Lecture - 30

Dimension Theorem

(Continued)

Gyanam Paramam Dhyeyam: Knowledge is Supreme.

Now, second one. We want to prove that. For second one, we many assume, V is nonzero. Because V is zero then we have made the convention so that they are equal. Both are -1 and let us put s equal to $s(V)$. So what we want to prove, second inequality is, we want to prove that $d(V)$ is smaller or equal to $s(V)$, this is what we want to prove. To prove this we assume, V is nonzero, because if V is zero both are -1 . So now, if we call s to be $s(V)$ that means, I have s elements, a1 to as in the maximal ideal of a, such that length of $\frac{V}{\langle a_1,\ldots,a_s\rangle V}$, this length is finite. That is the definition of $s(V)$.

Therefore, because the length is finite the support of $\frac{V}{\langle a_1,\ldots,a_s\rangle V}$, this consist of only singleton m because if your module is finite length this support consist of the maximal ideal and let us put now, let us put this ideal to be q, ideal generated by a_1 to a_s and this, actually let us put q is equal to this ideal plus annihilator of V. This is contained in m. And now we claim that this q is primary. q is m primary. Okay, for that it's enough to prove that if. So enough to prove that if q is contained in some prime ideal p, then this p must be maximal that will prove it is m primary.

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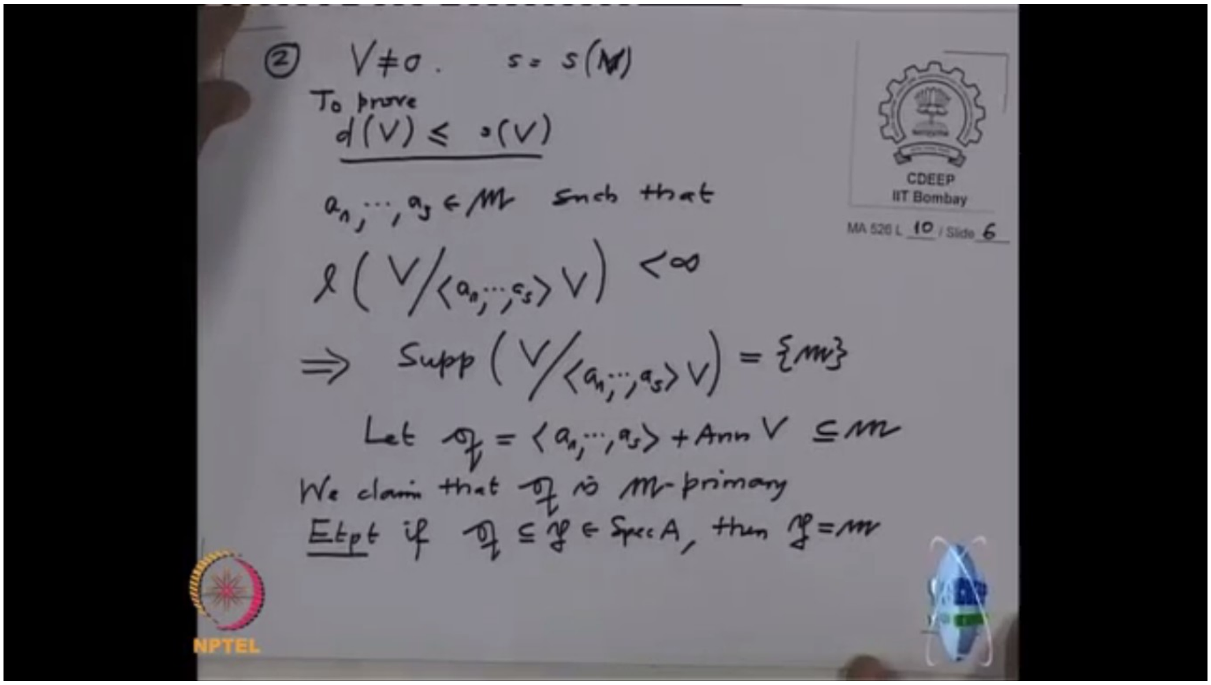

So suppose, on the contrary that we are in such situation, q is contained in p and p is not the maximal ideal. Suppose on the contrary that q is contained in p and contained in, properly contained in m, then p is not in the support of $\frac{V}{\langle a_1,\ldots,a_s\rangle V}$ that is, at localization this module is zero, so that is V_p is same as ideal generated by $\langle a_1,\ldots,a_s\rangle V_p$. And again by Nakayama Lemma V_p is 0. That means p is not in this support. That means p is not in this support of V but support of V is, V of the annihilator. p is not there means, p doesn't contain in annihilator. Annihilator V is not contained in p. But this is a contradiction. Because we have assumed that q is generated by the annihilator of V and along with the a_1 to a_s.

Okay, so we have proved q is primary and $d(V)$ therefore is degree of Hilbert-Samuel polynomial, I can use now this q primary this. Now, if you put $A^{'}$ with the residue class ring of A by the annihilator V and $m^{'}$ to be the ideal, the image of the maximal ideal m in $A^{'}$. And $q^{'}$ to be the image q in $A^{'}$, then the Hilbert polynomials $P_{q^{'}}(V)=P_q(V)$, they will not because we have gone module to the annihilators. So the annihilator kills the module m, therefore the p will not change and degree of. So $d(V)$ which is by definition degree of $P_q(V)$ which is degree because of this equivalent to degree of $P_{q^{'}}(V)$ which is less equal to $\mu(q^{'})$, which is. Now $q^{'}$, q was generated by a_1 to a_s along with annihilator of m. But we have gone mod annihilator so that disappears and this $\mu(q)$ prime is less equal to s which is $s(V)$. So we have proved the second inequality. So the second inequality it was $d(V)$ is small or equal to $s(V)$.

The third and last inequality, now we want to prove is, that $s(V)$ is small or equal of dimension of V. This is to prove. So first of all note that the dimension of V is finite because we have proved inequality one that dimension of V is small or equal to the degree of the Hilbert-Samuel polynomial and the degree is the integer. So, therefore the dimension is finite. And so to prove this again we may assume V is nonzero. So suppose if the dimension is 0 then, if the dimension is 0 that means the chain has only one element in a support because the supremum is 0, so there is only one element in any chain in the support. So that will mean that support of m. Support of V can only be the maximal ideal.

Support of V can only be the maximal ideal. Because if it is not maximal ideal you will have at least, two elements in the chain in the support. So dimension will be non-zero in that case. So if dimension is zero support of V will be exactly one element namely the maximal ideal and therefore the length of the module V is finite then because support of V consisting of the maximal ideal the length is finite. So that will mean, that so that is, S is zero then because I can simply take empty sequence a_1 to a_s and then mod that V mod that is V and the length is finite so by definition of the early dimension I can take S to be zero. So in this case, so we proved if dimension is zero then this is also zero and then it proves this inequality for dimension V equal to zero. Now I assume the dimension is positive.

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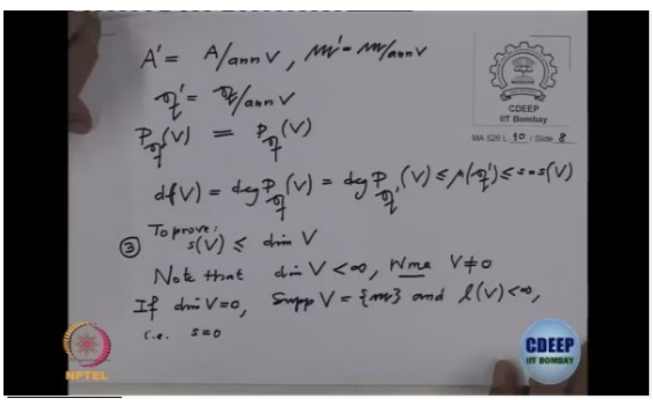

Assume that dimension of V is positive. And choose and let p_1 to p_r be elements in the support such that dimension V is dimension of $\frac{A}{p_i}$ for i equal to 1 to r. So what I mean saying is, we know that the dimension is finite, so that means all chains of the prime ideals in the support of V they will have finite lengths and in any maximal chain which gives the dimension in that I chose the left end point and that is called p_1 for another chain they may be another one, so p_2 and so on. So I call p_1 to p_r all, all those prime ideals for which in the dimension V is precisely the dimension of $\frac{A}{p_i}$. These are in the support of V and they will be by, automatically they will be the minimal primes in the associated prime. And therefore they are finitely mini. Okay. No, I just call that r. r is some number, right?

See, they are finitely many of them so I call that number to be r. So because now we are assuming dimension is positive none of this p_i will be m. So p_i is not m for all i equal to 1 to r, since dimension V, we are assuming is positive. And now, look at their union p_1 to p_r and m, this m cannot be contained here in the union because if m is contained in the union then it will be contained in one of them and then therefore m is maximal it will be one of the pi, but that is not the case. So m is properly contained in p_1 to p_r. This is also called prime avoidance of Lemma. So that means I can choose an element in the union which is not in m. So chose so there exist a in m which is in none of the primes p_1 to p_r union p_r.

And take $V^{'}$ equal to V by $\langle a \rangle V$, ideal generated by $\langle a \rangle V$. And now note that the support of $V^{'}$, will be contained in support of V and none of the p_1 to p_r are in the support of $V^{'}$ because when I localize $V^{'}$ at any one of them this a is not in p_i therefore this will become zero. At this a will become unite and therefore this module will become zero. So that implies this, therefore when I want to compute a dimension of $V^{'}$ s, we have to take the chains of the prime ideals in the support of $V^{'}$ but none of this guys are there. So any chain will not contain this, so therefore this will be at most dimension V -1. And now by induction, $s\left(V^{'}\right)$ is smaller equal to dimension of $V^{'}$ and obviously $s\left(V\right)$ is smaller equal to $s\left(V^{'}\right)+1$ because at most one element I will add to that then we will get the number of generators for this other dimension of V will be at most one more than the dimension of $V^{'}$. And this is plus equal to dimension of $V^{'}$ +1 but this is dimension of this less equal to dimension of V. Shifting the earlier inequalities -1 to the other side. So that proved the third inequality that therefore it proves the theorem.

So now I will deduce few consequences. So Corollary one, dimension of module is finite, remember or assumption V is a finitely generated module over a neotherian local ring. And we have approved that these dimension is same as the degree of the Hilbert-Samuel polynomial. And degrees always finite so therefore this is finite. In particular dimension of a local ring is finite. Dimension of local ring is finite.

Corollary two, if q is m-primary then the dimension of V is bounded of $\mu(Q)$. μ is the minimal number of generators for q which is by Nakayama lemma, this is the dimension of the, $\frac{A}{m}$ vector space $\frac{q}{mq}$. In particular, dimension of the local ring is less equal to $\mu(m)$ because m is m primary ideal and $\mu(m)$ is by definition or by Nakayama lemma, it is the dimension of the $\frac{A}{m}$ vector space $\frac{m}{m^2}$, this number is also called embedding dimension of A emdim A. That is the definition of embedding dimension of A. I will write on the next page.

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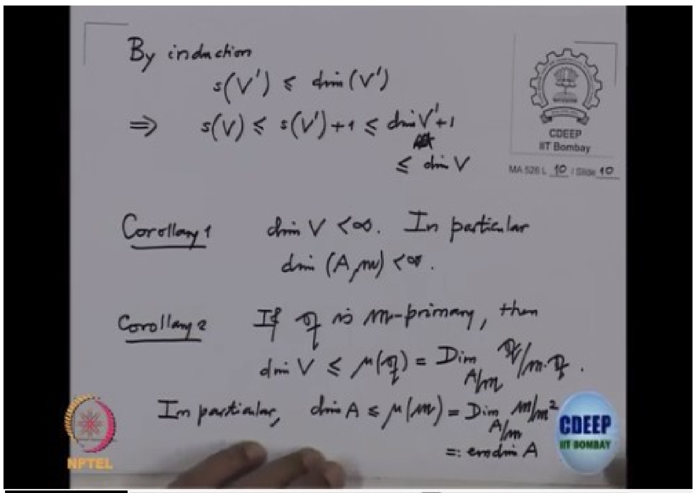

Embedding dimension of the local ring is by definition $\mu(m)$ which is equal to by non-common element of the k vector space $\frac{m}{m^2}$. So dimension is bounded by the embedding dimension. If m is non-zero, V is non-zero.

And if the dimension of V is, let's call it d. this is bigger equal to zero, because the non-zero. This means there exist. Now I'll use the definition, show you the dimension that exist d elements a_1 to a_d in the maximal ideal m, such that if I go modulo of this d elements length of $\frac{V}{\langle a_1,\ldots,a_d\rangle}$, hence V, this is finite. So such a system of elements of a_1 to a_d , such a system a_1 to a_d is called a system of parameters for the module V. So number of elements in a system of parameters for a module, degree of the Hilbert-Samuel polynomial and supremums of the chains of the prime ideals in the support V. There all same numbers that is what the content of the dimension theorem is.

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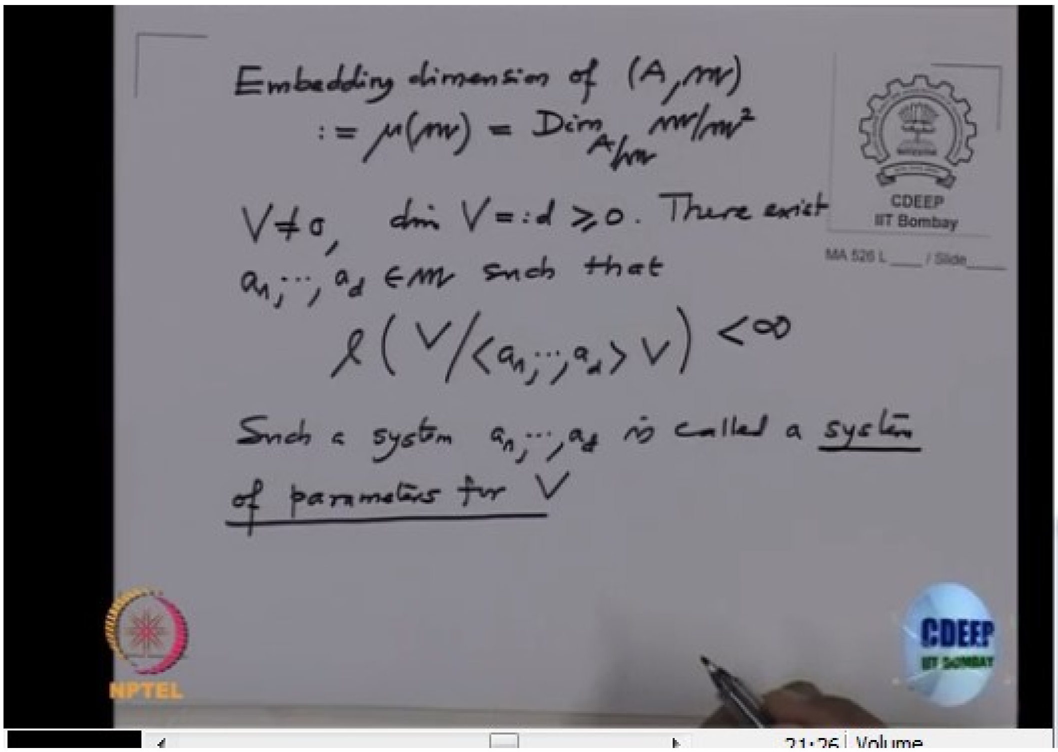

Okay, the next is, d is the dimension of d, right? So, corollary 3. This is very important theorem in local algebra. Okay, if r elements in the maximal ideals a_1 to a_r, then, if I go modulo this a_1 to a_r, then a module that means if I considered a module V, and consider residue class module $\frac{V}{\langle a_1, \ldots, a_r\rangle V}$, this module. Then the dimension of this is bigger equal to dimension of V - r. So proof, dimension can drop at most by r elements. Okay, moreover, equality holds if and only if so equality here holds if and only if these elements a_1 to a_r, this system a_1 to a_r this can be extended to a system of parameters for V. Then only the equality will hold. Okay, so proof. So let us put V' equal to this quotient module of this residue class module.

And V equal to dimension of V'. We want to prove that dimension of V' is dimension V - r. Okay. If this d is dimension of V' that means by a definition, Chevalley dimension of V' will be d that means there will d elements in the maximal ideal so that if I go modulo V', those d element, the length will be finite. So that, this means there exist elements b_1 to b_d in maximal ideal M such that length of $\frac{V^{'}}{\langle b_1, ..., b_d \rangle V^{'}}$, This length is finite. But look this module, this factor module, quotient module, $\frac{V^{'}}{\langle b_1, ..., b_d \rangle V^{'}}$, this is same as V module because $V^{'}$ is $\frac{V}{\langle a_1, \ldots, a_r, b_1, \ldots, b_d \rangle}$, this length is finite. But then, by definition of equality, dimension of this will be less equal to $r+d$. So that implies dimension of V which is $s(V)$ which is less equal to $r+d$, but this is equal to r, plus dimension of $V^{'}$. So prove equality. So that proves dimension of $V^{'}$ is bigger equal to shift this r to the other side. Dimension r, - r.

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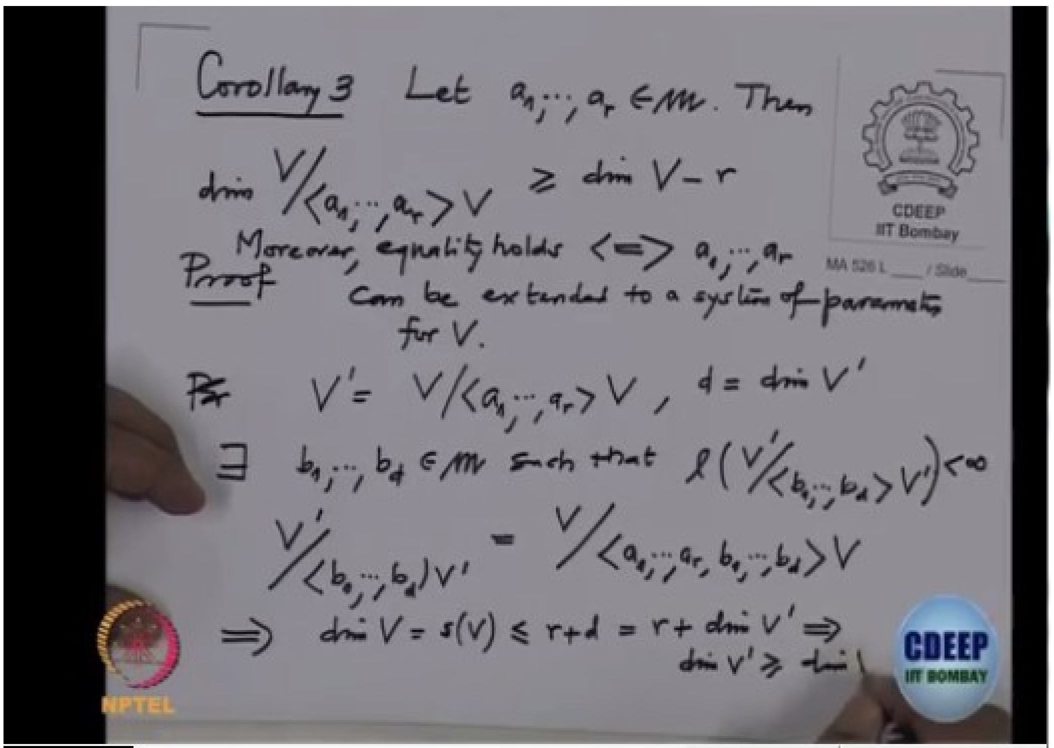

Now, the moreover part. Suppose the equality holds. If the equality holds, equality means dimension of V equal to $r+d$, where r and d as above. Then this will mean that $a_1, \ldots, a_r$, along with $b_1, \ldots, b_d$, these is a system of parameters for V. Because dimension is r and this is number of elements are also correct and mod those number of elements in dimension or the length if finite. Therefore by definition of system of parameters. $a_1, \ldots, a_r, b_1, \ldots, b_d$ in the system of parameters. So if equality holds, then this $a_1, \ldots, a_r$, we have completely into system of parameters so that to the one implication conversely. If $a_1, \ldots, a_r$, can be completed or can be extended to a system of parameters. Let's call it $a_1, \ldots, a_r, c_1, \ldots, c_s \in m$. The system of parameters for V. Then the length of $\frac{V^{'}}{\langle c_1, \ldots, c_s \rangle V^{'}}$, this is finite. Where $V^{'}$ is $\frac{V}{\langle a_1, \ldots, a_r \rangle}$. But this residue class module is same as. This is same as length of the inside, if the residue class module is $\frac{V}{\langle c_1,\ldots,c_s\rangle V}$. This length is finite.

Therefore by definition of Chevalley dimension this proves that, this s at least d because d was the dimension of V' . This is dimension of V' . But then, dimension of V, will be bigger equal to $r+s$ which is bigger equal to $r+d$ which is bigger equal to dimension of V. So all equality, all inequality they're equalities. In particular, we get dimension of V, equal to $r+d$, which is r + dimension of V' .

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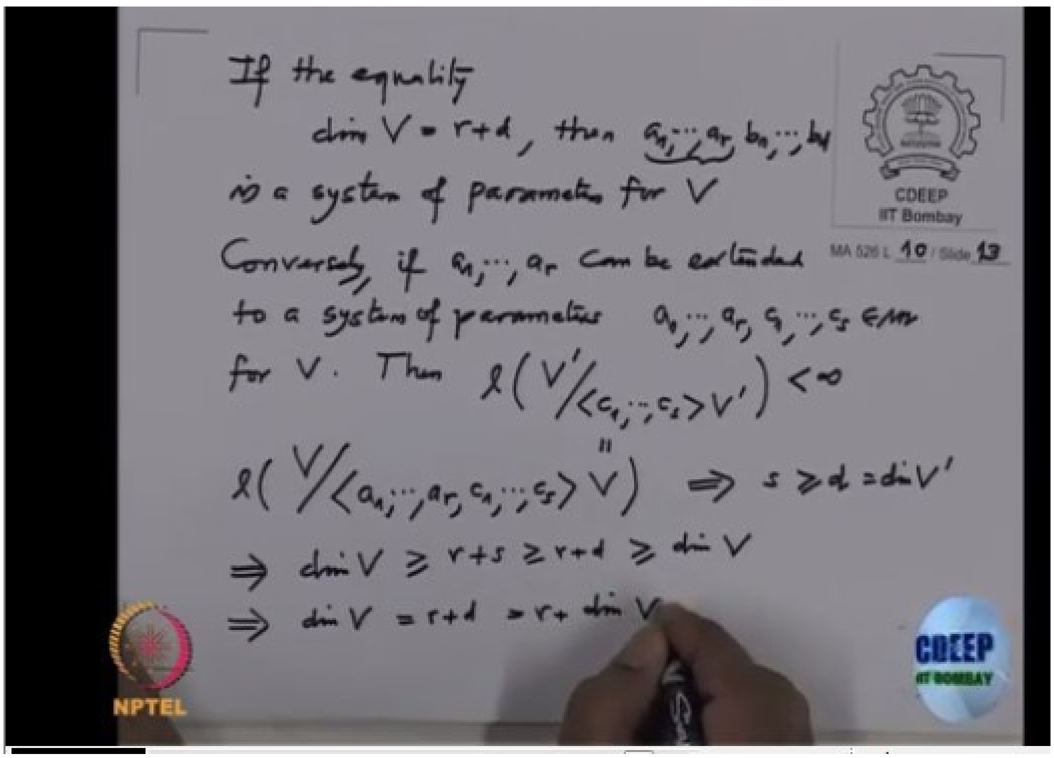

So that proves the equality. So, there are some more corollaries but I will do that in the next time.