INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

CDEEP IIT BOMBAY

COMMUTATIVE ALGEBRA:

PROF. DILIP P. PATEL
DEPARTMENT OF MATHEMATICS,
IISc Bangalore

Lecture No. – 33

Second Proof of Krull's Principal Ideal Theorem

Last time we finished the proof of the Krull's principal ideal theorem and even generalize principal ideal theorem of Krull. Today I want to give a second proof which will not use dimension theorem, so in principal one could have learned this proof much before, so what we want to prove is if A is, so this is the theorem we want to prove if A is Noetherian and P is a prime ideal in a, then and P is minimal over ideal generated by n elements, minimal means it doesn't contained, P contains a and in between there is no other prime ideal, then we want to prove that height of P is less equal to L, n = 1 this statement is called Krull's principal ideal theorem and arbitrary n it called generalize principal ideal theorem,
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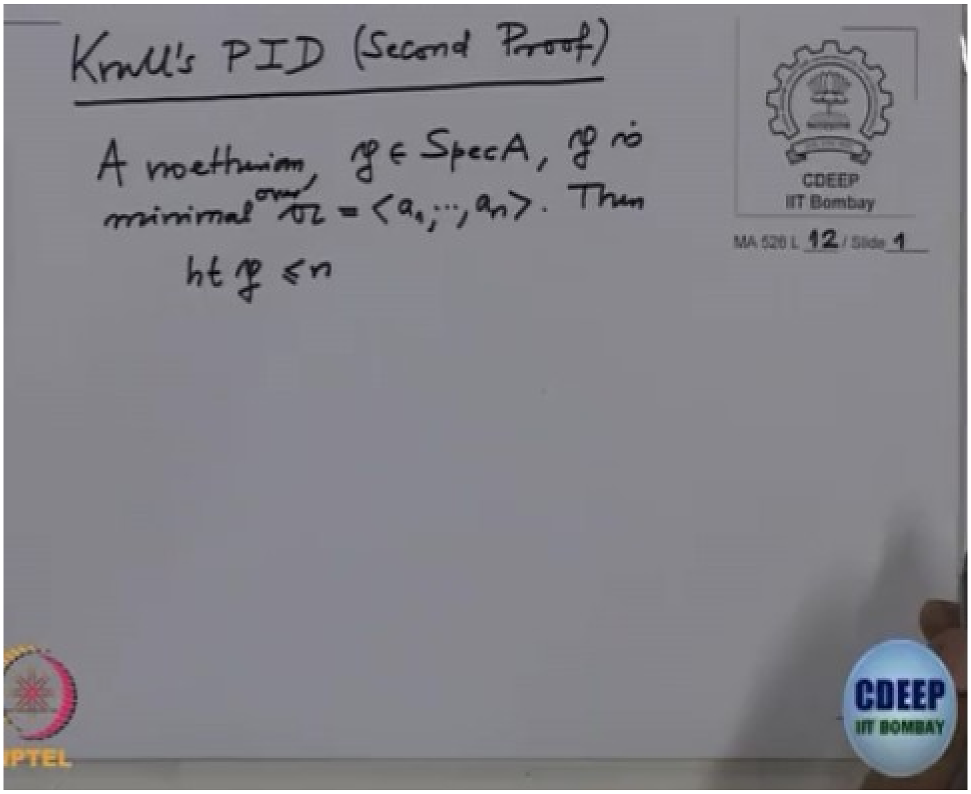

but in the literature people keep calling or writing principal ideal theorem only.

So we are going to prove as usual by, so proof as in the earlier case we may assume A is local and P is the maximal ideal of A. And again I'll prove by induction on n, proof by induction on n, so first let us take the case n = 1 we have an element a and this maximal ideal is minimal over this a, this is minimal over a or we want to prove that, to prove height of P is less equal to 1, this is what we want to prove, alright.
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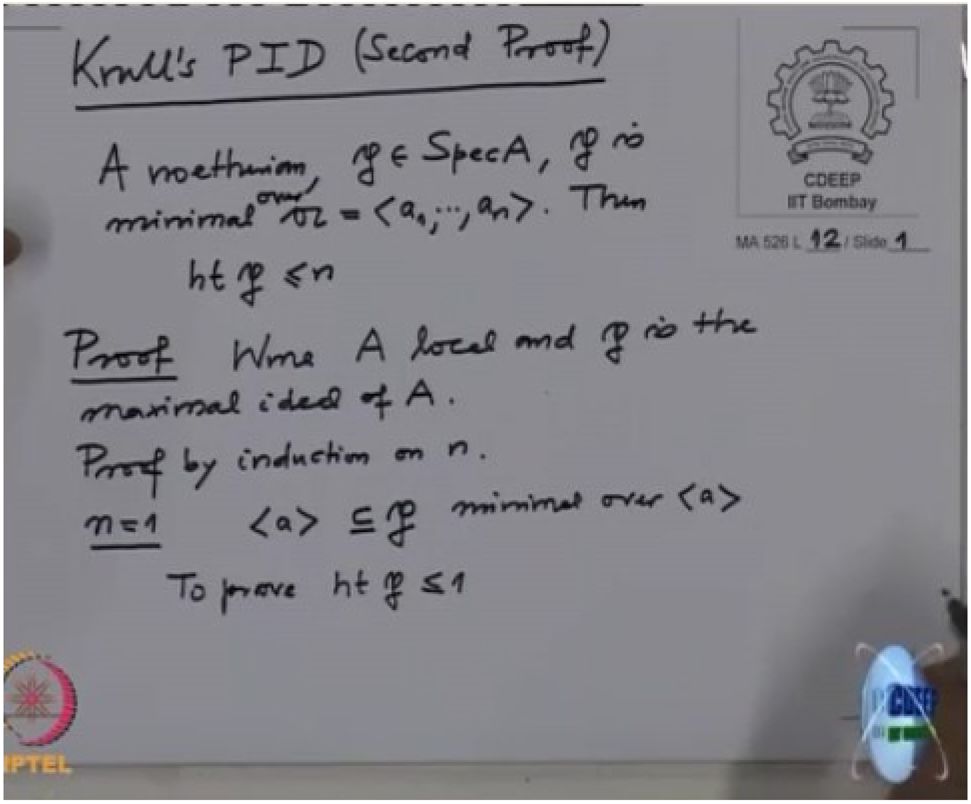

So this means, this minimal means in fact go mod a ideal generated by in this ring P the only prime ideal, so spec of $\frac{A}{\langle a \rangle}$ it's just the singleton $\frac{P}{\langle a \rangle}$, but there is no minimal prime ideal in between, there is no prime ideal in between a and P, and this is the maximal ideals of , $\frac{A}{\langle a \rangle}$ so this first of all shows that this residue class ring A by ideal generated by the Noetherian ring,
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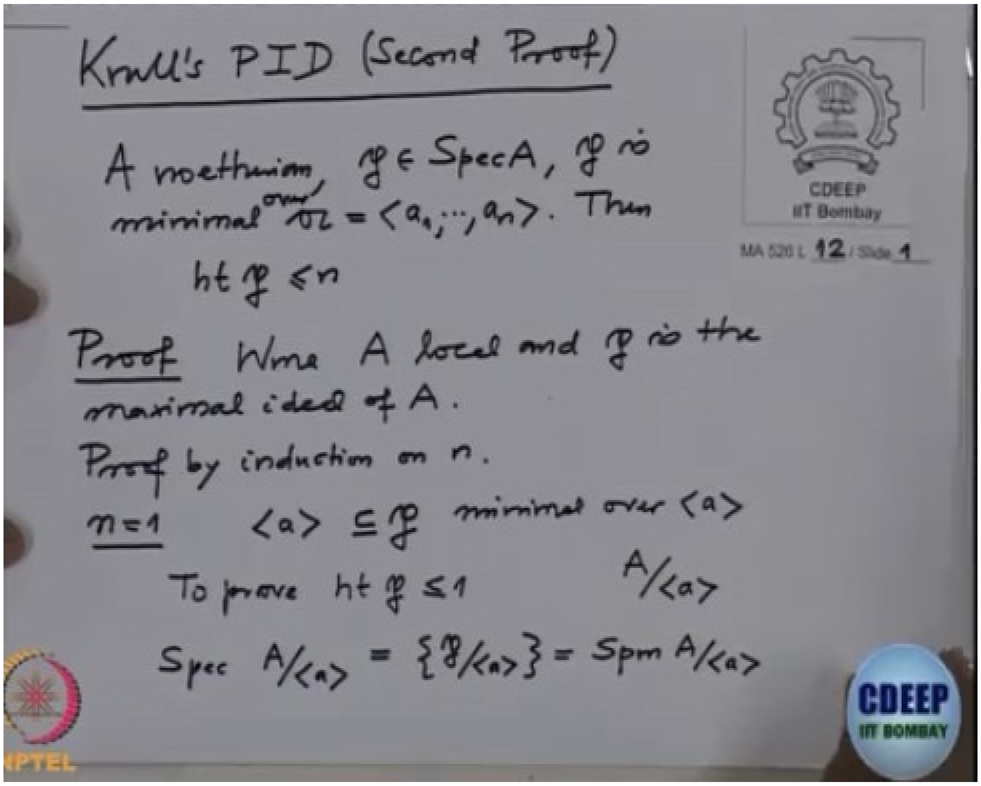

but it's Noetherian and every prime ideal is maximal therefore it is Artinian, so this ring $\frac{A}{\langle a \rangle}$ is Artinian.

And I want to use this Artinian fact to prove that now the height is less equal to 1, okay, so we will transport the Artinian property of this residue class ring to the localization at small a, A suffix a this means it is a localization of a and the multiplicative set generated by this a, $S^{-1}A$ etcetera, a^n so, it's a multiplicative set generated by A and we take the localization, what I am saying is if this ring is Artinian this localization is also Artinian, this is what we will proof first.

So look at the diagrams, we have two natural maps A is our given ring and from A to A modulo ideal generated by a this is the residue map, and to the localization also there is a natural map, this map is the natural inclusion map this will I will call it iota, this is any b goes to $b/1$, so both these are the homomorphisms, and what we want to prove? We want to prove this ring is Artinian that means what?

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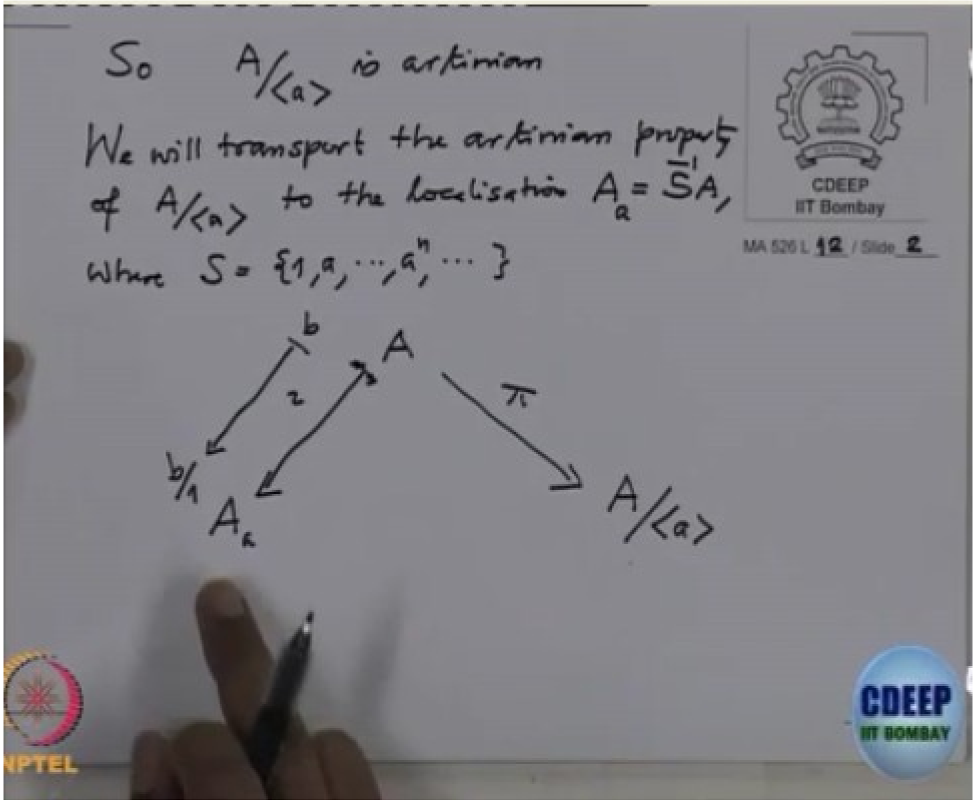

We want to prove that if I take any descending chain it should become stationary after sometime, so we will have to analyze, suppose I have two ideals in this ring a and b ideals in this ring with a contained in b, a contains b this.

So first note that if a is contained in b, if I pull then back to a this inclusion will still be valid it's inclusion preserving map on the ideals so this will still be valid, and if I go here it will still be the same, so therefore what I'm saying is if I take π, if I take $\iota^{-1}a$ and $\iota^{-1}b$, and I apply π to this, this, and π to this, this containment will be valid, okay.

And we know this ring is Artinian, so when we would have taken a descending chain, after some, after a finite state it will become a stationary, so I want to prove that same thing will happen here, it may happen before also but certainly if equality holds here then equality should hold here, you see this is what I need to prove, this simplification I need to prove, if equality holds here then the equality should hold here, that will prove this ring is Artinian, is that clear? Because when you take a descending chain here, pull it back and push it to by π, here it becomes stationary, so at that stage onwards or even before that it might become stationary, in any case this will prove that this is Artinian.
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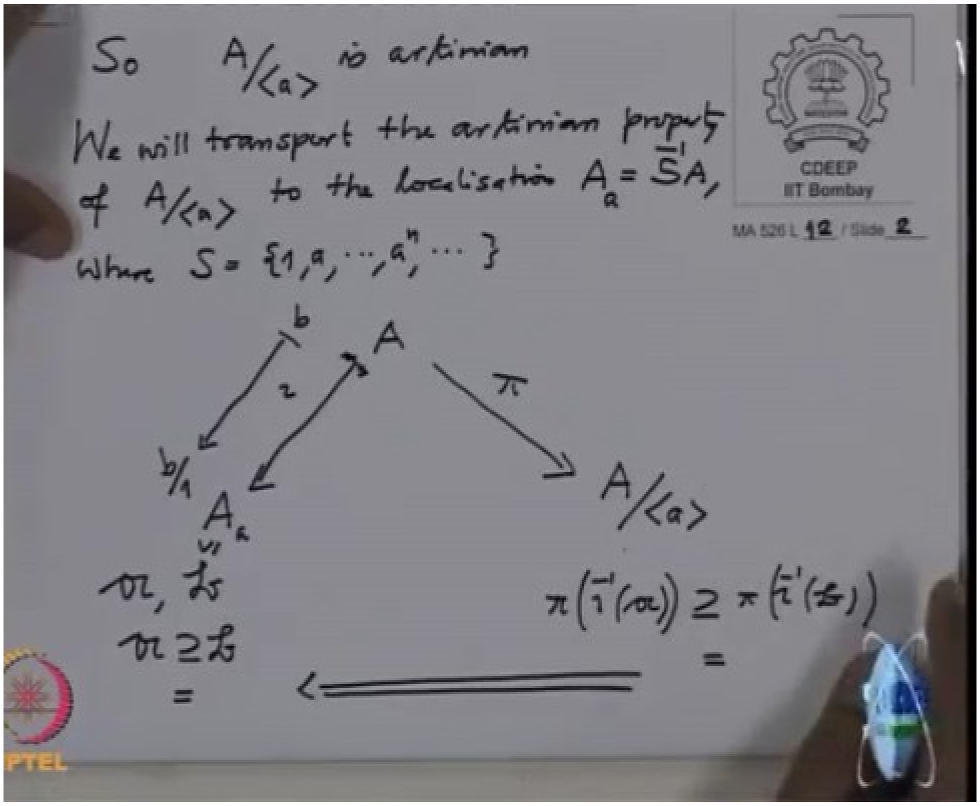

So we will start with the assumption now a and b two ideals in localization, and if it equal happens here then I'll prove the equality holds here, so therefore suppose I have two ideals b contained in a, contained in A localization at a such that $\pi(\iota^{-1}a)$ equal to , $\pi(\iota^{-1}b)$ and from here I want to prove a = b this is what we need to prove, so start with an element, so what I will prove? I want to prove that this, okay, we will see what, I want to prove that actually they are equal, so we will actually prove $\pi(\iota^{-1}a) = \pi(\iota^{-1}b)$, if I prove this then you know localization this map is very well, when we have a ideal in localization when you pull it back and push it back again to the localization you get back the equality,

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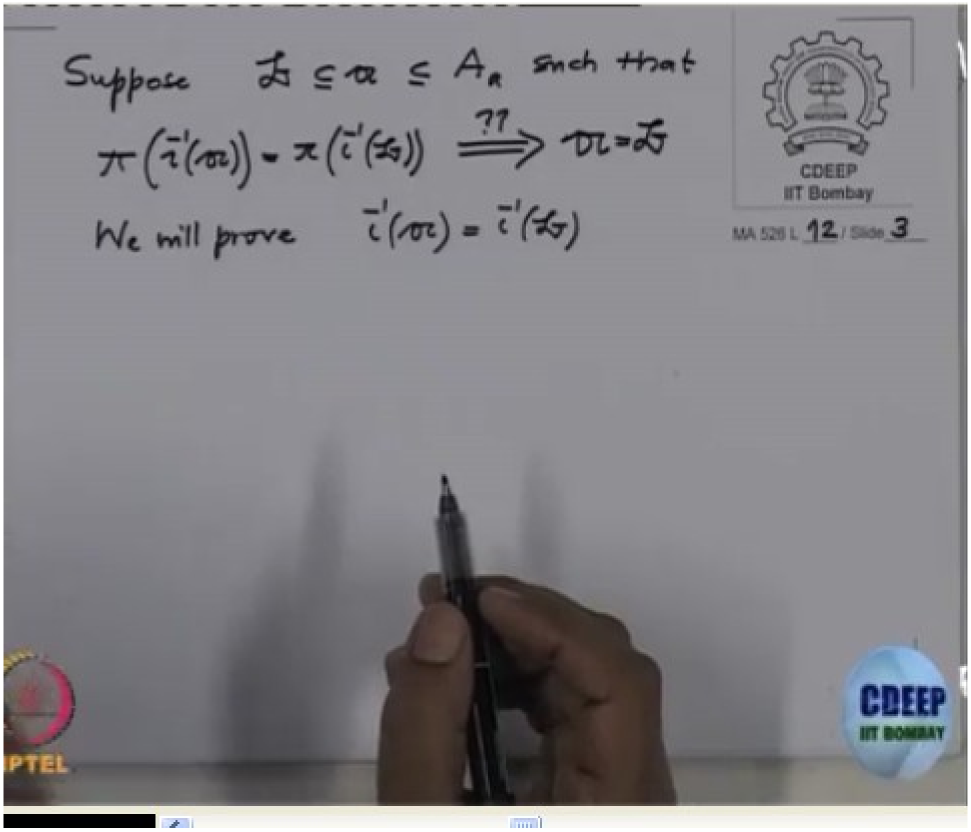

so once I prove this that is enough that will imply then a will be equal to b.

So I want to prove that and obviously one is contained in the other, this is obvious, this is given to us, so I will consider the quotient module, so we will consider V= $\frac{\pi(\iota^{-1}a)}{\pi(\iota^{-1}b)}$ and we are in a local rings, so I want to use lemma to conclude Nakayama lemma to conclude V is U, that is the plan, is that clear? Okay so this is what I will do so, that means I want to prove actually, I want to prove that V = PV, V is finitely generated module that is clear because all these ideals are Noetherian etcetera, this is finitely generated, P is the maximal ideal of the ring. So if I prove V = PV that will imply V is 0, and we will be done, okay, so we want to prove this is 0.

So start with an element in the numerator here, so let x belong to $\pi(\iota^{-1}a)$, so that means it is here π that is here, $\pi(x)$ is here, so and this equality is given, so that $\pi(x)$ will be π of somebody that means x and that somebody will defer by kernel of π which is the ideal generated by A, so given x there exists y where y in this such that x and y they will defer by an element of the kernel of π , what kernel of π is a principal ideal generated by a, so that means x-y will be a multiple of a, so this means it is +az for some z, z in the ring A, clear.

So now this implies, this is in, x is in $\pi(\iota^{-1}a)$, y is in $\pi(\iota^{-1}b)$, but $\pi(\iota^{-1}b)$ is contained in $\pi(\iota^{-1}a)$, so both are in x and y both are there, so this will be there, so az will also be $\pi(\iota^{-1}a)$, but this a is a unit,
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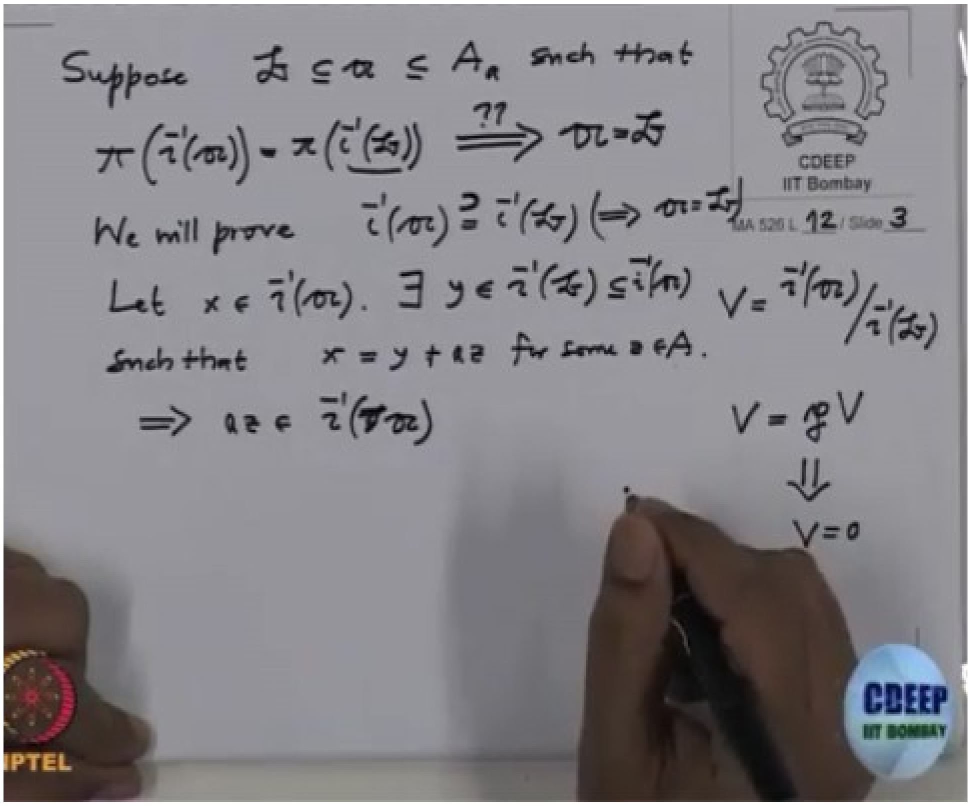

but since a is a unit in this localized ring that will imply z belongs to $\pi(\iota^{-1}a)$, because we cannot adjust by the unit so it belongs to $\pi(\iota^{-1}a)$, so what did we prove? Starting with x in $\pi(\iota^{-1}a)$ we proved that x belongs to, that is y, that is $\pi(\iota^{-1}b)$ + ideal generated by a this a, and this z $\pi(\iota^{-1}a)$.

We proved that x = a+az, but this y is in I inverse, $\pi(\iota^{-1}b)$ and this z is in $\pi(\iota^{-1}a)$ that's what we noted,

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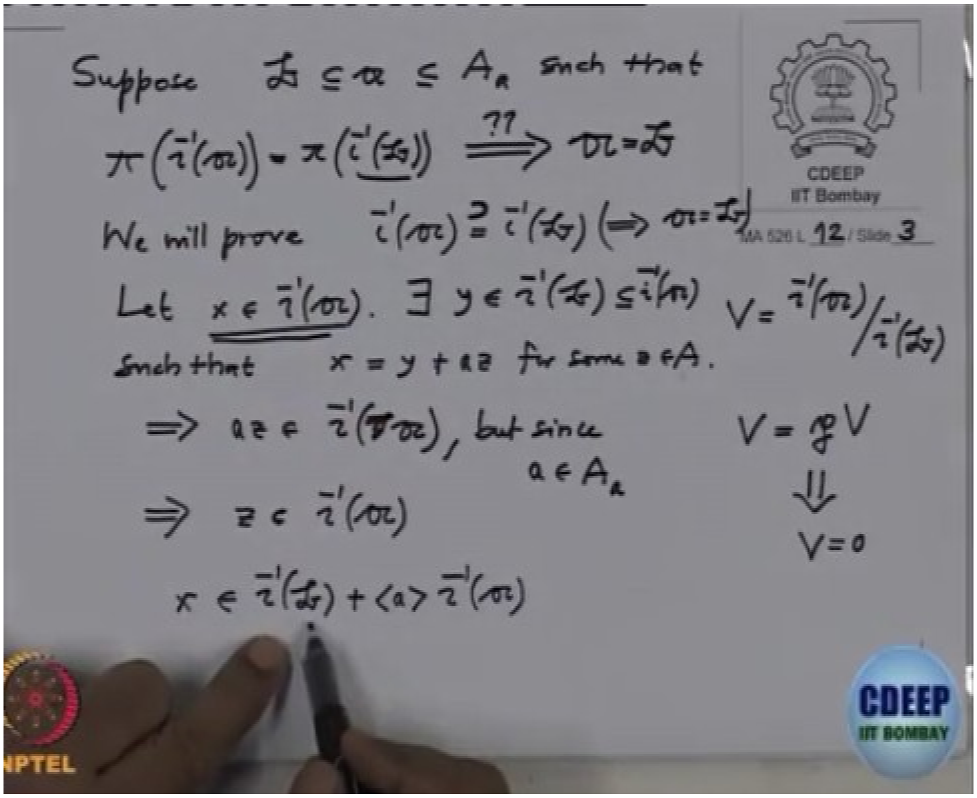

so therefore x will belong to $\pi(\iota^{-1}b)$ + the ideal generated by A times $\pi(\iota^{-1}a)$, but that will mean that is the module, this module V is contained in, when I go mod this, so every element x of this, so when you read this mod $\pi(\iota^{-1}b)$, V is contained in ideal generated by a times V, okay if this, look at this and go mod $\pi(\iota^{-1}b)$, so this will disappear so every element of $\frac{\pi(\iota^{-1}a)}{\pi(\iota^{-1}b)}$ will be ideal generated by a times, this mod $\pi(\iota^{-1}b)$, so this and this is obviously because a is non-unit this is contained in PV, but this is contained in V, so all the time equality happens,

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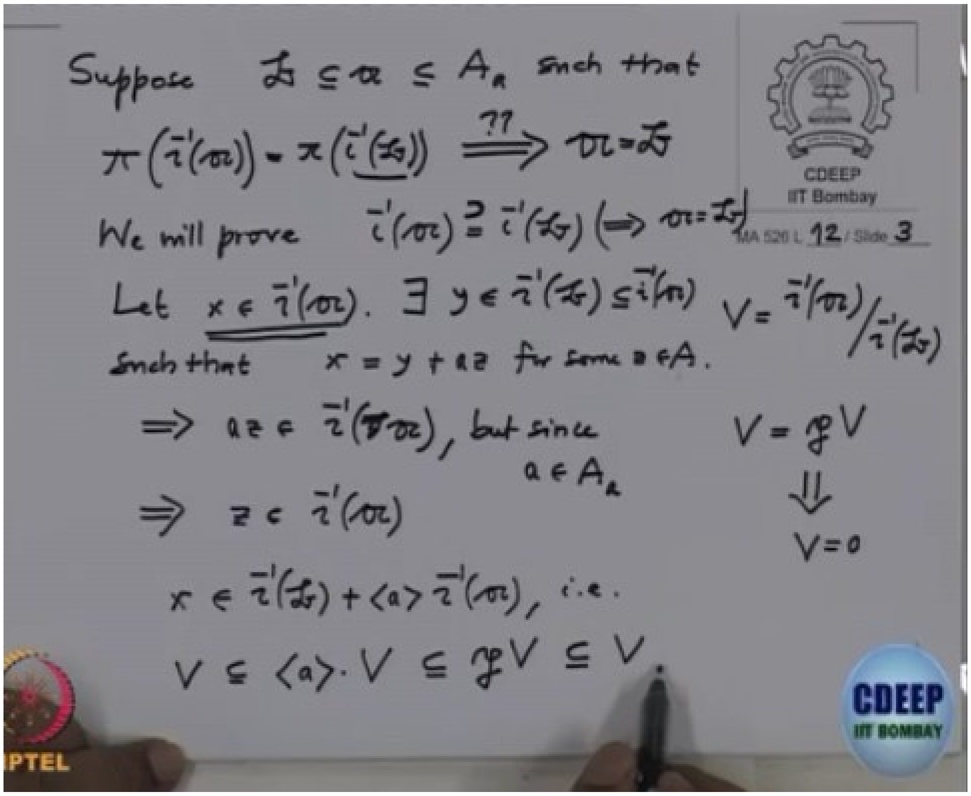

so therefore we proved that V = PV. And now use Nakayama lemma to say that V is 0 and then everything is done, so this proof is easy but tricky, right, okay.

So we have proved that a is, A localized at a is Artinian, so in particular Artinian, so see note that it could be, we have, I'm not saying that this $\pi(\iota^{-1}a)$ is a nonzero ring, it could be zero ring also, because we have no evidence for that concluding it is zero, so to conclude it is nonzero we need to check that a is not nilpotent, okay.

So in any case we have checked that we proved A localized at a is Artinian, in particular the dimension, dimension of A localized at a is less equal to 0, if it is nonzero ring then it will be dimension will be 0, if it is a zero ring by definition of a dimension it is -1, and therefore this, so but I want to rule out the possible, I want to claim that dimension is actually 0, so note that dimension of A localized at a is actually 0, because let Q be a prime ideal in A such that with Q is contained in P, so P is a prime ideal, given prime ideal and I'm saying just let Q be a prime ideal which is contained in P, maybe Q equal to P also, but take such a prime ideal, but we know because P is minimal over the ideal generated by A is contained these, and these is minimal so this Q cannot contain A, so A is not in Q, that means so this means so that is, A localize at Q is a localization of A localized at a, yes yes, so we want to prove. So this will also follow on the way, so let us start with Q contained in P, so what, which contains A then A cannot be in Q, what is your question?

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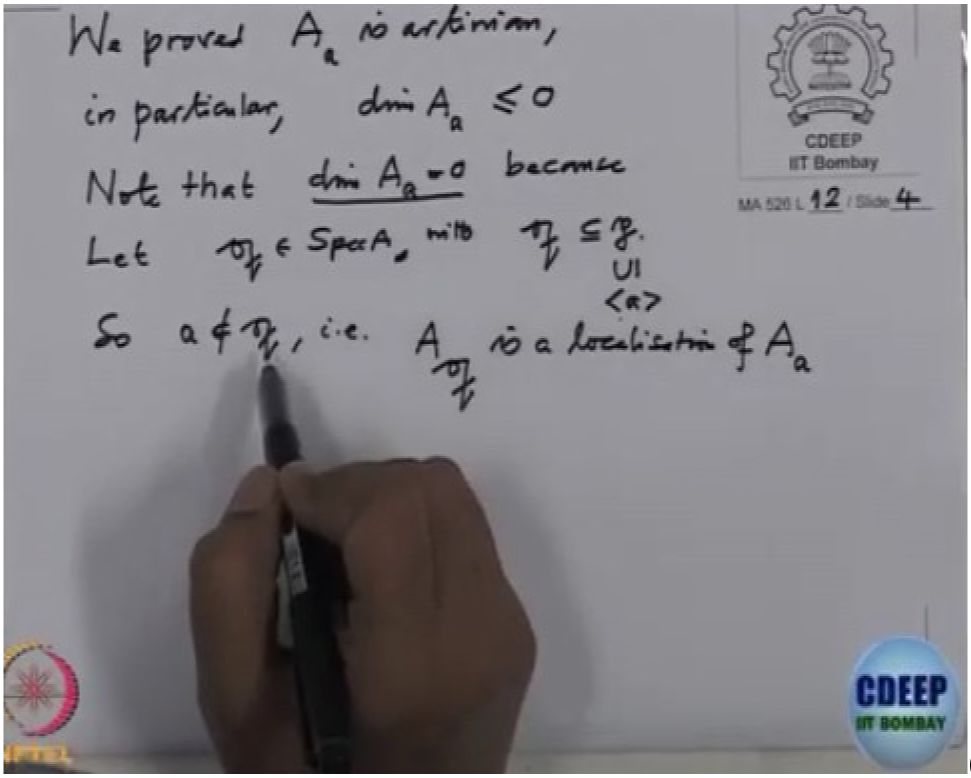

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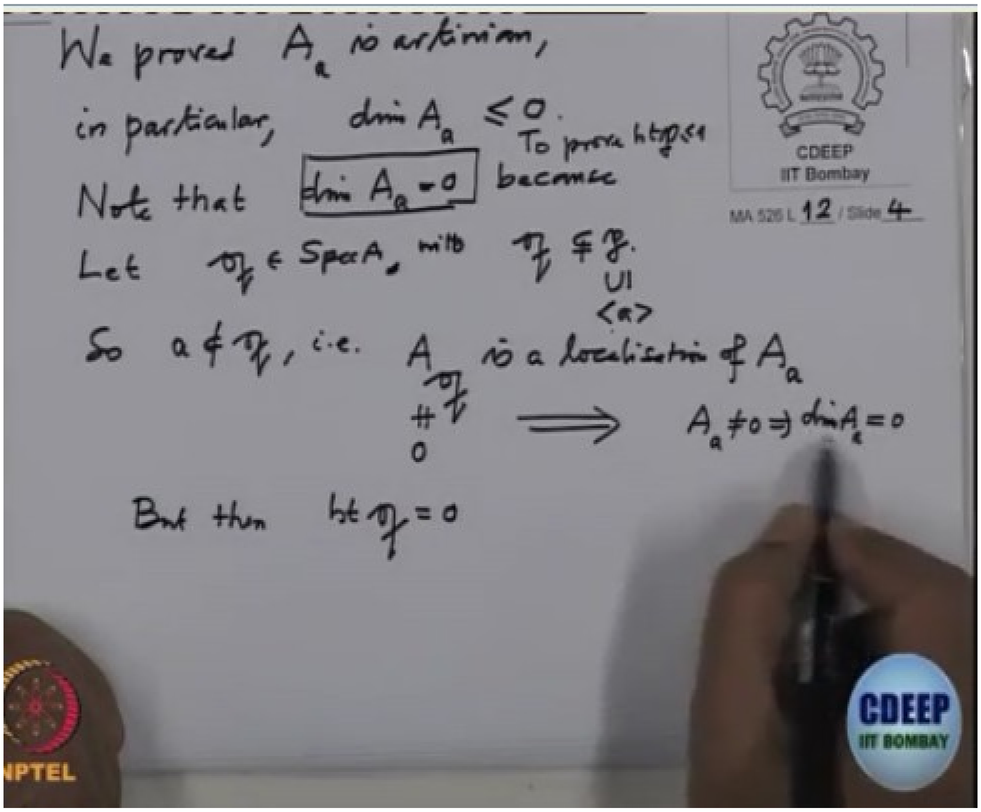
Yes, but that, we want to prove that height of P is less equal to 1, right.

So suppose not, so this will also follow this fact will also follow afterwards, immediately after this, so we want to prove, so to prove height of P is less equal to 1, so let's assume the contrary that means there is a prime ideal in Q which is properly contained in P, then A will not be in Q and, so when you localize at A means you are inverting the powers of A, and when you are inverting at Q means you are inverting all elements outside Q, so that means you are inverting all powers of A and some more, so that means this is a localization of A localize at small a, but this is already nonzero ring, because it's a localization at a prime ideal Q, so this is already nonzero, and therefore this cannot be zero, and because it is a localization of, the localization of this ring is nonzero, so the original ring cannot be zero, so that will imply A localize at small a is nonzero, and the dimension of $\frac{A}{\langle a \rangle}$ is 0, is that clear?

Still I've not completed the contradiction to this height P is not less equal to 1, we have assume that, we want to prove height of P is less equal to 1, so if there is a prime ideal which is properly contained in P then we have concluded that dimensions of this A localized at a is 0, but then the height of Q will be 0, because this is a localization of this ring and this ring has a dimension 0, (Refer Slide Time: 22:28)



therefore this will also have dimension 0, no, that means the height of this Q is 0, so what did we prove? We prove that assuming that any, we started with any prime ideal which is properly contained in P and check that this Q has edge 0, so that will imply height of P is a less equal to 1,

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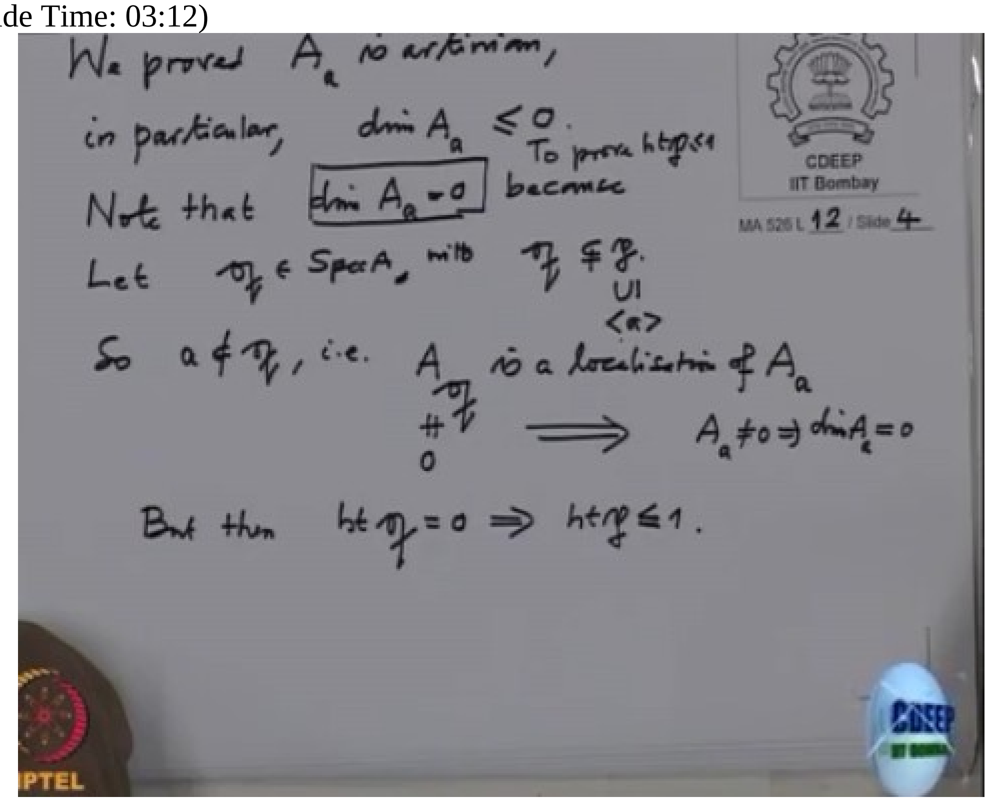

because we took any prime ideal Q which is properly contained in P, so that this now proves the theorem for n = 1.

And now we have two induct, so note that I have not used dimension theorem here, okay. Okay now induction hypothesis so now general n, now any n, n is big or equal to 1 arbitrary, okay. So let, I want to induct so I have to create a situation where somebody, some prime ideal is a minimal over ideal generated by n-1 elements, and n conclude that the height of that is less equal to n-1 by induction hypothesis and tie it up.

So let Q be a prime ideal which is properly contained in P, and no prime ideal in between with no prime ideal in between saturated chain of prime ideal, okay, so and we remember assumption that P is minimal over ideal generated by $a_1, \ldots, a_n$, so obviously all this a_i 's cannot be contained in Q because otherwise P will not be minimal over that, so then we may assume without any loss that one of them is not in Q, let's renumber them and call it a_1 , so a_1 is not in Q.

So look at now the ideal generated by this Q and this a_1 , $Q+a_1$ this is contained in P, because Q is contained in P, a_1 is contained in P, and residue is, this P is minimal over this ideal because if there is a prime ideal in between that prime ideal will be in between Q and P, so this is P minimal over $Q+a_1$, alright, but you remember this P is a maximal ideal in a ring, in a local ring, and this maximal ideal is minimal over this, so what will be the radical of this ideal? So therefore radical of $Q+a_1$, this is precisely the intersection of all prime ideals that it contains, but there is only one prime ideal it contains namely P, because if there is some other prime ideal then that P will not be minimal over this,
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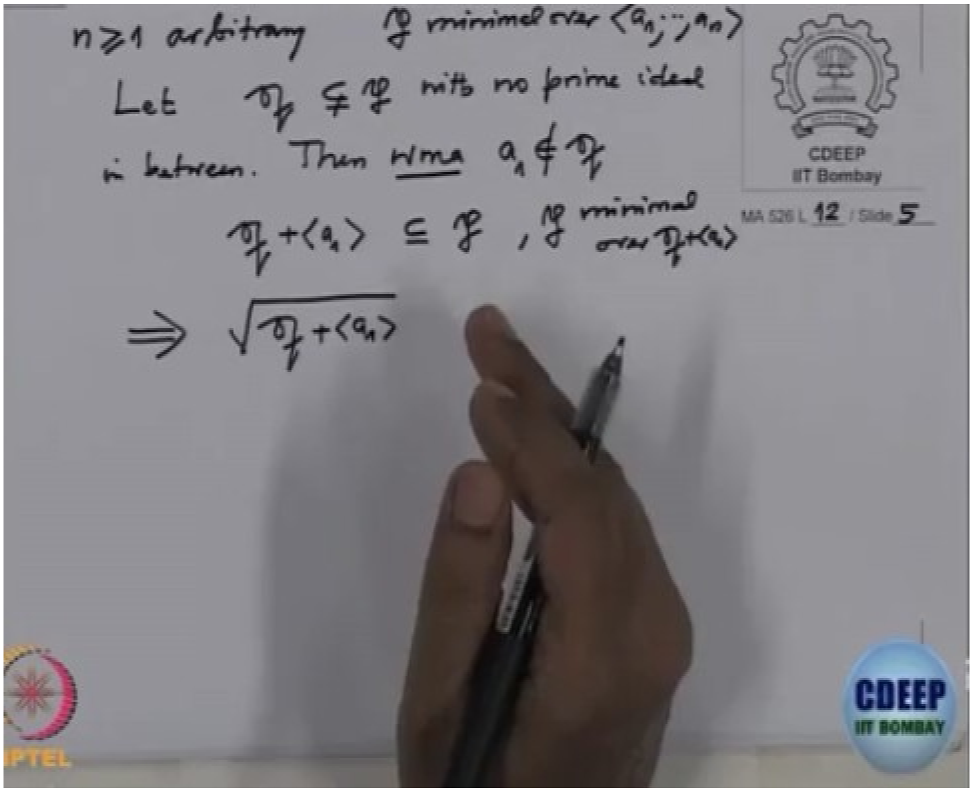

so this radical is precisely P, but this radical is precisely P means that every element here is here, so every element of this P some power of that would belong to this ideal, and all this a_i 's are elements here, so some power of a_i s will be in this ideal so that implies $a_i^{r_i}$ will belong to Q + ideal generated by a_1 , for every i= 2 to n, that means we can write this element $a_i^{r_i}$ as some of these elements, some of the element from Q, and some from this ideal, so that will be of the form, let me call here b_i + some multiple of this a_1 that is c_i a_1 where b_i belongs to Q, c_i belong to the ring and I should have said here r_i is big or equal to 1 integer, alright.

So when I shift now, okay, so therefore I say, if I look at the ideal generated by $a_{1,} a_2^{r_2},\dots,a_n^{r_n}$, this ideal and ideal generated by a_1 , $b_2,\dots,b_n$, both this ideals are same,
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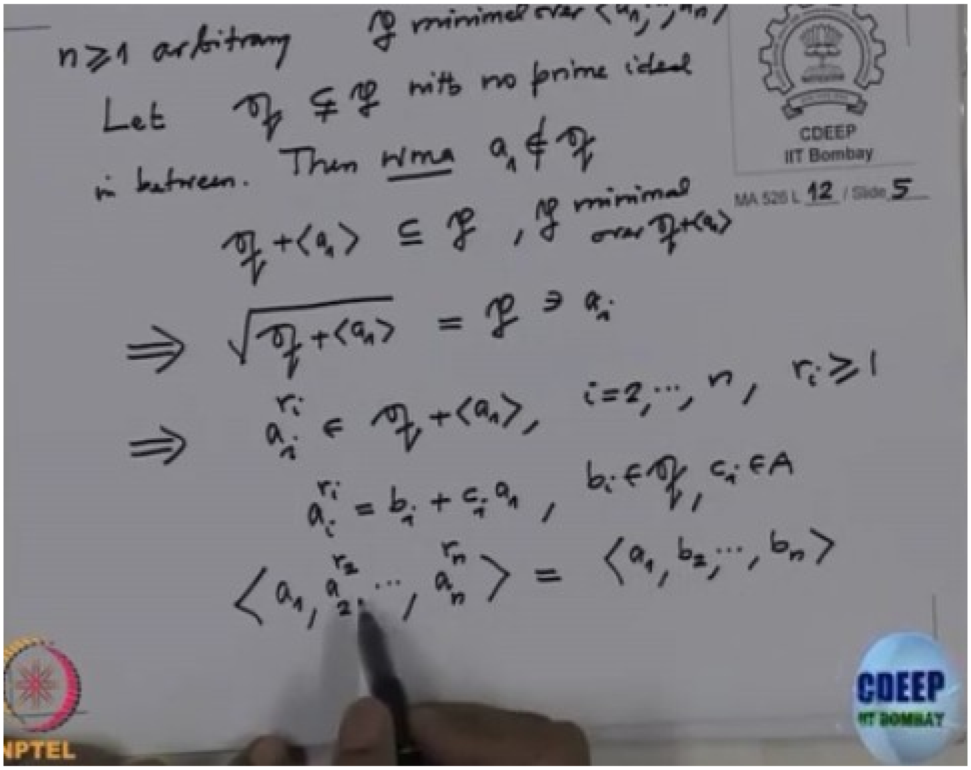

so to check that you see here this is, this one for example it’s, if a_1 is there b_2 is there, so therefore this two are equal ideals, is that clear? See I want to check this equality and the other equality, so I have to check all b_i ’s are here, but b_i if I shift this to the other side all b_i ’s are here, similarly the other way, so therefore this equality.

And P was minimal over a_1 to a_n therefore P will be minimal over this also,

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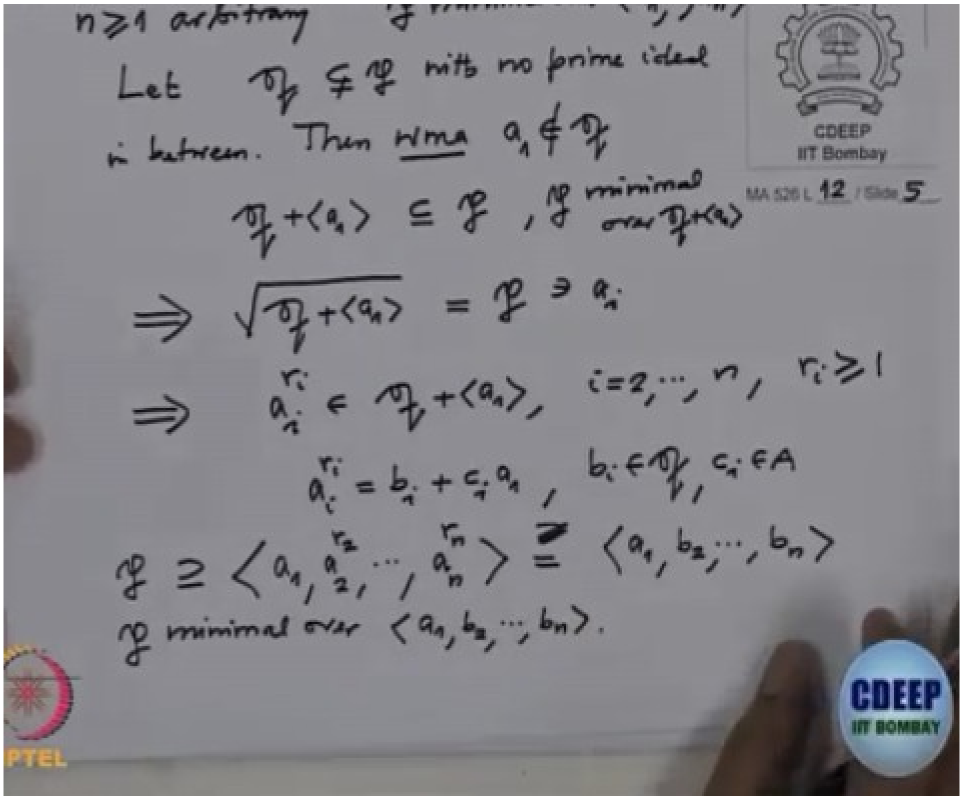

so P contain this and this P is minimal over this, and therefore this, is that okay? You see P contain this, and if there is a prime ideal in between that will contain $a_1, a_2, \dots, a_n$ because that is a prime ideal, okay.

Now once P is minimal over this I will go mod, if I go mod then the, so then P modulo ideal generated by $b_2, \dots, b_n$, this is minimal over ideal generated by a_1, $b_1, \dots, b_n$ modulo $b_2, \dots, b_n$, then B1 to $b_2, \dots, b_n$. In the residue class ring, in the ring $\frac{A}{\langle b_2 \rangle}$ ideal generated by $b_2, \dots, b_n$ this is a prime ideal, this is an ideal and this is contained in this, and it is also minimal over, because if it is in between then you lift the prime ideal, and then that will be in between.

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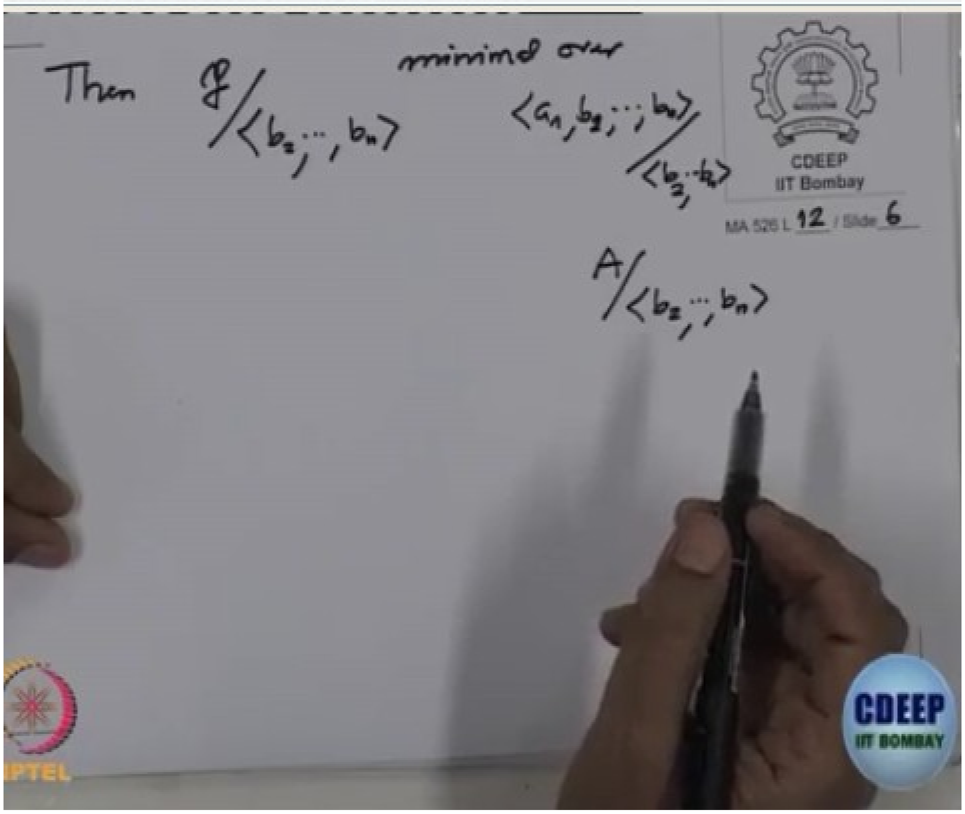

So now in this ring this is a principal ideal and this is a minimal prime over that ideal, therefore by the case 1 height of this, so n = 1 case, height of $\frac{P}{\langle b_2,\ldots,b_n\rangle}$ is less equal to 1, but then now look at, this one contains all this the advantage is now this B's are in Q, the one we started with, Q as a prime ideal so remember Q was chosen with this property, Q is contained in P and there is nobody in between, this was Q. And then we have chosen this b is in Q, so therefore this contains, P contains $\frac{Q}{\langle b_2,\ldots,b_n\rangle}$.

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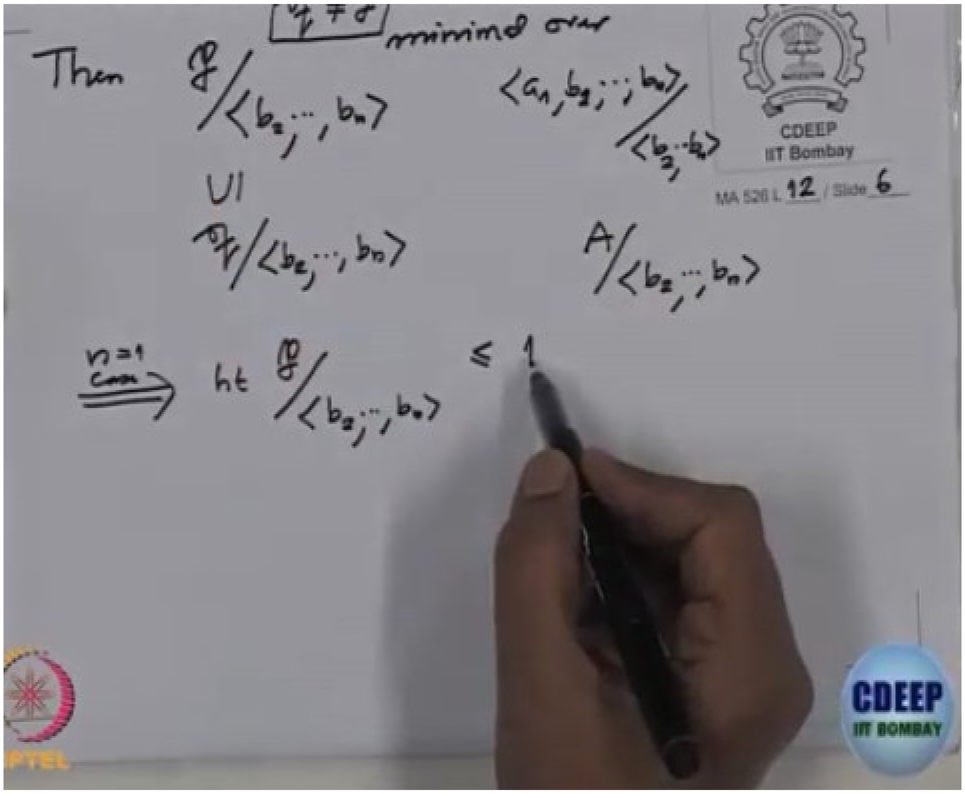

And in the residue class ring height of this prime ideal is less equal to 1, so therefore height of this and this also is not equal, because if it is equal here, lift it will be equal there also, Q equal o P then, you see we are using fact again and again, if I go to residue class ring then there is a inclusion preserving bijection between the ideals of the original ring which contain that the ideal modulo we are going and residue class ring, this is what we are again and again using it, and maximal ideal under that bijection maximal ideals will correspond to maximal ideal, prime ideal will correspond to prime ideal, and containment will always be there, so if the height of P is less equal to 1 then height of $\frac{Q}{\langle b_2,\ldots,b_n\rangle}$, this has to be 0, but that will mean, that will mean Q is minimal over ideal generated by $b_2,\ldots,b_n$, right,

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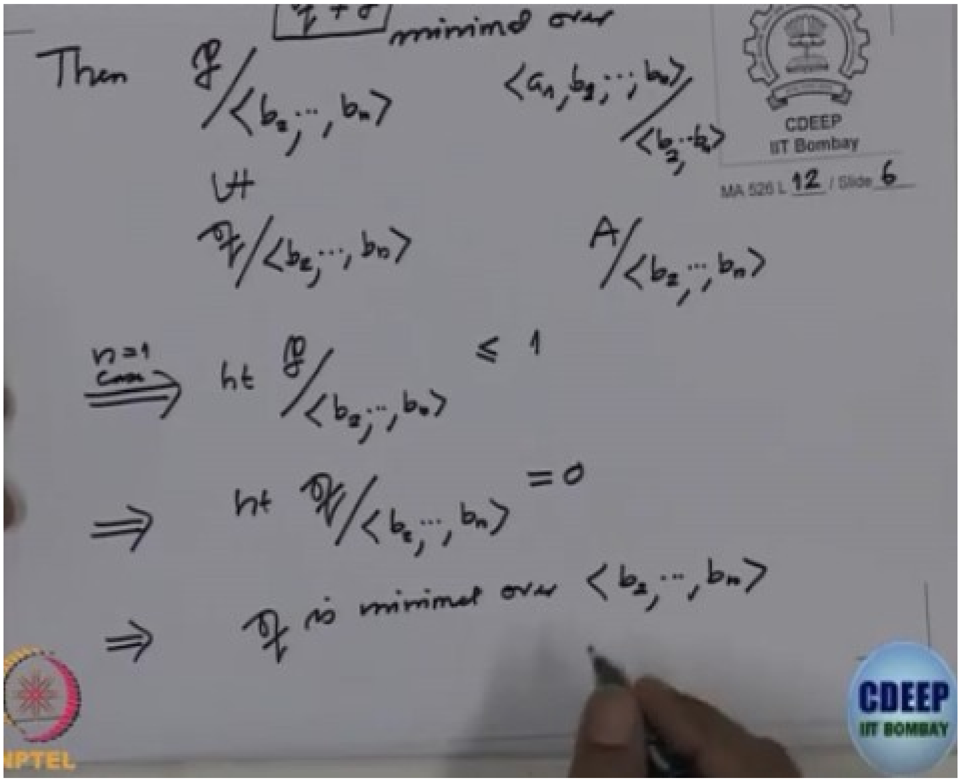

it has to be minimal over because if it is not minimal, the height of this will not be 0, it will be at least 1.

If it is minimal over 0, now induction, now the case n-1 case that height of Q will be less equal to n-1 by induction, induction hypothesis, therefore height of P will be less equal to n, so that proves, so this completes the proof.

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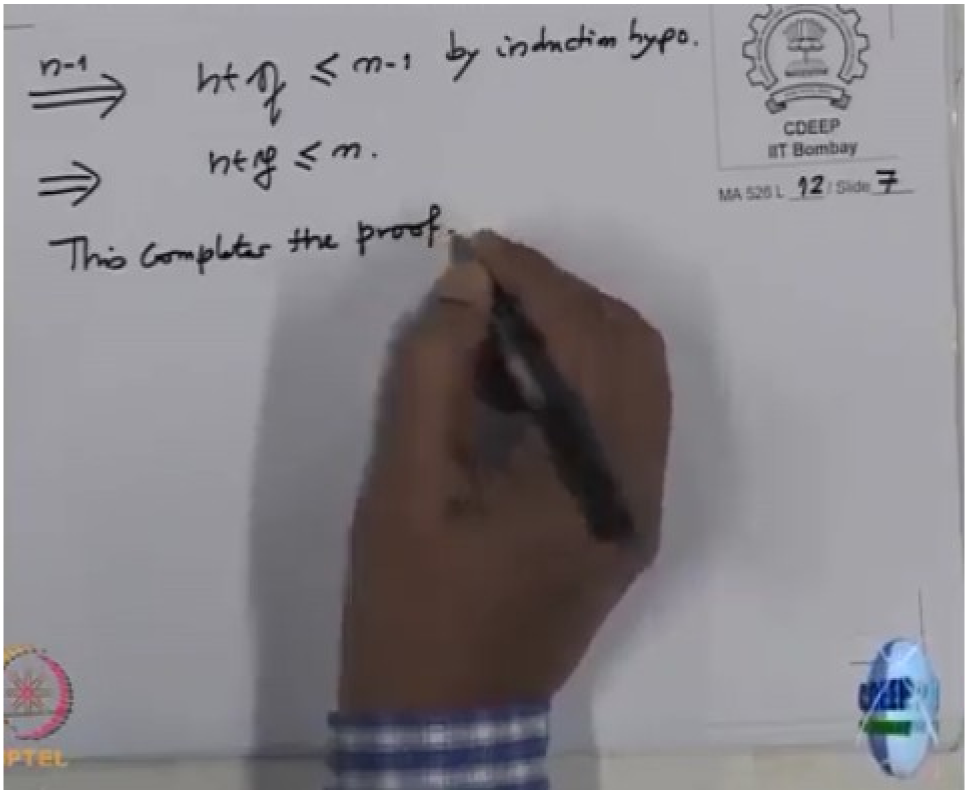

Prof. Sridhar Iyer

NPTEL Principal Investigator
&
Head CDEEP, IIT Bombay

Tushar R. Deshpande
Sr. Project Technical Assistant

Amin B. Shaikh
Sr. Project Technical Assistant

Vijay A. Kedare
Project Technical Assistant

Ravi. D Paswan
Project Attendant

Teaching Assistants

Dr. Anuradha Garge

Dr. Palash Dey

Sagar Sawant

Vinit Nair

Pranjal Warade

Bharati Sakpal
Project Manager

Bharati Sarang
Project Research Associate

Riya Surange
Project Research Assistant

Nisha Thakur
Sr. Project Technical Assistant

Project Assistant
Vinayak Raut