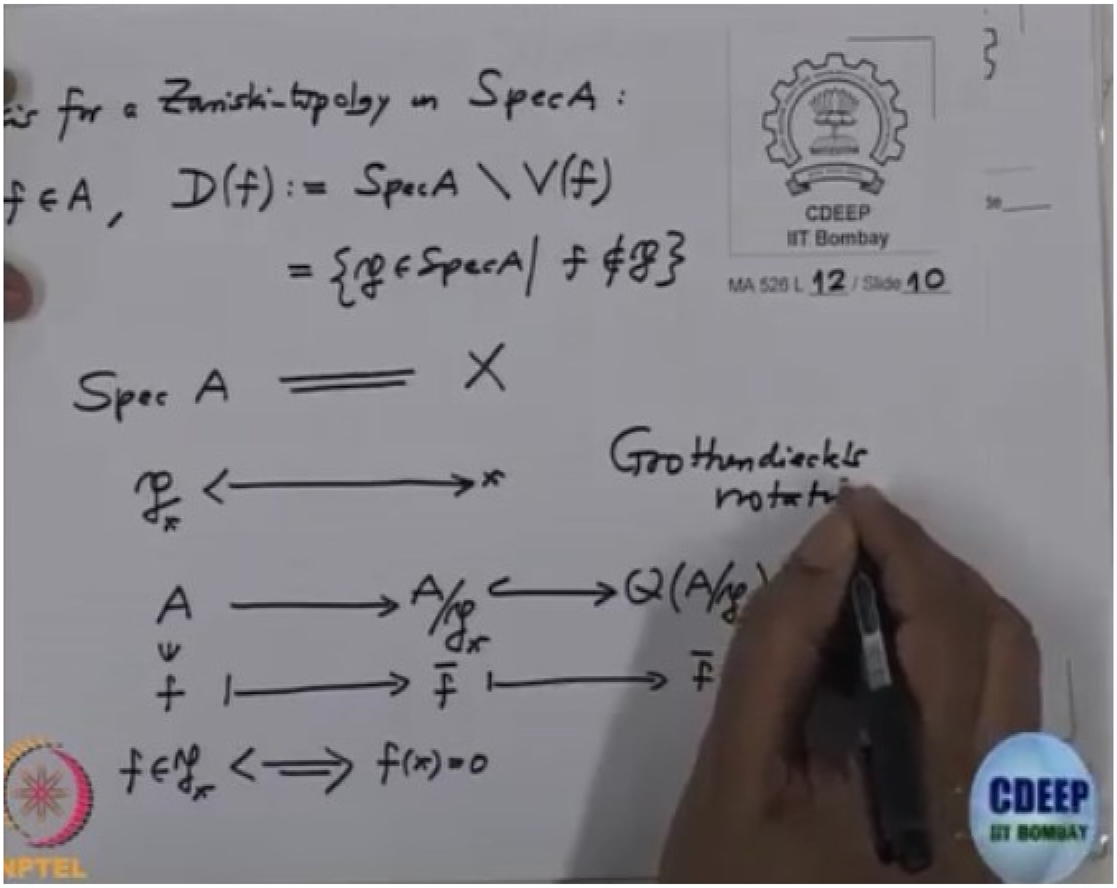

Okay so this means we have here category of rings that do you know what is a category? So that means it's a collection of objects, category means it's a collection of objects or the objects could be sets, groups, rings, vector spaces, algebras, algebra over a fixed ring or topological spaces or differentiable manifolds or any subject to studying those are the objects and the morphisms are the corresponding homomorphisms, like if you are studying groups, then group homomorphism, if you are studying ring then ring homomorphism, if you are studying differentiable manifold then the differentiable maps, if you are studying Lie group then the Lie group homomorphism and so on, so that together form a category if it satisfy certain rules like respecting composition and so on and so on, right, so category of rings on one side actually commutative rings we should say, and category of topological spaces, and we have a functor, functor means each object is associated to an object in such a way that if there is a morphism here, there is a morphism there and so on, so that data is the functor.

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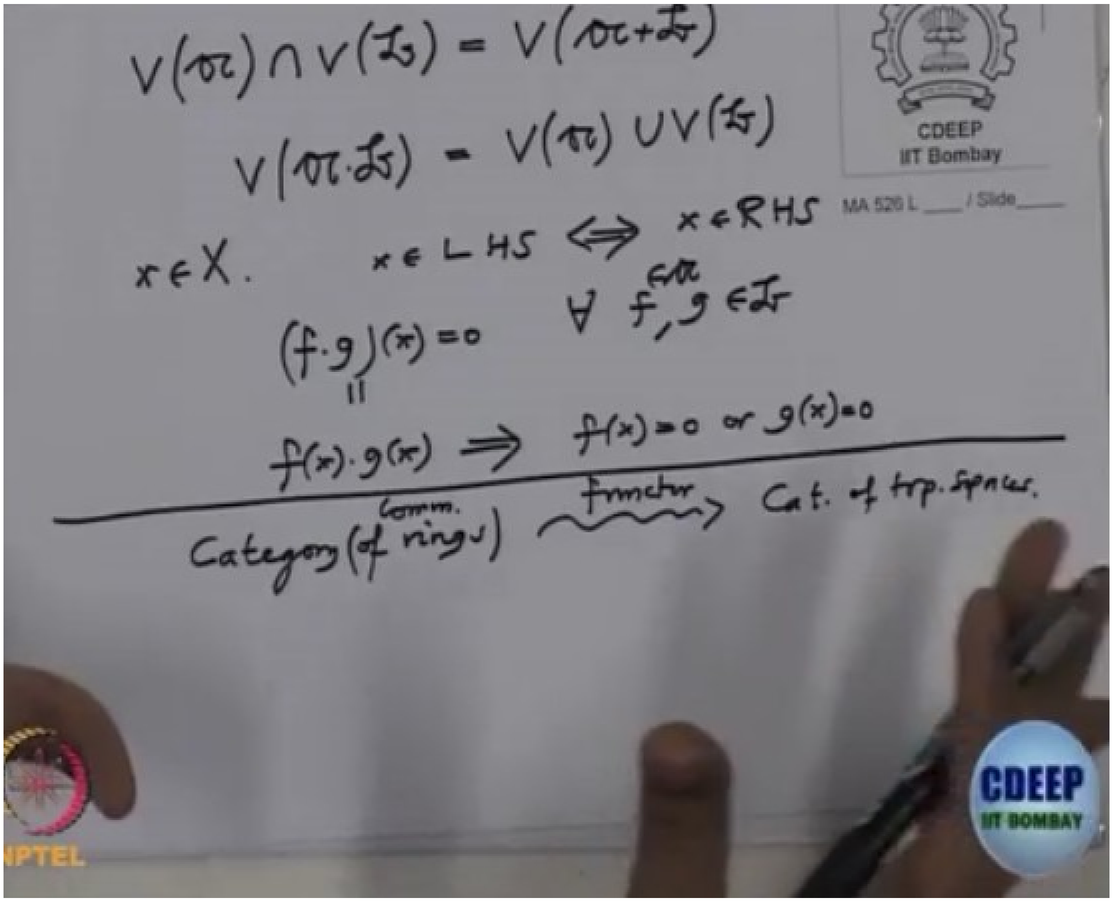

And the functor is called equivalence if it's like there is a other way direction also functor and so that each composition is identity and so on, so they are suitable definition, but that is just language, and here we have a functor namely spec, given a ring we have attached topological space to that, and what one should do is one wants to study, one wants to get information about the rings by using the topology information, and one would like to actually comeback also, that means this is a natural equivalence, so that means from this given topological space one wants to recover your ring also, and this interplay is precisely what is called abstract algebraic geometry, okay.

So this actually is not quite correct because what happens we have to put some restrictions and put some correct definition, for example if you take say very simple example $\frac{K[X]}{\langle X^2 \rangle}$, then what is a spectrum? Then spec of this, this is only one ideal, right, namely ideal generated by that small x, where small x is the image of capital X, see such problems you will also come with the notation conflicting, so one has to be the variable in capital D and small d and so on,

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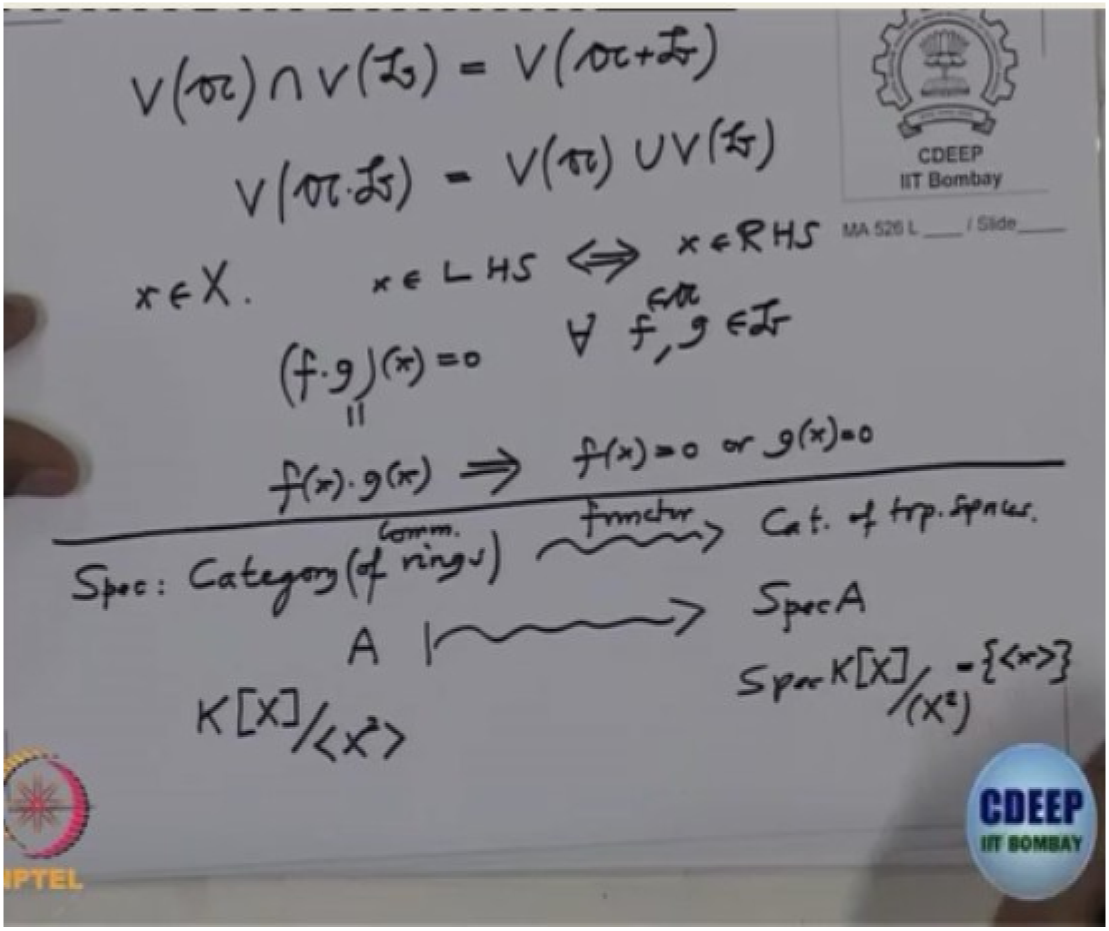

anyway, so this is an artinian ring with only one maximal ideal.

And if I would have taken field then also there is only one point namely the zero ideal, and the topology is clear, this is a discrete topology, so how are you going to recover, you get recover this or this? So there is, it's not quite correct you will have to assume the ring is reduce, then whether you can recover or not, so it's so all this, so whenever I have more and more time I'll keep adding this comments with respect to the Zariski topology and the category of rings.

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