Lecture - 32

Generalized Krull's

Principal Ideal Theorem

Gyanam paramam dhyeyam. Knowledge is supreme.

Dilip P. Patil: Okay, now let me state the theorem. This is what we want to prove. So, theorem. This is, this is called Krull's generalized principal ideal theorem. Krull's, he did it first. Krull's did it and proved first for the principal ideals and then later for obituary ideal. So, but somehow in the literature the same carries or principal ideal theorem. It's for the ideals in general. But first he did it for principal ideal but the theorem is really for obituary ideal. Okay, so the part one actually the, this prove theorem. So that is part one, that if, If P' ideal. No, as usual A Noetherian. Is minimal over an ideal A, which is generated by r elements. a_1 to a_r . Then height of P will less equal to $\mu(A)$. Which is less equal to r. Okay, before I gone. I want to first make few comments. So first of all r equal to1 is what is known as Krull's principal ideal theorem. Okay? Second one. If A is a proper ideal then height of A, height of this ideal A cannot be more than the minimal number of generators for A. Remember, height of A means, minimum of height of P for, where P is the minimal over A. Okay, the third one. Suppose P is prime ideal and one say that it has two ideas. Already, them also says height is finite. So, height of P's are, then there exist. r elements, a_1 to a_r in P, such that P is minimal over ideal generated by a_1 to a_r . Moreover, moreover, height of ideal generated by a_1 to ai is i. For each i, i is equal to 0 to r. You see, note that this three is actually the converse of one. This is converse of one. Okay, so let us prove it.

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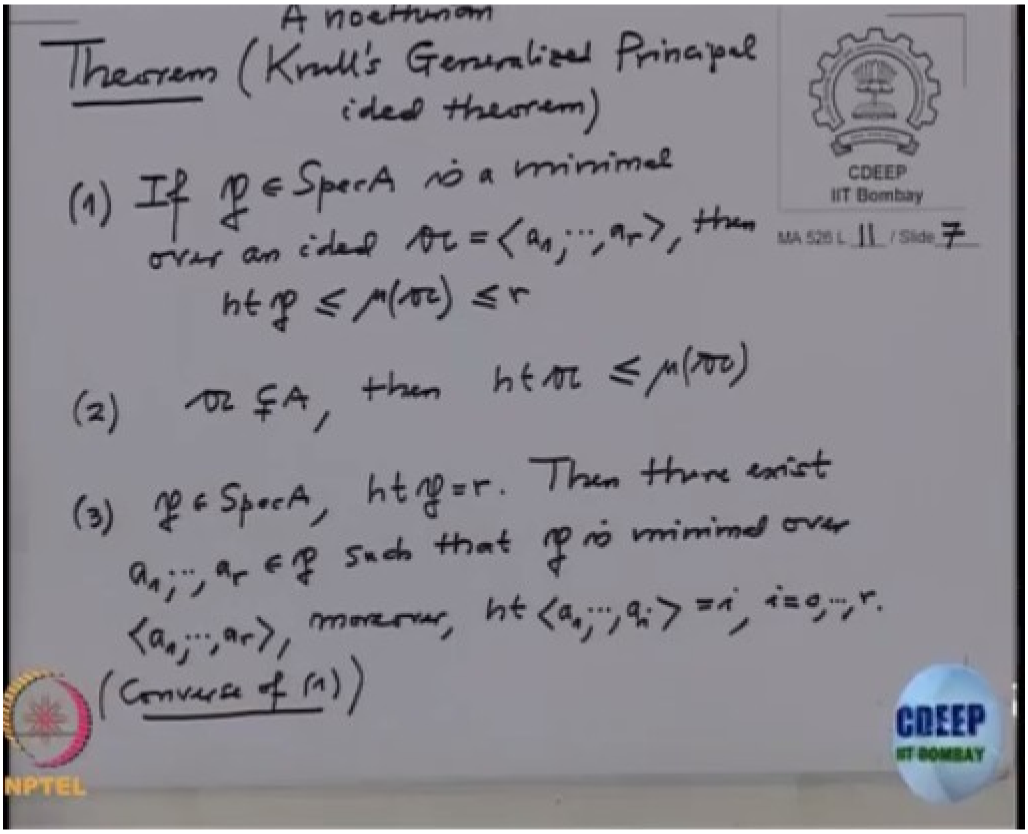

So, note this statement one, say that, minimal prime ideal over a given ideal has height, no more than the minimal number of generators for the ideally. So, if you talk in terms of geometric language, you have these V of A, this is a close subsidy in Spec of A and the prime ideals here. So the, this is the union of finitely, maybe, irreducible components. And irreducible components given by the minimal prime ideals. So, this says, that the co-dimension of the irreducible component of this is less equal to r. That is what the geometric content of this theorem. Okay, so we want to prove that, so, proof of one. Okay, P is minimal over A. That means there is no prime ideal in between, so therefore, at localization, this is minimal over A localized P. And therefore, when this is minimal, support of this A, mod A localized P this support, it just a singleton, namely the maximal ideal of AP. Because if there is somebody else, then it will be a prime ideal in between these two and then PAP will not be minimal or AP. So therefore, this means that the ideal in the localization, this is PAP primary. So in a local ring you have, this is a local ring with maximal ideal P and in that we have a primary ideal. And then our dimension theorem says that. We are applying dimension theorem to the local ring, A localized P. And this primary ideal, A localized P.

So, this dimension theorem say, that the dimension of this ring will be equal to the degree of the Hilbert-Samuel polynomial defend by this primary ideal. And that will be less equal to the minimal number of generators for A. Okay, so therefore, we know height of P. This is equal to dimension of A localized P and this is the dimension theorem here. It say that, it is a degree of $H_{A A_P}$, but this degree is less equal to minimal number of generators at the primary ideal. But this number is obviously, small or equal to the minimal number of generators. So the ideal in A original P. Because localization will not increase the minimal of generators, if at all they can go down. Okay, two is obvious on A because, say that the height will achieved by the, it's a minimum of the, heights of the minimal prime ideals. And height for each minimal prime ideal by one is less equal to μ. Therefore, all the heights will be less equal to μ. And therefore, height of the ideal will be less here. So, clear from definition of height of A and 1.

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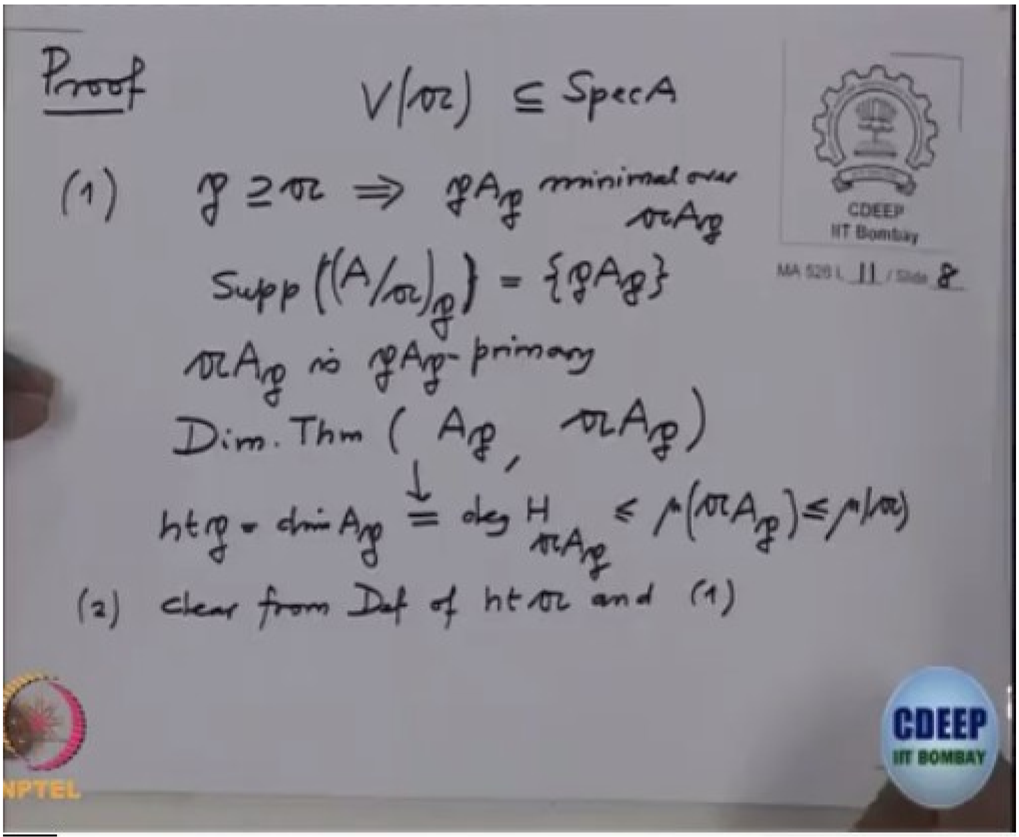

Oh, three. So proof three. Three is a construction and this construction will use avoidance, prime avoidance lemma. Okay, so what do you want to construct? We want to construct a chain. So we are assuming height of P's r and from here I want to construct r elements, a_1 to a_r in P. Such that P is minimal over the ideal generated a_1 to a_r. And more over part is, if you take any part of this a_1 to ai, the ideal generated by that will have height exactly i. Okay, that is the problem. We are looking for r elements in a prime ideal of height r so that p is minimal or the ideal generated by a_1 to a_r. Okay, these I am going to do it induction on r. Proof by induction on r. r equal to 0, it's clear. r equal to 0, the assertion is clear. Because r equal to 0 means what? Height P is 0. Height P is 0 is, it is minimal prime ideal in a ring. So I could take empty set. I could take a zero ideal. So I got a minimal prime over the zero ideal. So r is equal to 1 there is nothing. So assume r is at least 1 and choose, we know height is r, therefore there is definitely a chain of length r with ends at P. So by lemma, part 2 in a lemma, there exist a chain, like this P_0 contained in P_r equal to P in Spec A. Then first of all note that height of the earlier one P_{r-1} has to be $r-1$. It is bigger equal to is clear. Because there is a chain of laying $r-1$ which ends at P_{r-1} and it can't more because otherwise I will put that chain before this P_{r-1} and then height of P will increase by more than r, so therefore this is correct. So therefore by induction and hence by induction there exist r minus elements. a_1 to a_{r-1} in P_{r-1}, such that P_{r-1} is minimal or a_1 to a_{r-1} and also the property that if I take i of them, the height of and height of ideal generated by a_1 to a_i is i if i is on 0 to $r-1$. Okay, now we are looking for one extra guy.

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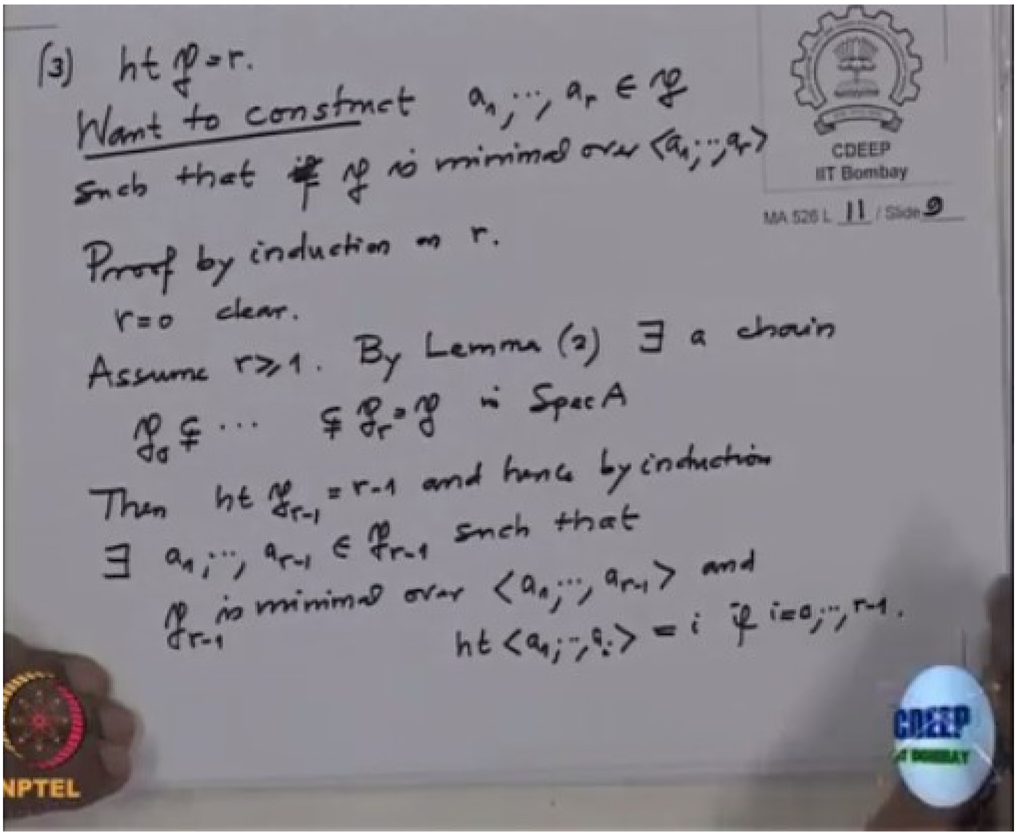

If I take this ideal generated by a_1 to a_{r-1} and I take it's minimal primes. So I take minimal associated primes of A modular this ideal. Minimal elements in this associated primes. That means they are associated to this ideal and they are minimal. So this is the finite set. This finite set, I want to give the label to them, these are the some prime ideals from q_1 to q_s. And by 1, what did we prove in one. If I take any minimal prime over this, the height will not be more than $r-1$. So by one height of each one of them, height of q_j will be less equal to $r-1$ for all j from 1 to s. qs are the minimal primes over the ideal generated by $r-1$ element. Therefore height of q_j 's will not be more than $r-2$. And height of P we are assuming, height of P is r. This is by assumption. So P cannot be contained in this. In any one of them for all j from 1 to s because if P were contained then height of P which is r and then height of q_j will be at least r. But height of q_j

we know or by 1 it is $r-1$, therefore P cannot be contained there. Therefore P cannot be contained in the union. So that is Prime Avoidance. Therefore P cannot be contained in union of these qs. This is Prime Avoidance. So I can choose an element in P which is not in any one of them. So choose, that element I will call it a_r which is P and not in any one of these. Q1 union union union qs. Then I want to prove that this is the required set a_1 to a_{r-1} . So what do we want to prove, we want to prove that P is minimal over this and any a_1 to a_i ideal generated by that, that has height i. This what we want to prove. Okay, so take any minimal prime over this. So first take arbitrary prime ideal P, P' which contain this. I want to show that, we want to conclude something about the height of P' . a_1 to a_r is contained in P' therefore I say, therefore each q_j will be contained in P' . Because P' contains a_1 to a_{r-1} and therefore it will contain one of the minimal prime for some j = 1 to s. Moreover I want to say that this is proper inclusion because a_r is an element here and a_r is not here. That is how we have chosen a_r . So therefore it is not equal.

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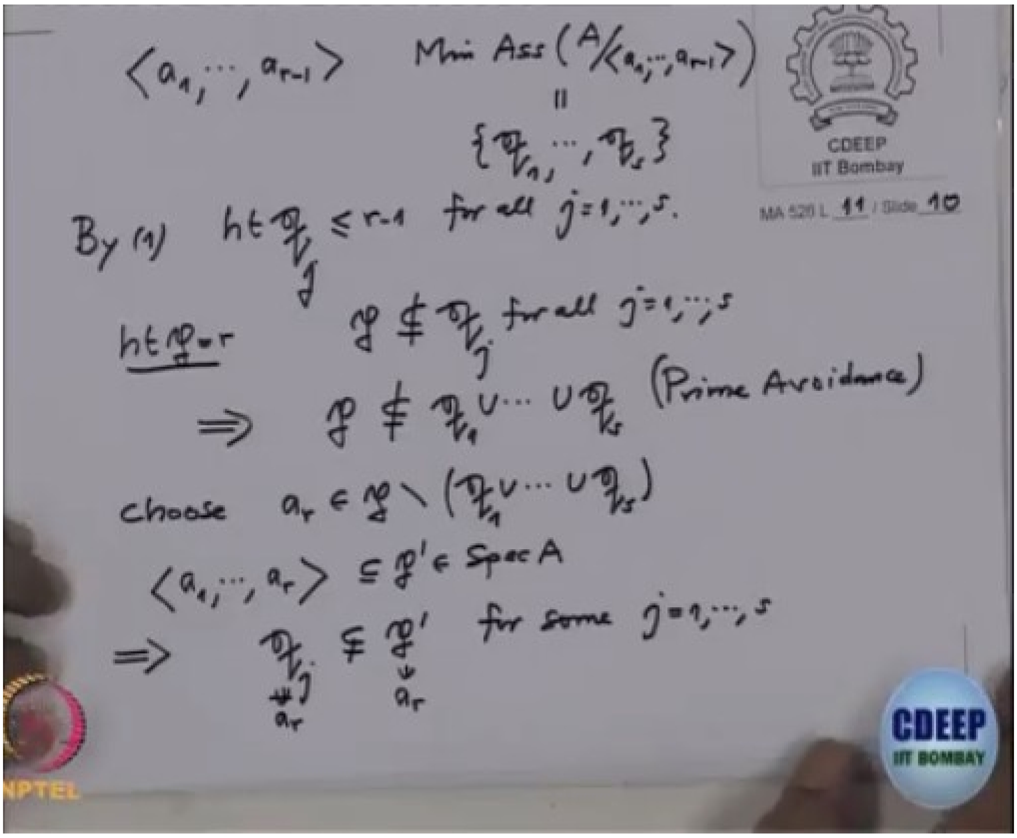

So therefore the height of P' , so therefore height of P' will be at least 1 more than the height of q_j . But we have seen the height of q_j is, okay. Also I will choose, actually I should have chosen correctly. So in these actually, among the minimal primes I will choose not only these I will choose the one which has exactly height $r-1$ or i many not needed. This is equal to r. Yes, because it's part of over induction hypothesis that we have chosen a_1 to a_{r-1}, so that this height is also r minus. Therefore all minimal primes, there is at least one minimal prime of that which has height $r-1$ and I will only choose, so that is correct what I said. I will choose among these guys. The one which has height exactly $r-1$. So these are the minimal primes of a_1 to a_{r-1} with height of q_j equal to $r-1$. So for the later, when I choose a_r, so that show the height of P' is at least r. On the other hand. Okay, height is r and it contain this r elements, so therefore height of minimal over. So that implies P' is minimal over a_1 to a_r, if this then height of P', it's actually r. Actually why do I need really that. I don't even need this, right? I don't need this. Yes, I don't even need. So you take all minimal primes in of the ideal a_1 to a_{r-1}.

Male Speaker: it is one plus height of P at least

Dilip P. Patil: Yes.

Male Speaker: Instead of equality.

Dilip P. Patil: Yes. Instead of equality. This one here, right.

Male Speaker: Yeah. You can.

Dilip P. Patil: Yes. Okay, that's enough. So therefore what we checked is. P is, okay. No, no, no, this is also, I don't need this also. So height of P'. what did we proved? We are proved that if I take any P' which contains a_1 to a_r then we proved that P' has height r, bigger equal to r. Okay, so therefore I claim now that given P is minimal over a_1 to a_r. This is what we wanted to prove, right? And height of a_1 to a_r is r. Okay, let us first prove that P is minimal over a_1 to a_r. If not then there is a prime ideal in between q'. Otherwise we are in such a situation, it's not minimal, so this is not equal. This is a prime ideal q' which contain a_1 to a_r and it is in between. But then this a_1 to a_{r-1} are contained in a q', so one of the q_j will be contained in q'.

So that will imply their exist q_j which is contained in q', contained in P. This is not equal, we have noted that this one has height $r-1$ that was part of the induction. And then this one, then this says height r we know by given assumption. So it has to be equal here. What do I want to conclude, no this is not equal here. This is not equal because a_r is here and a_r is not here. We have chosen a_r so that that a_r is not in any minimal prime or a_1 to a_{r-1}. Therefore this cannot be equal, so this cannot be equal, this cannot equal therefore but that is a contradiction. So that shakes that P is minimal over a_1 to a_r. Let me repeat once again. I want to conclude that P is minimal over a_1 to a_r. So suppose not. Then they will be a prime ideal in between strictly contained in P. but a_1 to a_{r-1} are contained therefore, in q', therefore it has to contain at least one of the minimal prime of a_1 to a_{r-1} and that is in our notation that was q_j and in that q_j , a_r is not there but a_r is in q' because q' contain this ideal. So this cannot be either equal and we have also checked that q_1 to q_j has height exactly $r-1$ that is because we are using induction. Induction hypothesis say that the ideal generated by part of the a_1 to a_r , in particular a_1 to a_{r-1} has height exactly $r-1$ that means all minimal primes will have a height $r-1$. So therefore, but this cannot happen be height of P will increase then. Height of P are given, so that is not possible. So that proves that P is minimal over this. Yes, and that proves everything, because then and also. So the proof is complete by induction hypothesis.

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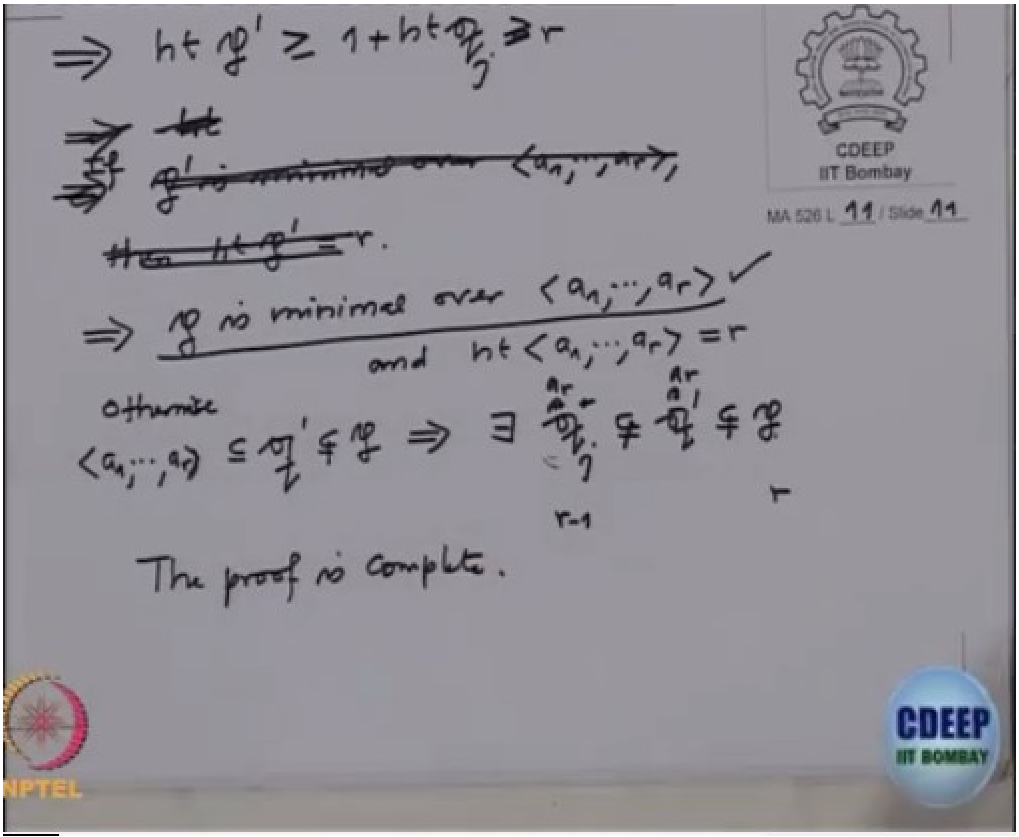

Because if I take any part of a_1 to a_r , the height is i when the last stage a_1 to ar the height is i because P one of the minimal prime, height P is r, therefore, the height of a ideal is a minimum of, so therefore, it can't be lass. So it is r. so that complete the proof. Okay, now few corollaries. Corollaries and also I want to give another proof. So this proof, I want to give another proof which is not as complicated of this. And also it avoids dimension theorem. So in that sense it will be better but that I will do it next time but I want to deduce one corollary at least or two corollaries. Corollary 1, if I take any A noetherian and if I take any element in A which is a non unit and any prime ideal, P minimal over this then height of P is less equal to 1. This is in particular height of a principal ideal is less equal to 1.

This was actually proofs principal ideal theorem. This is the case r equal to 1 from the statement 1. One more corollary I want to write, Corollary 2. Suppose I have a prime ideals P_0 contained in not equal to P_1, contained in P_2 in Spec A. So that means what, what I am assuming. Given any think of it like that. Given two prime ideals one contained in the other. Suppose there is at least one prime ideal in between then I want to conclude there are infinitely many; then there exist infinitely many prime ideals q in A such that with P_0 contained in not equal to q contained in not equal to P_2. If there is one then they are infinitely many. This is very astonishing statement but its true. Okay, proof. Okay. So, we may assume P_0 is 0. And also assume and A is local with maximal ideal P_2. This is the usual trick when one wants to concentrate in between the chain, you go mod the first one and localize at the biggest one. Okay, and now let us take, so let a belong to P_2 then there exist definitely a prime ideal q such that height of q is less equal to 1. Namely you can take this a and take its minimal prime, minimal prime ideal over that A then the height of that q will be less equal to 1. So this is true for every a in P_2.

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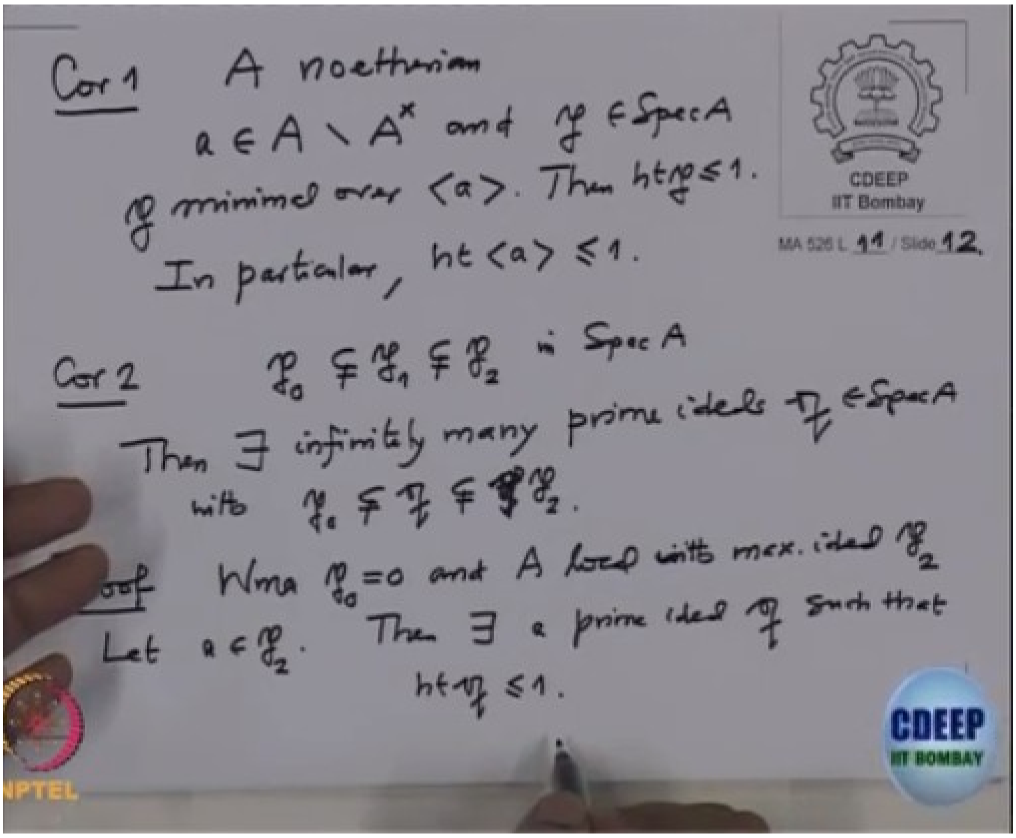

So that means P_2 is a union over prime ideals q in Spec A such that height q is less equal to 1 and union of such qs. Clear? Because if you take any element there is at least one prime ideal less equal to 1 which contain that element and therefore P_2 will be union of this. So if there finitely we need qs. So now let us look at this set. We are interested in this set, q in Spec A such that q is in between 0 and P_2, this. And we want to show this set is not finite. If it was finite then this will be finite union. And then by again primary avoidance P_2 will be contained in one of them. So in particular P_2 will be equal to that. Right? But then that is not possible. Then the height of P_2 will be less equal to 1, but height of P_2 is 2 because there exist at least one prime in between. So this is, if this set is finite, then P_2 will be contained in q for some q in this set. Let us call this set as X. But then P_2 has to be q of height less equal to 1 on the other end P_2 has, but height of P_2 is at least 2. So hence we have P_0, we have 0 and P_1 here and P_2 here and this we have given not equal. So height is at least 2. So that's a contradiction and therefore this set is. So contradiction, so X is infinity. Okay.

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