Lecture – 31

Consequences of Dimension Theorem

Gyanam Paramam Dhyeyam: Knowledge is supreme.

So, good afternoon. I want to continue from the last lecture, we proved dimension theorem and we were deducing some consequences of dimension theorem. Today, actually I will concentrate on proving so called Krull's principal ideal theorem. It has many applications also it is very important in algebraic geometry for many applications but before that I want to deduce two Corollaries from the dimension theorem. So we have deduced Corollary one, two, three. Now, today is Corollary four.

Remember dimension theorem is for the noetherian local ring, three definitions of dimensions are equal. One is the supremum definition, the other is the degree of the Hilbert–Samuel polynomial and the third one is Chevalley dimension. So, actually this Corollary, actually I have deduce in Corollary three, but this is slightly different flavor. As usual our notation is (A, m) local. Local always includes noetherian and a is an element in the maximal ideal m, and V finitely generated finite A-module. Then we've seen, if I go mod on this element a, that is if I consider a module $\frac{V}{aV}$ in the dimension of this module can drop at most by one. What extra line I want to add here is, if here is a non-zero divisor for the module V. Then equality will hold here. So proof, remember when you say an element of a ring is a non-zero divisor for a module that means the left multiplication by a on V , we need expand to ax, this map is injective. That is a definition of when you say, a is non-zero divisor for V. So suppose a is a non-zero divisor for V, that means this a can not belong to-- Okay. So, we have associated, look at the associated primes of V. This is a finite set where p1 to pm and obviously the dimension of V is then Sup of dimension of A by pi. Where i is running, i is equal to 1 to m. Because the dimension of V is this length of the chains of prime ideals inside the support of V. And any element of support of V, V will contain one associated prime and among the associated primes also if somebody is minimal then we would go down to get the maximum length of the chain. Okay, so among them actually I will choose p1 to pn, which actually gives the dimension, such that dimension of V is attained at all these guys. That is dimension of V by pi, I is from 1 to n. So this is a sub choose and because this a is non-zero divisor, a cannot belong to any one, any associated primes here. In particular any one of this. So since, a is a non-zero divisor for V, a can not belong to any one of them, a can not belong to union i is from 1 to n, pi.

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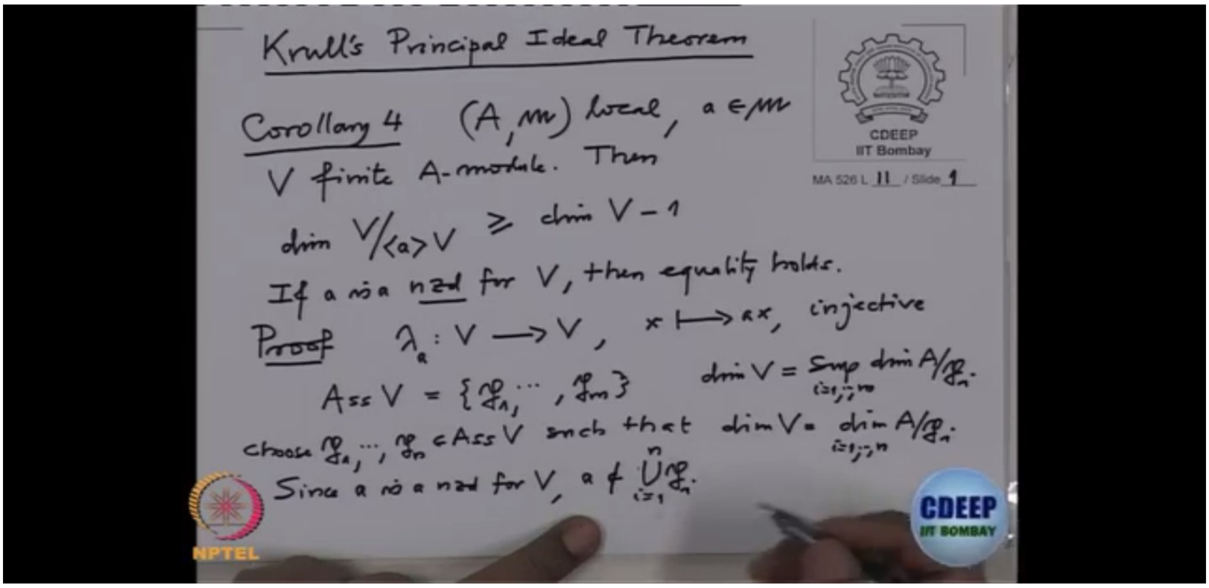

So, therefore, if I look at V' let us call V' to be V by a V, then support of V' will definitely be containing the support of V and removing p1 to pn because p1 for example is not in support of V, V' because when I localize this at p1 then that a will become unit, therefore this module will be zero. So it will not be the support. So, therefore the support of V' is actually will not contain to p1 to pn. And therefore the dimension of V' will not come from any one of this. Right. It will be bigger-- It will be less equal to dim of V minus 1because it will drop at most by 1. On the other end it is, this is inequality so equality holds. So that proves the Corollary. Okay. And this is a probably the appropriate time to define. So I want to make a definition. That suppose we have a non-zero module V and a_1 to ar are elements in the maximal ideal of A, so remember all the time our assumption is A is local and V is finite A module. Because in all the proofs we've been using associated prime support et cetera or this will make sense when the module is finitely generated over a neotherian ring.

Okay. So, this element even to ar is called a_1 to ar is called an A-regular sequence for V if, for each i from 1 to r, a_i is non-zero divisor for the module V by ideal generated by the earlier guys a_1 to a_{i-1} , V. So, this means for example, start at i equal to 1, a_1 is a non-zero divisor for V . So a_1 this means, so that is a_1 is a non-zero divisor for V, next a_2 is a non-zero divisor for $\frac{V}{a_1 V}$, a_3 is a non-zero divisor for $\frac{V}{\langle a_{1,} a_2 \rangle V}$ and so on. So such sequence is called the regular sequence. Now, it did not clear from this definition at all that this whether the permutation of a regular sequence is regular or not. Or how does one test somebody is a regular sequence. So this is a next topic I am going to take up with a lot of thing which are called, some homological algebra will come, and some other stuff will come. Bu just know I want to make a definition because I want write one more Corollary to the dimension theorem. That if I go regular sequence so the dim of $\frac{V}{\langle a_{1,} \ldots , a_r \rangle V}$. This will drop exactly by r, if a_1 to a_r is a regular sequence. So dimension of V - r if a_1 to a_r is an A-regular sequence for V. Already in Corollary four we've approved this assertion for r equal to 1. And obviously we are going to prove this assertion for by induction on r. By before the proof.

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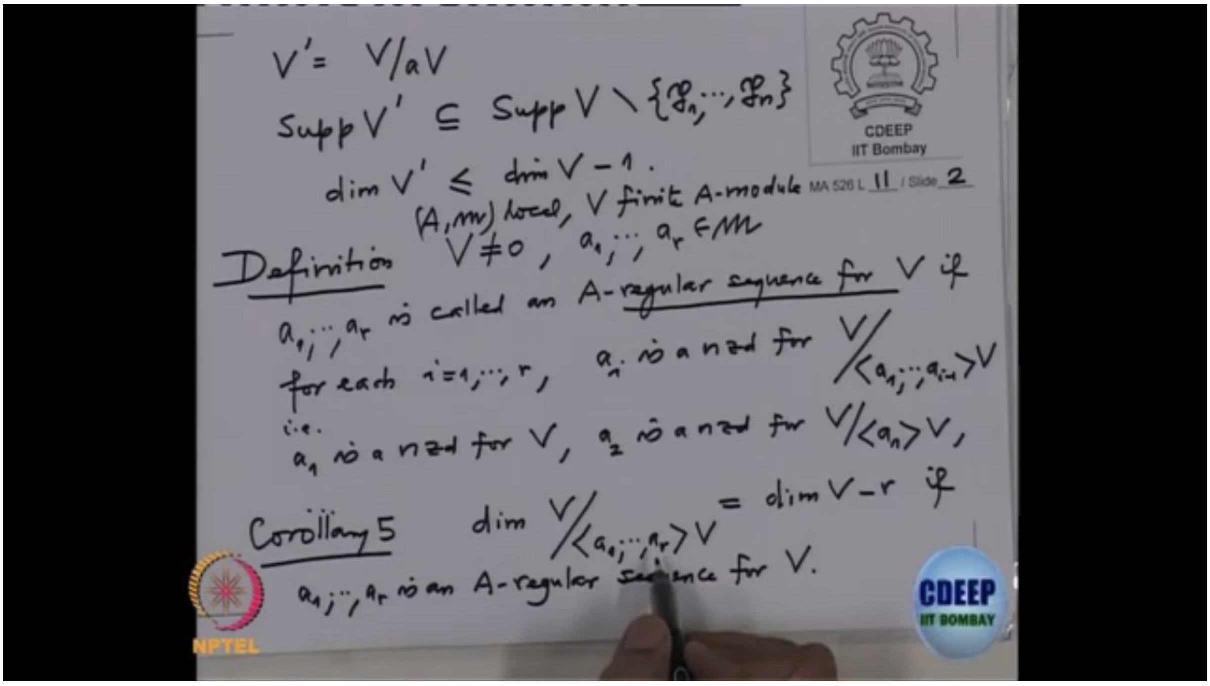

Also in particular I want to write down. In particular, any A-regular sequence for V is a part of system of parameters for V. That means any regular sequence I can extend it to system of parameters. And system of parameters means the d elements where, when you go mod them, the length is finite.

So the proof of Corollary 5, that is by induction on r, r equal to 1 is Corollary 4. So the way the definition is made we know the first $r-1$ elements will be, will form a regular sequence of V of length $r-1$. And by induction hypothesis dimension of V by ideal generated by a_1 to a_{r-1} and V, the dimension will drop exactly by $r-1$. And now I am going to apply once again Corollary 4, now apply Corollary 4 to the module V' is equal to this $\frac{V}{\langle a_1, \ldots, a_{r-1} \rangle V}$. Then by definition of a regular sequence this element ar then by definition a_r is non-zero divisor for V'. So by Corollary 4 dimension of $\frac{V'}{a_r V'}$ this dimension drops exactly by 1 from V'. So, now, let's put it together and then you get the Corollary effect.

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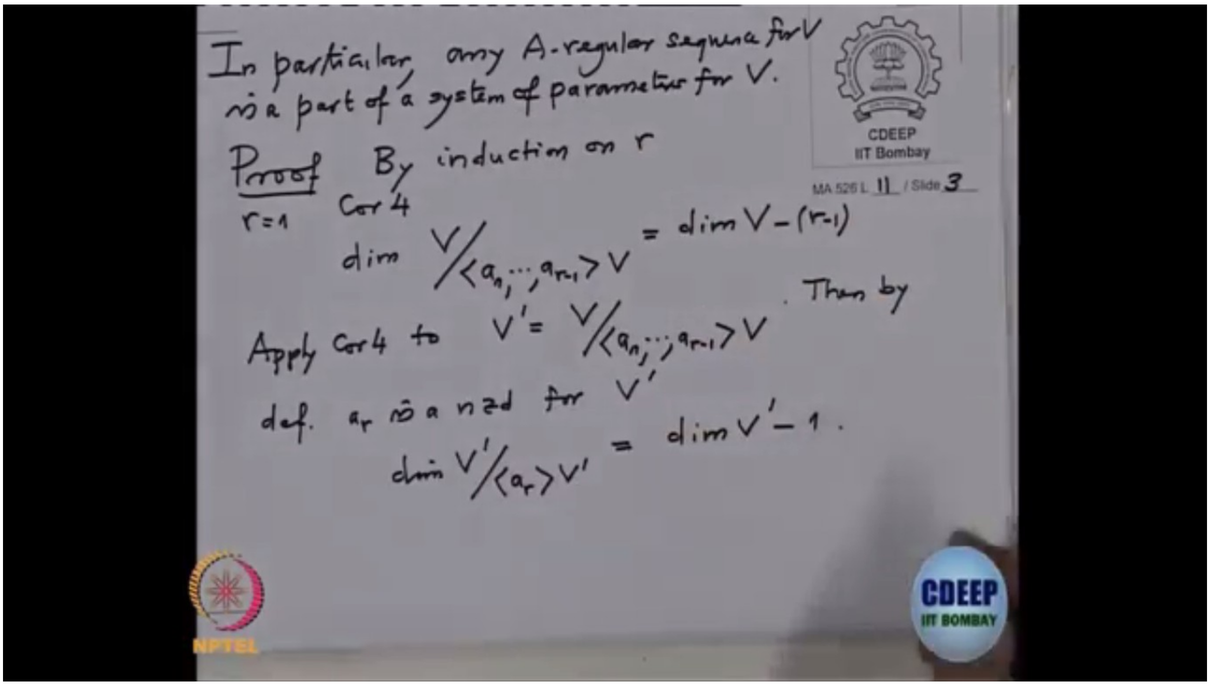

Okay. Now, getting back to principle ideal theorem, as I said this is very important theorem and actually I want to give two proofs. One by using dimension theorem and the other is not using. So, for example, how did we prove that if I have an noetherian.

A noetherian ring and p is prime ideal. And then we have defined height of p this is by definition, if the Sup of r, such that there is chain of prime ideals comtaining p. So that is p_0 et cetera, et cetera, et cetera, p_r this the chain of length r and these are all containing p in Spec A. Such a supremum we call the height of p. Or sometimes also in a geometric language I think it is better to co-dimension. So this-- I will just mention now but later on when we switch to some geometric statements it would be better to quality co-dimension. Okay. But that's not important now, but from this definition apriori it is not clear then this hieght is finite. And we have noted that this hieght is also noting but the dimension of the localization. Because we now the correspondence between the prime ideals of Spec, prime ideals of localization and prime ideals of A. These are the, those prime ideals which will remain here prime and they are containing P. And if they are not containing P, they will contain the unit and therefore it will not be a prime ideal. So we have a good connection, so actually this you can identify here, with that. So after this what we did was we proved dimension of a local ring is finite and that is, that follows from the fact that it is by the dimension theorem, because we have proved that this Krull dimension is same as the degree of the Hilbert-Samuel polynomial and degree of the polynomial is an integer, natural number, so it's finite that is how the proof was. But I will prove today by using Krull's so called ideal theorem that this is finite. So let us observe some facts may be some repetition, but let us do it for the sack of complicity, so let me write it Lemma. So first of all height of P is how with the move I will give by using the dimension theorem and the second part I will not give the dimension theorem and prove height is finite. So first, first statement is height of P is less equal to $\mu(P)$. Now remember my ring is not local, so one has to be little careful when one says μ of. So remember here when I have a module V, V then A-module. What do you say about $\mu(V)$ this is by definition? Look at all generating sets of V and among them take the one which as the least cardinality. That is called minimal, that number is called minimal number of generators for V. This is called minimal number of generators for V. And remember so this is slightly different from seeing that you take a minimal generating set and take the cardinality. So minimal generating set by definition means, the generating set, which you cannot remove any element from there. So that such a set is called a minimal generating set. For example, if I take A equal to ring of integers and V is also integers. So here {2, 3} is a generating set for V, and it's a minimal to because you cannot drop 2, you cannot drop 3. So this actually the minimal generating set, 1 is also minimal generating set. So the two minimal generating sets may have different cardinality. But among them we are using the minimal one. So in this case, $\mu(\mathbb{Z})$ as a $\mathbb{Z}$ -module is 1. And you would have seen probably that give an any integer r, you can write down a minimal generating set for $\mathbb{Z}$ as a $\mathbb{Z}$ -module, for arbitrary r. So, for arbitrary r, natural number, there existence of minimal generating set of cardinality r, for $\mathbb{Z}$. Okay. So I gave you example for two elements, three elements also you can win, four elements also you can do, you cannot drop any one of them this is a nice exercise probably you will need chinese remainder theorem are playing with the prime numbers. Okay. So the first statement is height of P is less equal to $\mu(P)$. Where $\mu(P)$ is the minimal number of generators for P, and because we are in a noetherian ring every ideal is finitely generated. So in particular P has a at least one set which is finite state of the generator. So, in any case $\mu(P)$ will be less equal to this is a finite number. So, this also proves height of this-- proves in particular height P is finite. Okay, so this is a statement of the Lemma.

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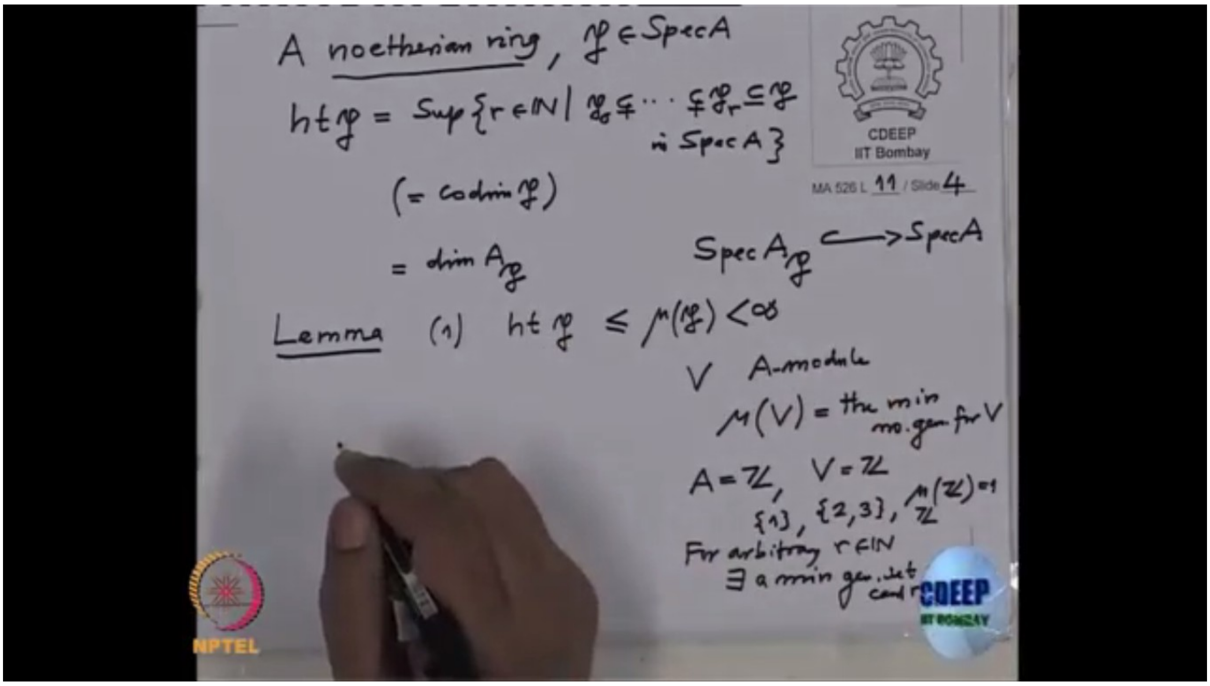

The second, if r is height of, Now it's finite, so I will call r is the-- if r is the height of the prime ideal P, then there exist a chain of length, r length this, which is P, in spec. So that means if you see that this height, it is r and then the other side is supremum that means supremum is attend. Right? Okay, third one. If r is the height of P, then for each r, for each i from 0 to r, there exists a prime ideal P_i with-- P_i which is containing P and height of P_i is exactly i. Okay, let us prove this. Prove this lemma first. So, proof of the lemma. One, I want to prove that height P is less equal to the $\mu(P)$. So, we have noted about height P is nothing but dimension of A localized at P. Now, A is the local ring. This is a local ring, with the unique maximal ideal PAp. And we have noted that this dimension is the degree of the Hilbert-Samuel Polynomial. But this degree, we have noted that it is-- the degree is less equal to the number of generators for this ideal. You remember, if you have taken a Q to be M primary ideal in a local ring and then taken a module over that and defined a Hilbert function using that, then we have check that the degree of the Hilbert function is less equal to the number of generators for the primary ideal. So, here the primary ideal it is M, Pap itself. So this is less equal to $\mu(PAp)$. Minimal number of generators for the maximul ideal. But this is obviously less equal to $\mu(P)$. Because if you take a generating set for P then after going to localization some of them may, may not be required. So, therefore the minimal number of generators will be less equal to $\mu(P)$. So this is where we are use the dimension theorem. That dimension of a local ring equal to the degree of the Hilbert-Samuel Polynomial, okay. So, that proves one. So, 2, since height is finite by 1, where the 2 is clear, 2 is immediate. Because it's finite number and so. Okay, so 2 is clear and 3, you want for each i, some prime ideal which is containing P of height i, but you see, 2 says, if r is height they change like these of length r and the last one is p. So, if I take P_i here the height will be i. that is obvious. So, clear from 2 or take P_i equal to the i the element appearing P_i equal to the i the prime ideal appearing in 2.

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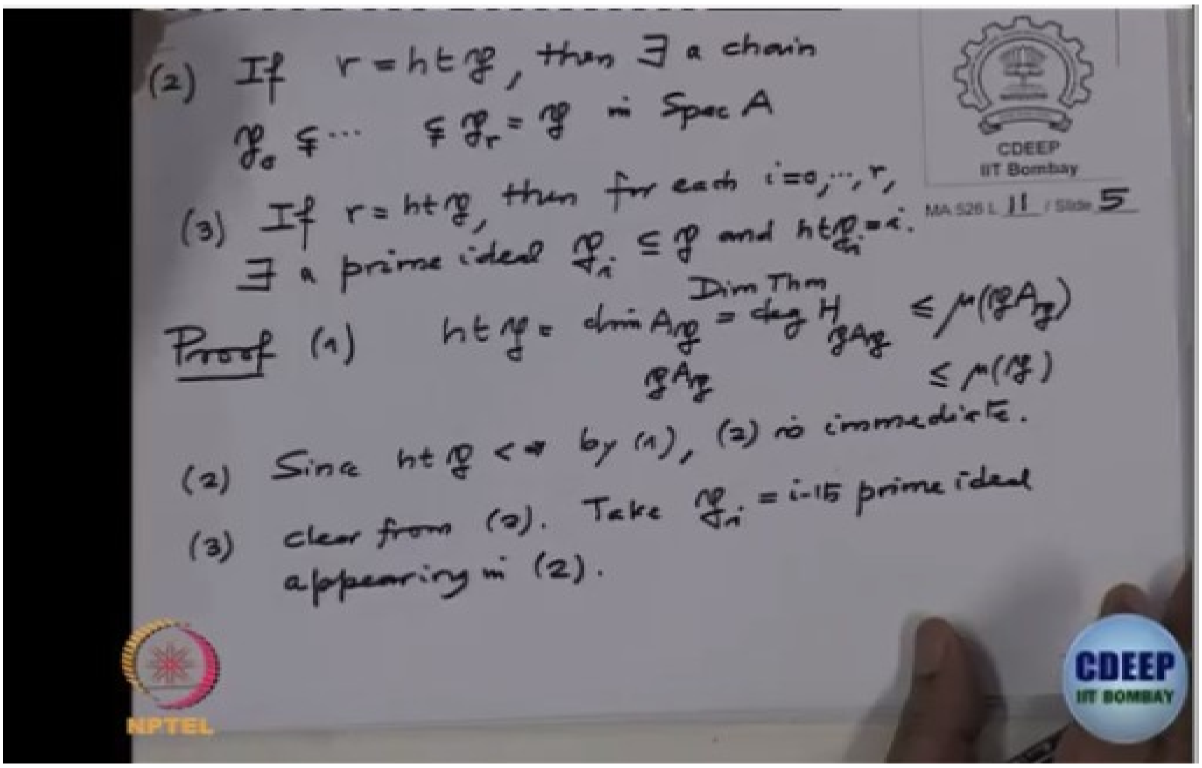

Okay, so that proves the Lemma. Now the Lemma was prime ideals. So, in general if you have an ideal A, in noetherian then 1 puts height of A by definition, this is infimum of heights of prime ideals P, such that P belong to Spec A and A is containing. But obviously, these are all prime ideals of the ring which contain all of A. But now, it is clear that we only have to concentrate on the minimal elements of the support. And minimal elements in the support is same as minimal elements in the associated primes. So what I'm talking about is support of $\frac{A}{a}$, and associated primes of $\frac{A}{a}$. This containment is clear and the minimal sets are same. Min support of $\frac{A}{a}$ is same as Min associated primes of $\frac{A}{a}$. When I say mean, you see, these sets are ordered by natural inclusion. Min means minimal elements. There would be many. It's not a chain. So therefore, height as an ideal A is minimum of. Now I have to use a minimum word because in sum what is usually used for infinite set. So when the set is finite then when you say the word minimum. So minimum of height p, where p is minimal over a. Minimal over a means with an element of the support which is a minimal element there. That means, so this means, there is no prime ideal in between A and P, there is nobody in between. If somebody is there, then it has to be equality here. That means the minimal, there's no prime ideal in between A and P. Okay. In particular, their finitely any element, therefore in particular at least one element. So, in particular, their exist, their exist P, prime ideal p, which contains A and height of a is equal to height of p. Such a P will be necessarily minimal over A. Because if there's in between the height of p will increase. Height of that whichever is contained, his height will increase. So therefore, it's minimum. Okay.

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