INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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Lecture No. – 03
Motivation for Krull's Dimension

In this lecture we will see that how that dimension of a commutative ring is defined in general, for general commutative ring the concept of dimension was first defined by Krull, and Krull's motivation to define the dimension came from actually topology and geometry, so first I'll set up the algebraic geometry language I will use that precisely means the following, so K is our base field and it is usually assume K is algebraically closed, and we saw in earlier lecture that so called affine algebraic geometry, this is precisely the study of algebraic sets in K^n , this is a study of algebraic sets in K^n , and we saw that algebraic sets in K^n they are in bijective correspondence with radical ideals in the polynomial ring, in n variables, and the correspondence is if you have an ideal A, so ideal A radical ideal we can assume that is, using that we have defined this V_K , this is all the common 0's of all the polynomials in A, A in K^n such that f(a) is 0 for all F in A, so this is a subset of K^n and we have saw the properties of this sets and those are precisely the closed sets and those are precisely the closed sets of Zarisiki topology on K^n .

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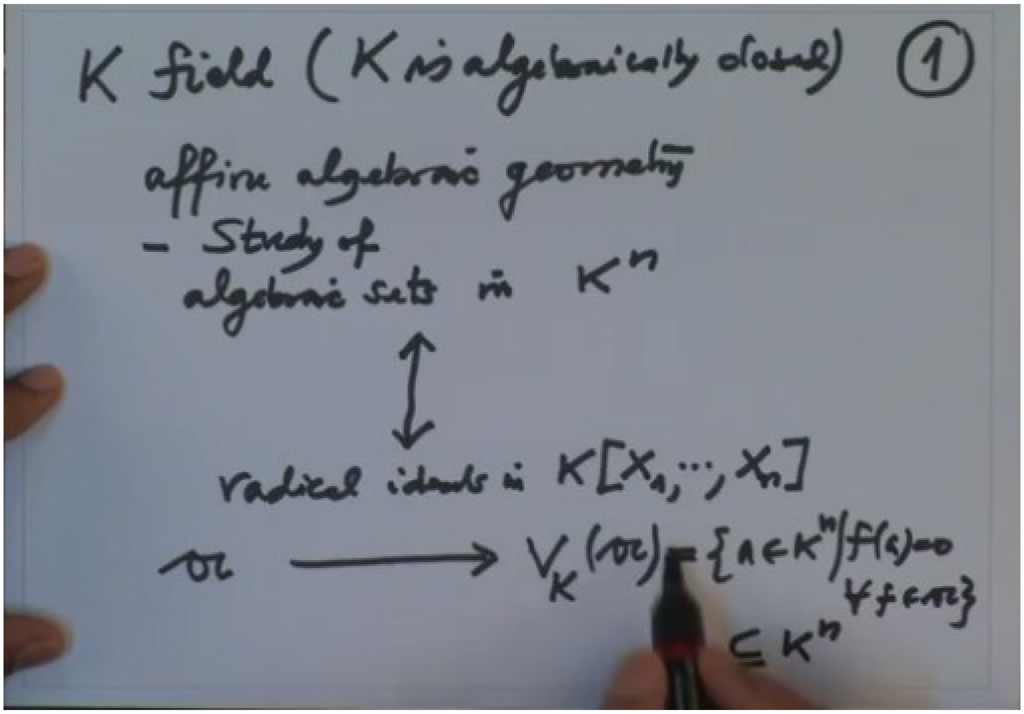

And other way map is if you have arbitrary subset W that would define an ideal of W, $I_K(W)$ to be all polynomials which vanish on W

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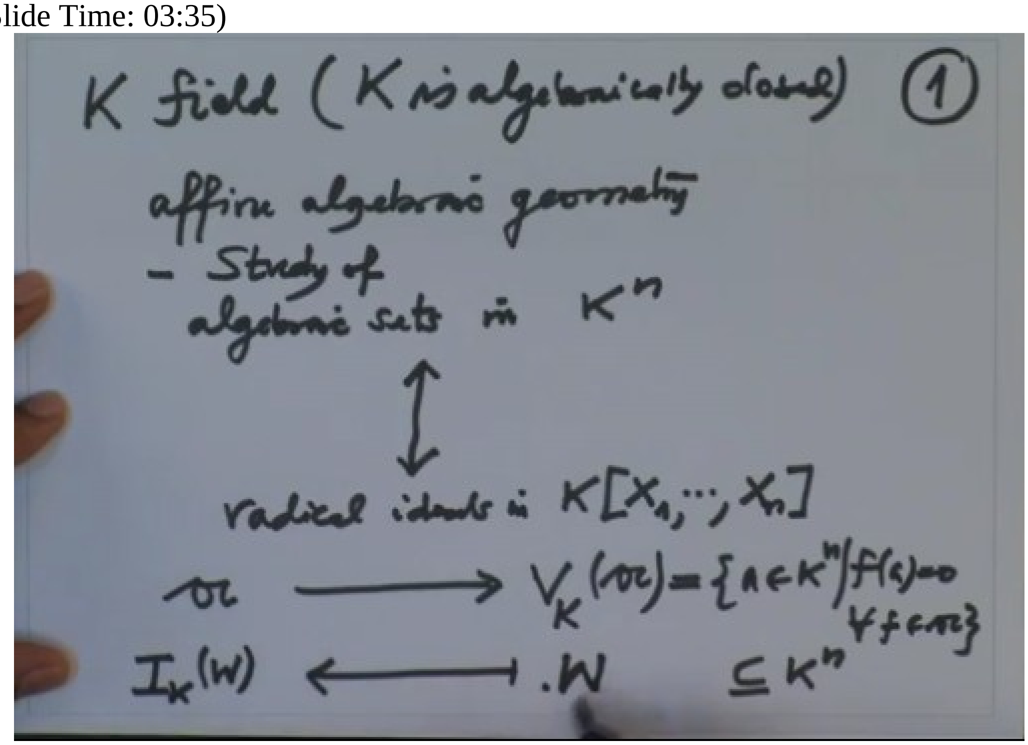

so this is precisely all polynomials $f \in K[X 1, \ldots, X_n]$ such that f(a) is 0 for all a and W, and this correspondence is a bijective correspondence, this are the inverses of each other and they are inclusion reversing, all this was the starting point was precisely the Hilbert's Nullstellensatz that I will keep calling HNS.

Hilbert's Nullstellensatz says that if I take V_K of an ideal and I take the ideal of that closed set then you get back your ideally, so this means this maps V_K and I_K are inverses of each other, and they are inclusion reversing, so one can study the algebraic sets by using the ideals in the polynomial ring in n variable that is called commutative algebra, so here is affine algebraic geometry to study that is commutative algebra, so these are geometric objects, and these are algebraic objects, so for example the points here they corresponds to the maximal ideals, and the closed sets they corresponds to the radical ideals,

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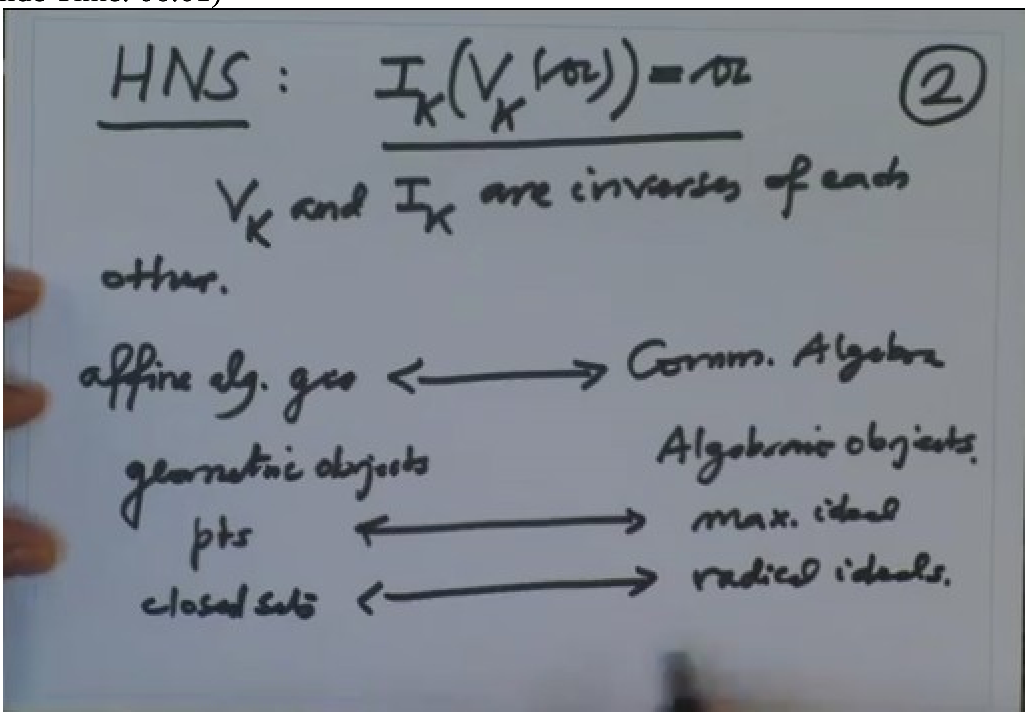

so the objects here have the structure of geometry, so for example so first of all I want to also say that every algebraic set is of the form $V_K(f_1, \ldots, f_n)$ only finitely many polynomials are needed to define our given algebraic sets because this correspond to the fact that the polynomial ring in n variables, this ring is Noetherian, so ideals are finitely generated, so that is ideals are finitely generated.

So but I'll still keep writing V_K of an ideal, so this is contained in K^n and this is a closed set, it has more structure, so it also has a topology which is coming from induced topology, so this is also topological space, with induced Zariski topology, that means the closed sets here, closed sets in $V_K(A)$ are precisely closed sets in K^n which are contained in $V_K(A)$,

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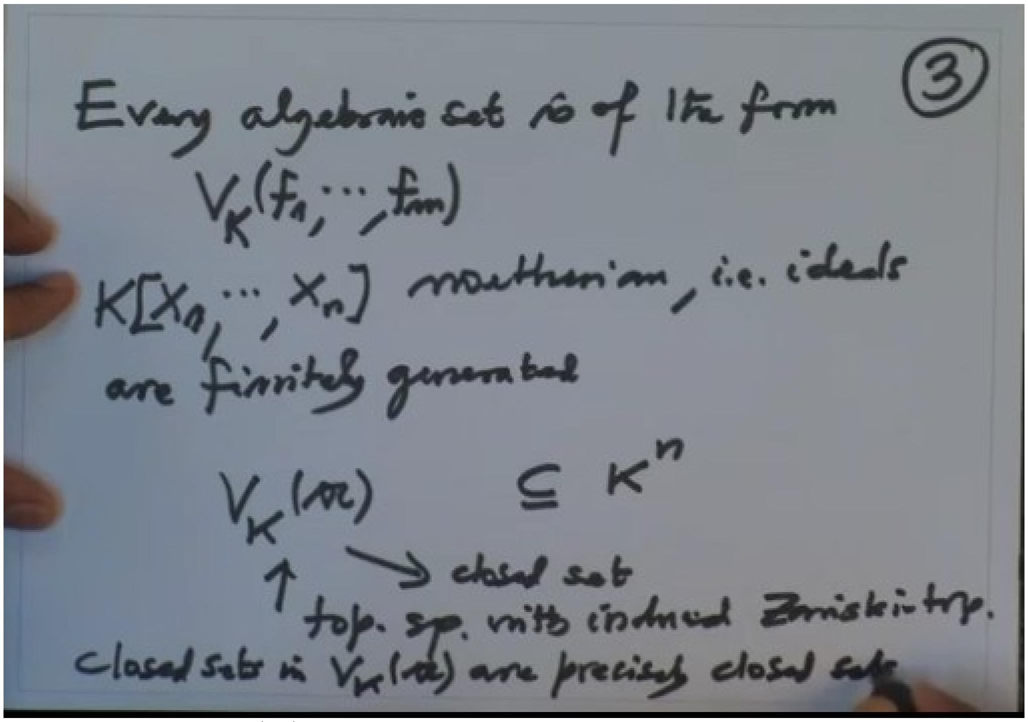

which are contained in $V_K(A)$, these are precisely the closed sets there.

Also to this V_K(a) we have an attached an ideal that $I_K(V_K(A))$, this is the vanishing ideal, all this polynomials is vanish on this, and therefore also we have attached a ring namely $\frac{K[X1,\dots,X_n]}{I_K(V_K(A))}$, so in any case this A may not be radical ideal, but here it will be radical ideal but here it will be radical ideal and therefore this ring will be reduced ring, reduced K algebra of finite type over K, this is called also the coordinate ring of $V_K(A)$,

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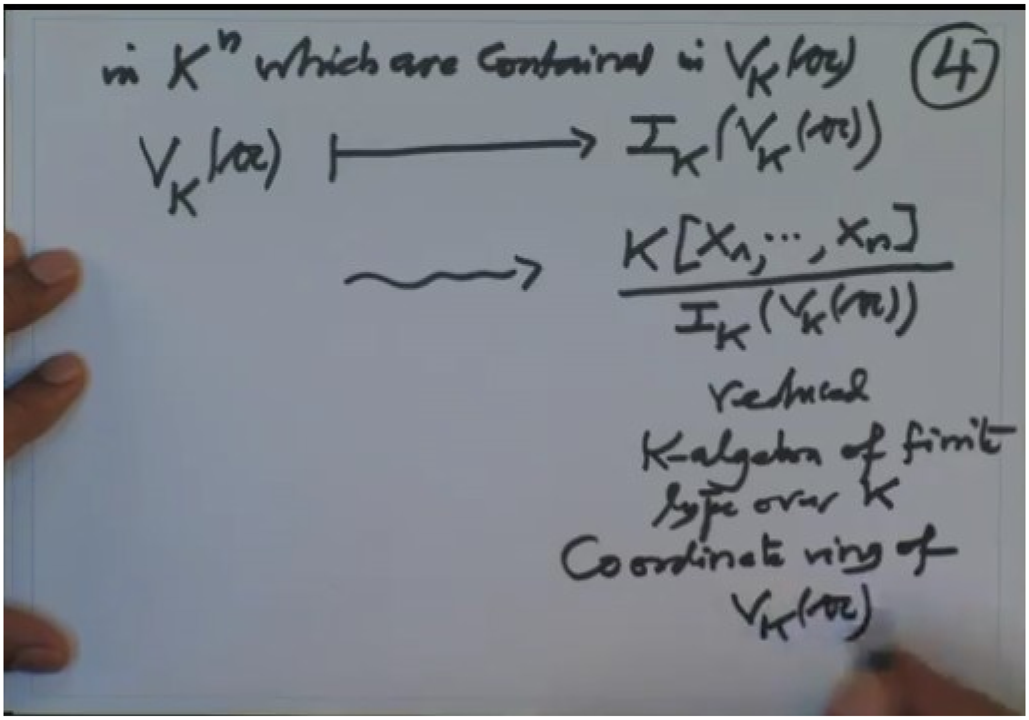

so for example if I take the ideal A to be generated by X^2+Y^2-1, this is two variables now K(X,Y) I'm taking, so if I have to draw the pictures I will draw in the plane, so what will be, so this is A, so what is V? I'm drawing, I'm taking K = R for drawing picture, so $V_{\mathbb{R}}(A)$ this is precisely the circle, this is $V_K(A)$.

So if I would have taken two linear equations, say $(aX+bY-c)(a^{'}X+b^{'}Y-c^{'})$, this is my f and what is $V_K(f)$ that will be pair of two lines, and I'll be passing to origin so they have two lines, okay and so on,

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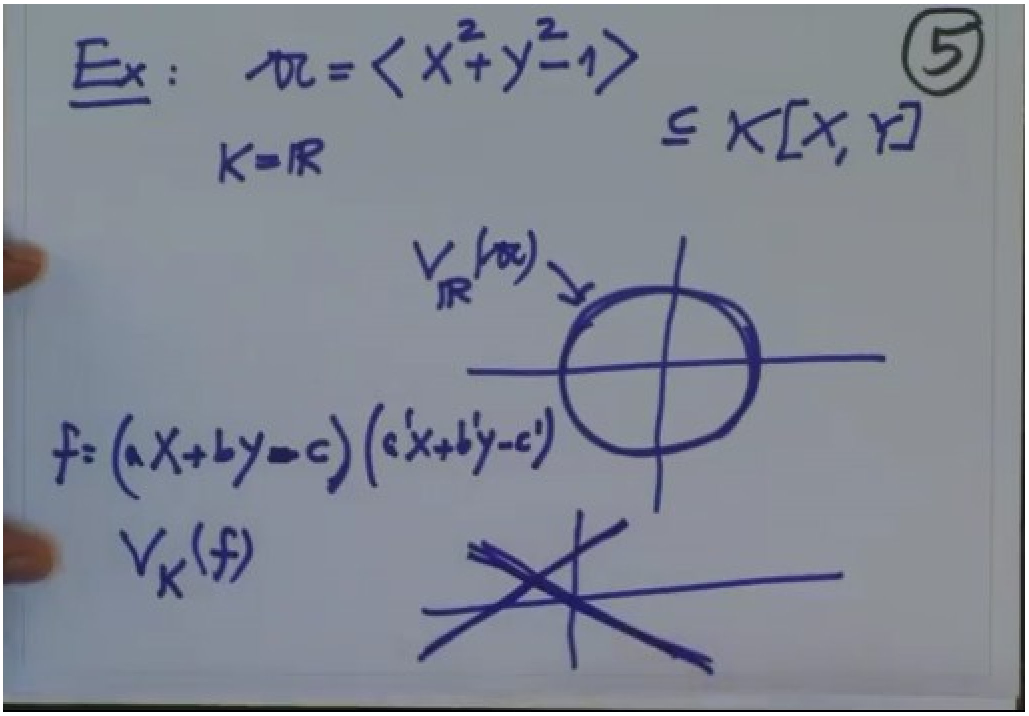

so we actually study all this, the study begins when we start studying analytic geometry in a college where we study the 0 set of polynomials of degree 2 into variables, and then we analyze then and we only do degree 2, and we analyze them and say they are hyperbola, parabola, circle and so on, so that was the beginning of this, alright, so our problem is to attach a number, we would like to assign to each algebraic set V, V is let me abbreviate V as $V_K(A)$ to this we want to attach the number called the dimension of V, so dimension, which should have desirable properties.

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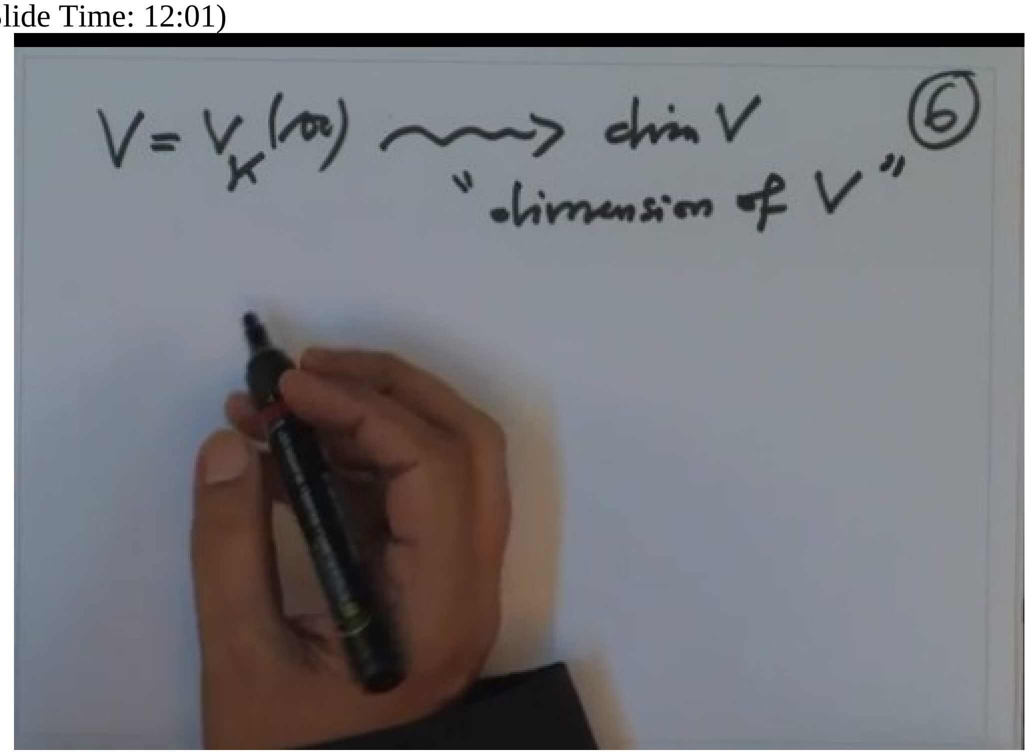

What are the desirable properties? So let us see two examples they are very important to understand how we have assigned the number dimension, so examples one of them suppose V is defined only by one equation, where F is polynomial in N variables, so suppose f is a nonzero constant, F actually is a constant A, A is in K and nonzero then what will be the vanishing elements in $V_K(f)$ that we will just, then $V_K(f)$ is empty set, because no matter which points are taking K^n when apply getting this A, it will never be 0 therefore this is empty set, this is when F is a nonzero constant.

If f is 0 then $V_K(f)$ is the whole K^n, because every point when I plug it in this equation it is 0, so it is this. So suppose f is non-constant, non-constant polynomial then if, okay if n = 1 we know this is then the finite set of point, this $V_K(f)$ is a finite set of points, because it's a polynomial in 1 variable it has only finitely many zeros, therefore it is finite set of points.

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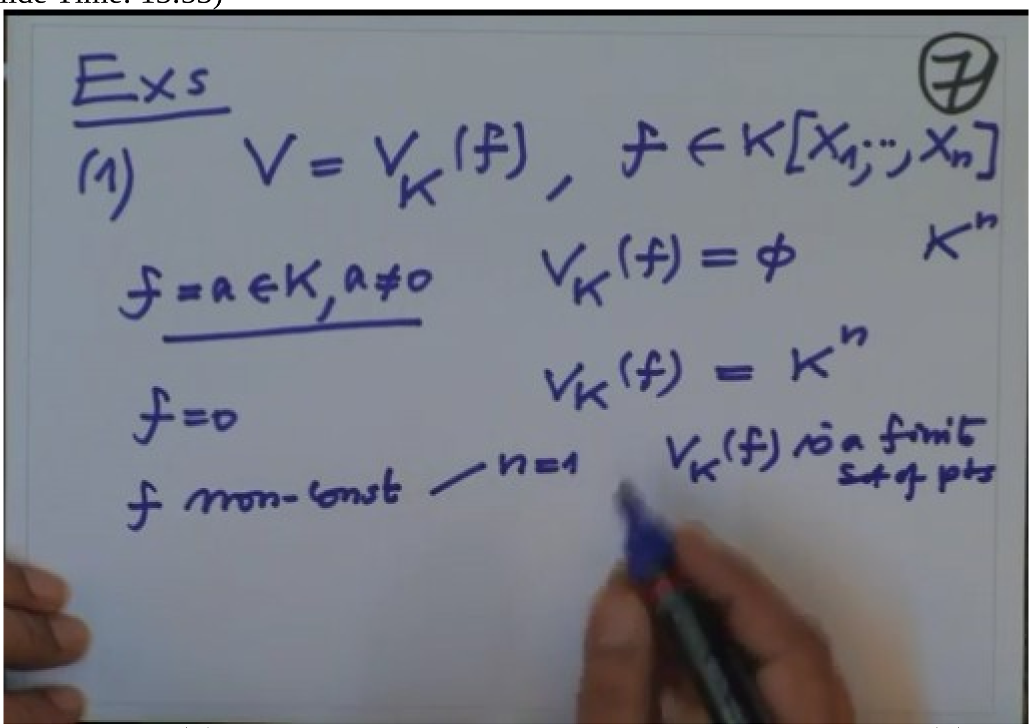

If n = 2 then this $V_K(f)$ is a curve when you try to plot the graph of this $V_K(f)$ that will be a curve, when n = 3 this V_K(f) is a surface, in general, for general n one call this $V_K(f)$ as a hyper surface, hyper surface is something which is defined by only one equation and you call it surface curve depending on whether n = 2, 3, if n = 4 it will be called 3, 4 and so on, alright, so this is one example

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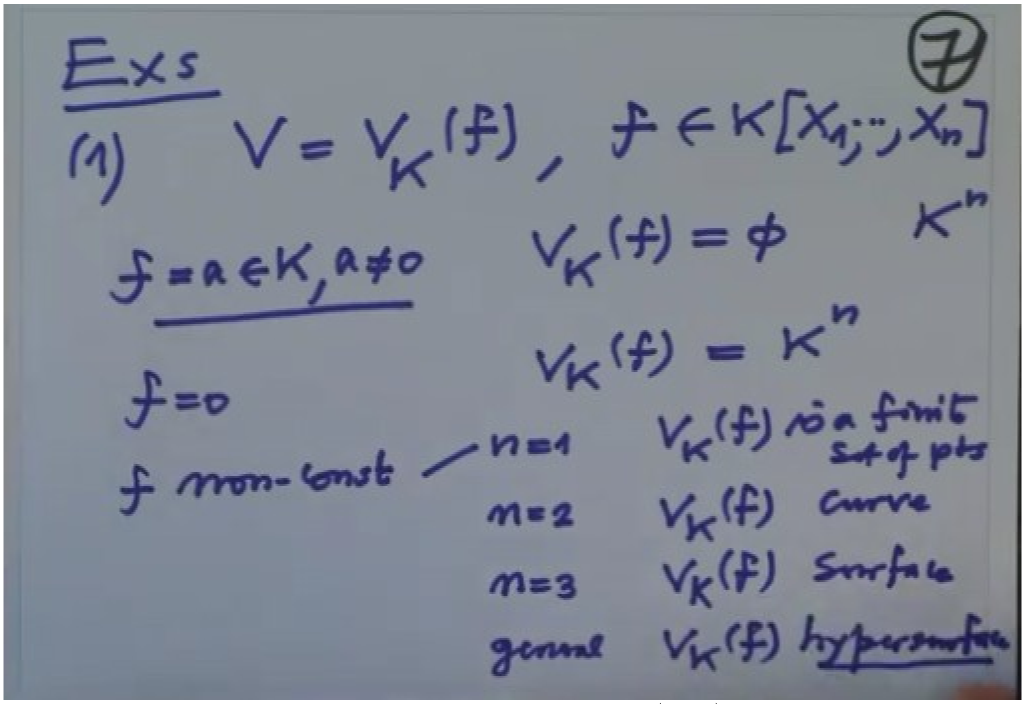

Second example, suppose now at n = 3, and I take $V = V_K(f, g)$ two polynomials in three variables and we are looking at the common zeros, so this is defined by two equations, two equations so f = 0, and g = 0 together, so that will give us the following, so what will be the analysis again? It is either K^3 I'm taking it in K^3, so 3 variable case, so either it is whole K^3 this will happen when this ideal generated by f and g is a 0 ideal, that means f and g both are 0 then you call this will be K^3, otherwise it will be the surface or a curve or empty set, so empty set will come this is precisely means the ideal generated by f and g is a unit ideal, so this ideal is the whole, so it's generated by 1 also so it's a unit ideal, this K is precisely when f and g this polynomials in 3 variables have common factor, have non-constant common factor, common factor, non-constant, that is that case or when will this happen?

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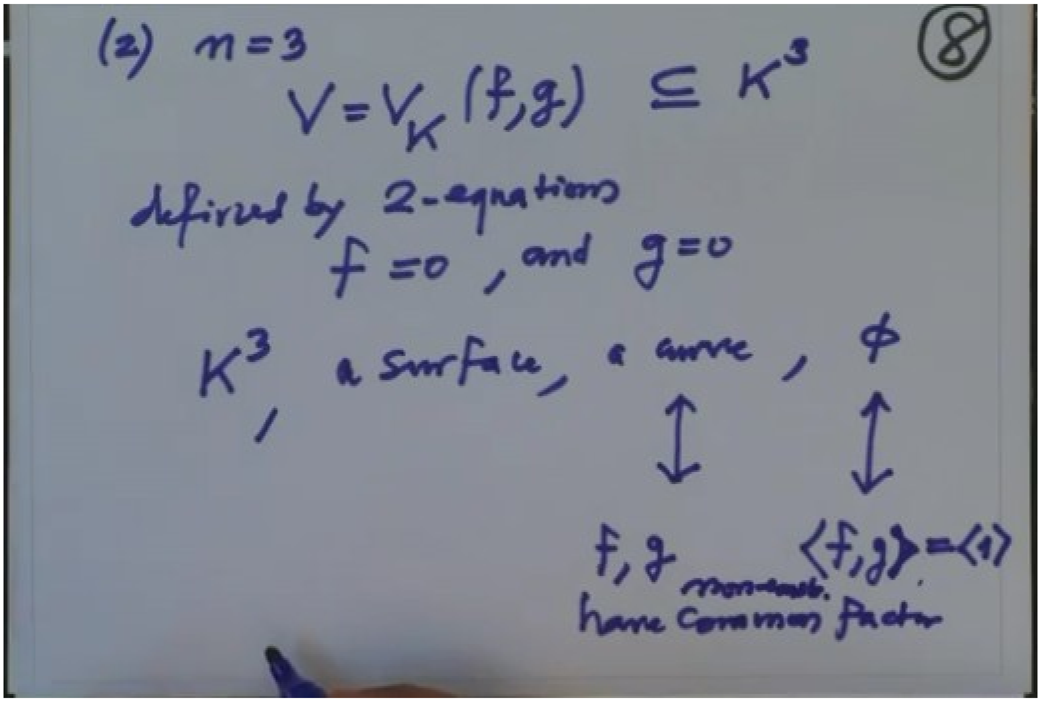

And so you notice that this are the geometry properties curve, surface or whole thing, and these are algebraic properties, so this K^3 will occur for example when this ideal generated f and g is 0 ideal, that means this is equivalent to saying both f = 0 and g = 0, okay.

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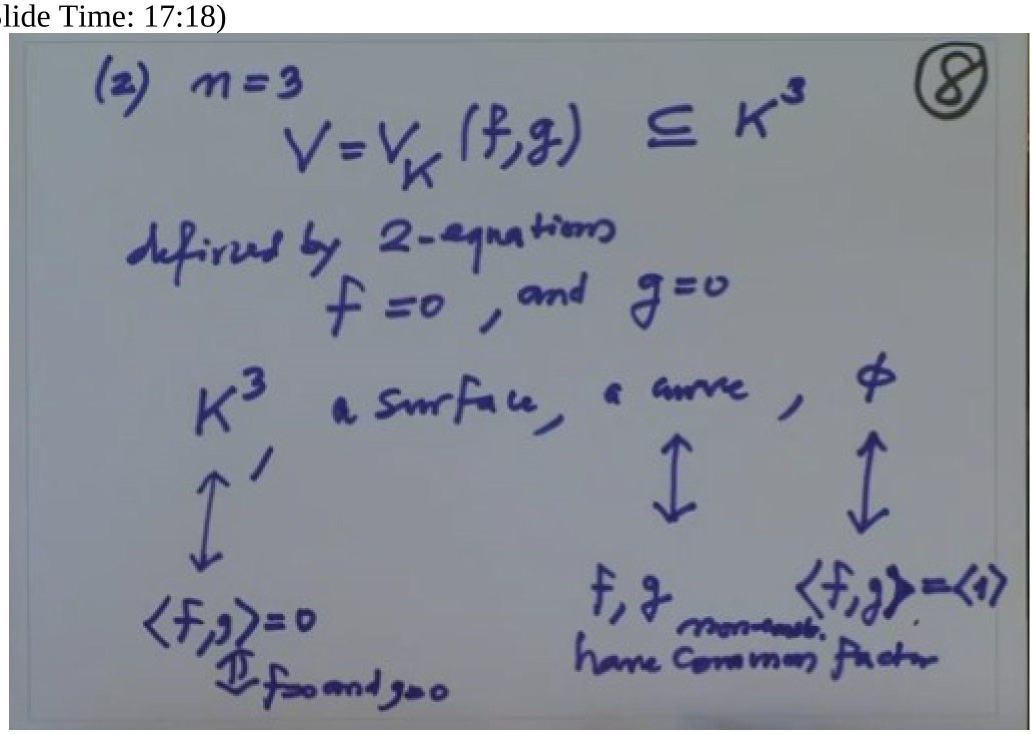

Now in this case the, it will never be finite state of points, so it cannot be finite state of point that needs a little proof but that I'll leave it for you to check it cannot be finite state of points, okay.

Now I want to attach a number, so before I do that I also, so here it is I want to attach the number so this page should be, okay, I want to attach some number to the algebraic set so that and we call it a dimension, and before that when do I say that one algebraic set, when do I say that V which is an algebraic set is irreducible subset, this means you cannot write V so first of all if V should be nonempty and V is not union of two proper algebraic sets, so that means V cannot be equal to V_K of some ideal B union V_K of some C and these are the proper subset that means $V_K(B)$ and $V_K(C)$ they are properly contained in V,

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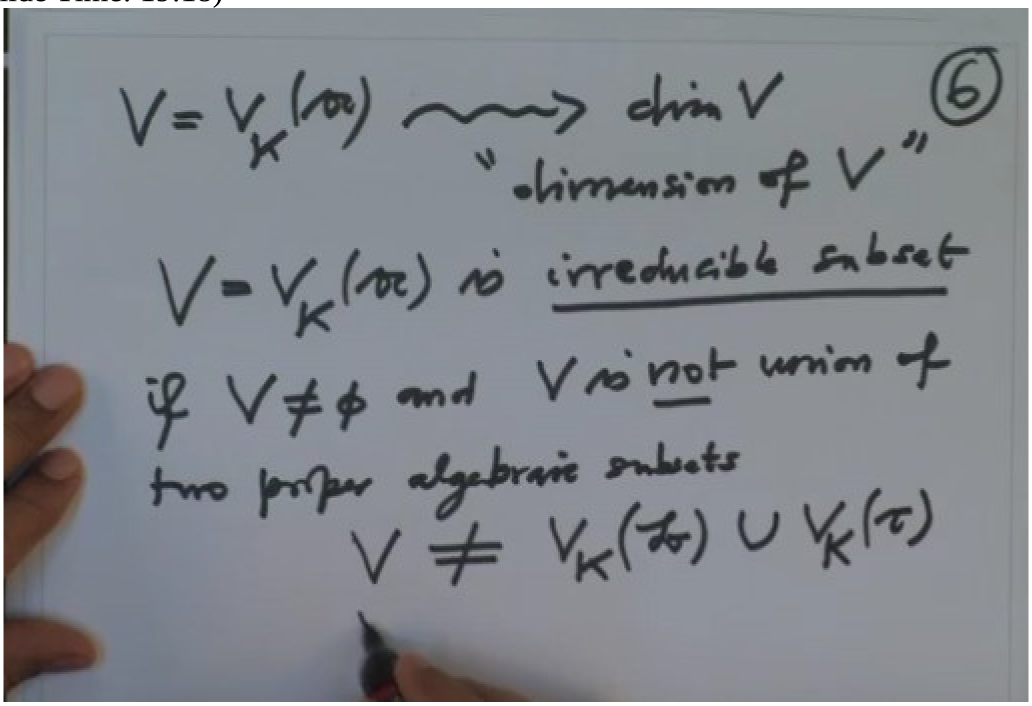

this can safety define for arbitrary topological phase and subset of a irreducible, subset of a topological space is called irreducible, if it is nonempty and it is not union of two proper closed sets then you call it irreducible,

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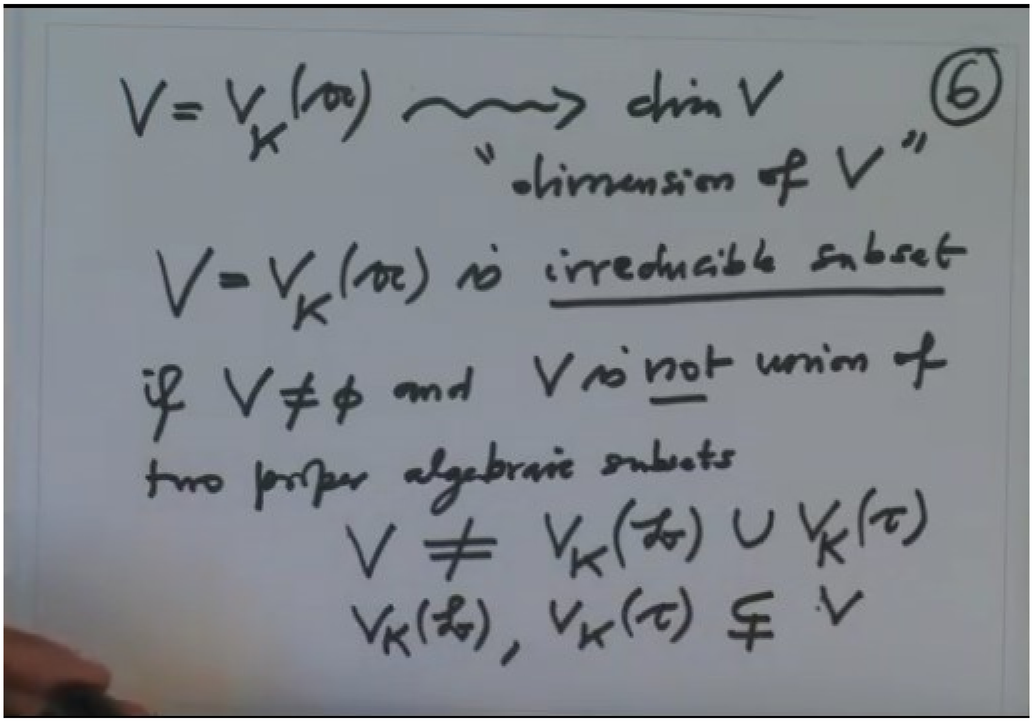

for example in this case, in this example in the plane I have taken two lines this is obviously not irreducible because this is union of this two proper closed sets, this is one closed set, this is another closed set so this is not irreducible, whereas the circle this is irreducible and later on you will see that this property irreducibility in algebraic sets that is equivalent to saying that corresponding polynomial, this polynomial X^2+Y^2-1 this is irreducible in, irreducible polynomial in two variables, so this is a geometric property irreducibility the geometric property but one can explain in terms of algebra this is what I'll do it in general soon, alright.

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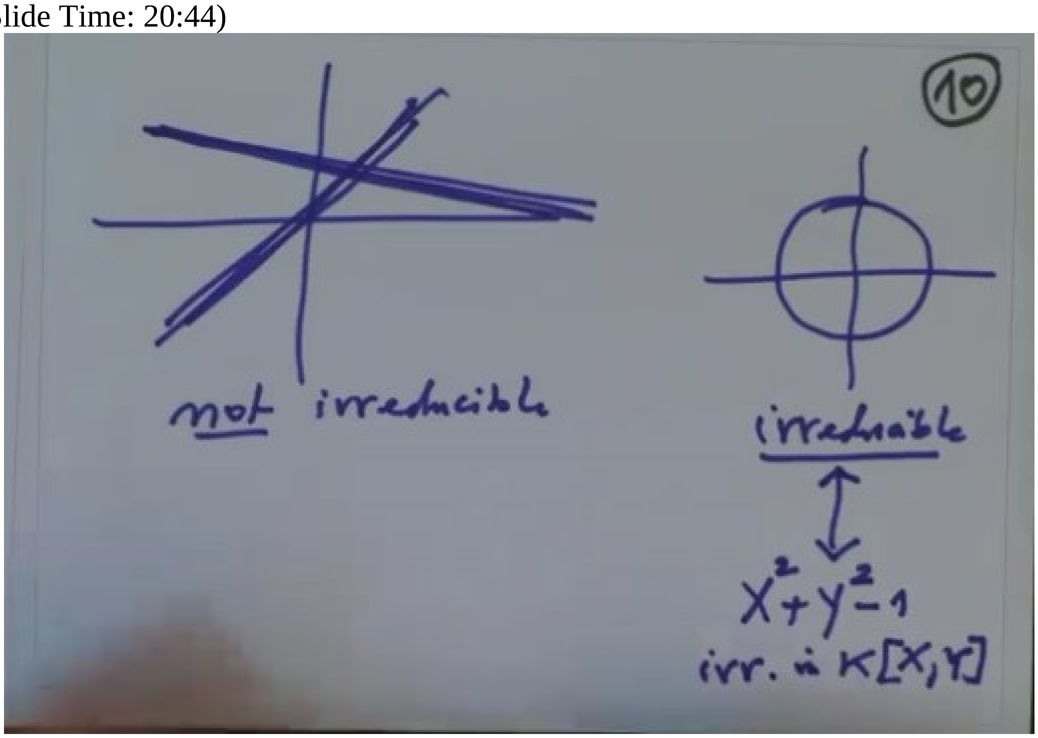

So what is the dimension? Okay, the dimension we want to attach the number, so what are the desirable properties? So these are desirable properties of V going to this function dimension V, I'm writing desirable properties so that we will know what can it be, and we will prove it should be that, so first of all number one that dimension of the affine space K^n should be n, this is also matching with our earlier study that for example as a vector space dimension is n and so on, in $\mathbb{R}^2$ we are saying that 2 dimensional geometry, 3 dimensional geometry and so on, so dimension of K^n should be n, two if V is union of finite union of closed sets V_i 's, i is from 1 to n then the dimension of V should be equal to supremum of dimension of V_i 's, so for example if you take a line union of lines then the dimension should be 1 only because dimension of each line should be 1 and therefore this. This desirable property 2 is based on the thinking that if I take finitely many points you will not make it a curve, if I take finitely many curves you will not make a surface and so on, so this is the one desirable property.

Third one if V is irreducible, V is irreducible and W is a proper algebraic set, algebraic set that the dimension of W should be less than dimension of V, a dimension of a proper subset should be strictly less than dimension of A that is when V is irreducible.

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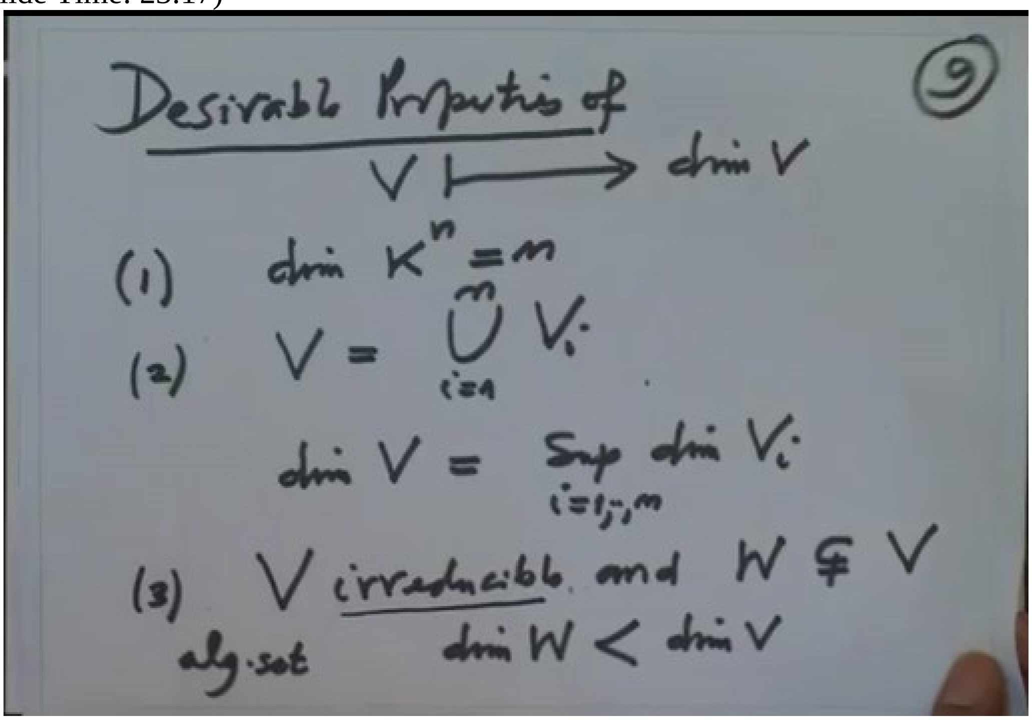

Now the next desirable property is if d is the dimension, if d is, this number is the dimension V and if it is positive then V contains nonempty algebraic set W such that dimension of W drops exactly at 1 which is d-1, so that means so this observation is based on the fact that every curves as points,

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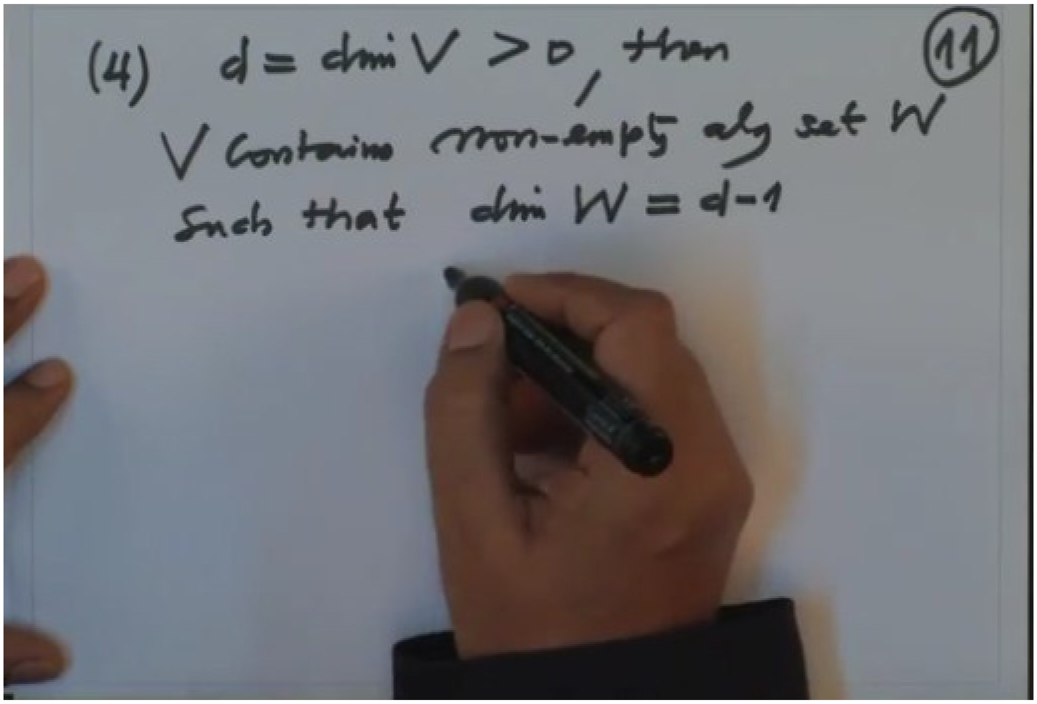

every surface has curves on that and so on.

So fifth, if V is an algebraic set in K^n and d is dimension of V which is positive then for most hyper surfaces, if I intersect V with H the dimension should drop by one, exactly by one, so that is the fifth desirable property.

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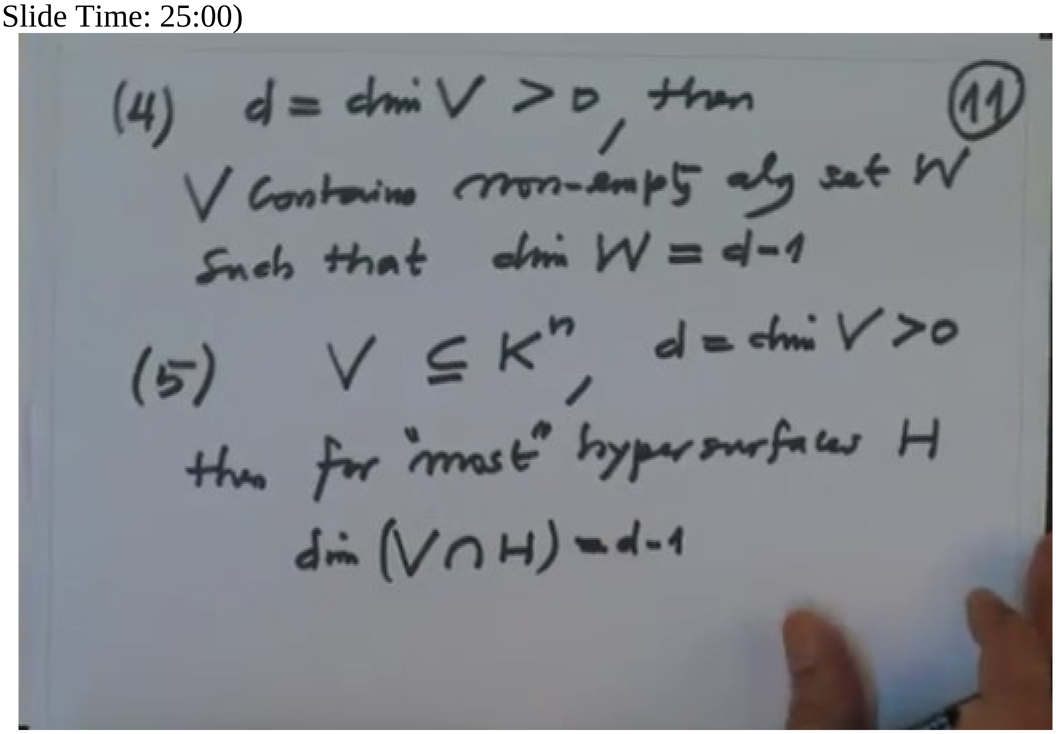

So sixth one, if I have an open subset U open subset of an algebraic variety V, algebraic set V and if it is dense, U dense in V, then the dimension of U and dimension of V should be same, alright.

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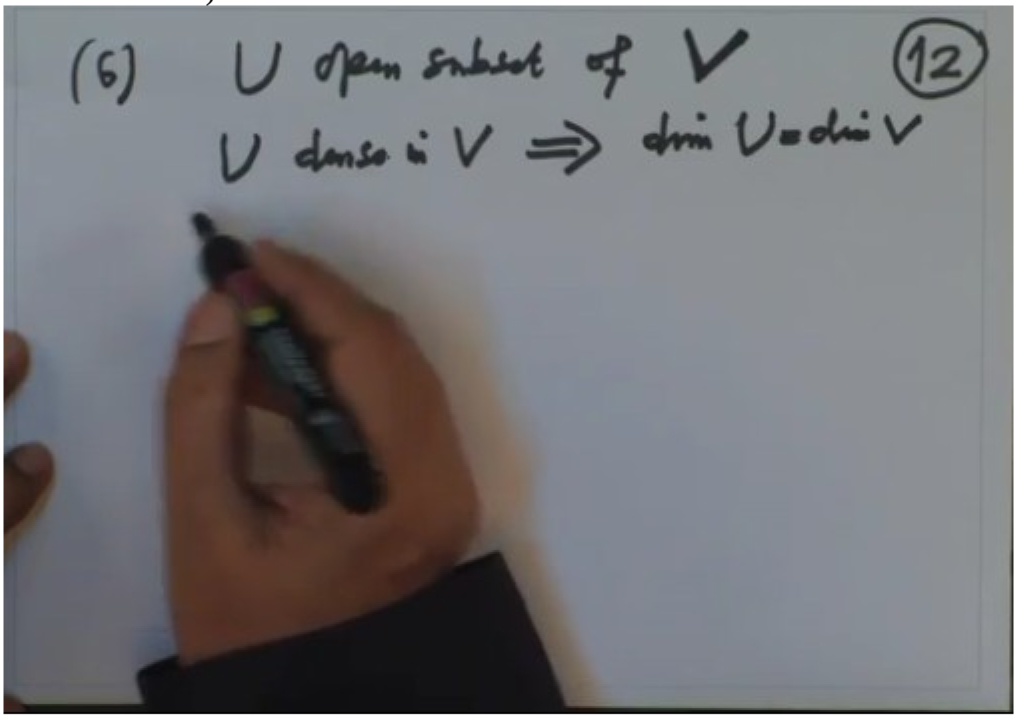

Two more, 7, if d is the dimension of V, then there are exactly d independent functions, functions on V, so for example if V where $\mathbb{R}^n$ then we have n coordinate functions and they are independent functions, so that is the motivation for this 7.

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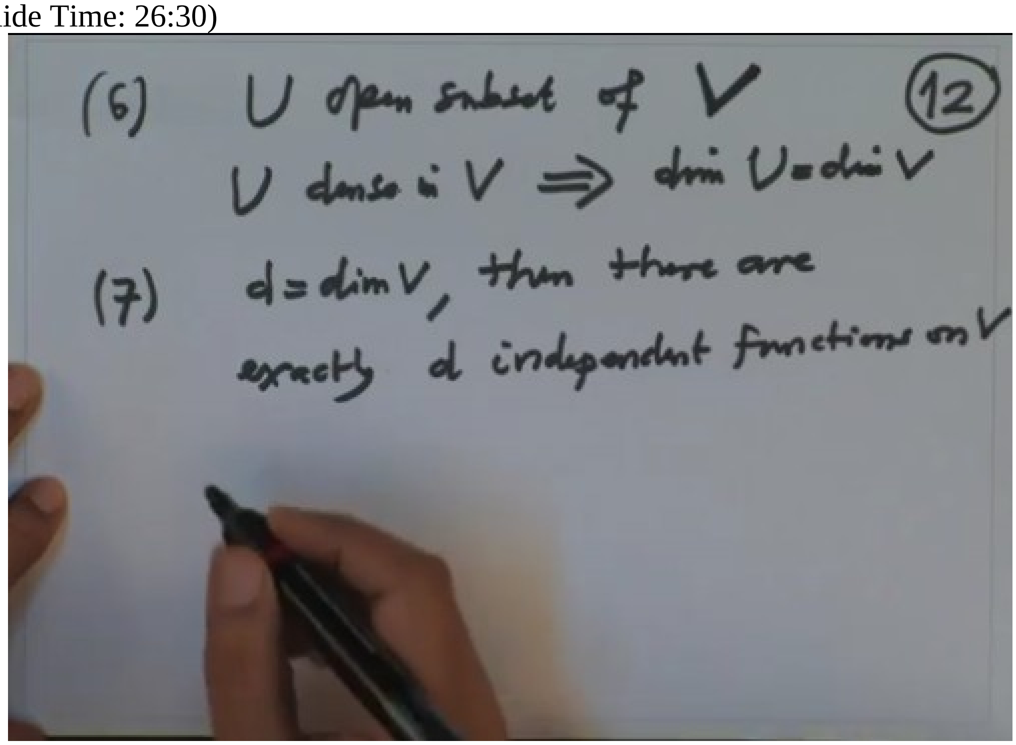

The lastly 8, if d is the dimension V then so this concepts will become clearer when we define them properly, then there are exactly d degrees of freedom on V, so this we will define,

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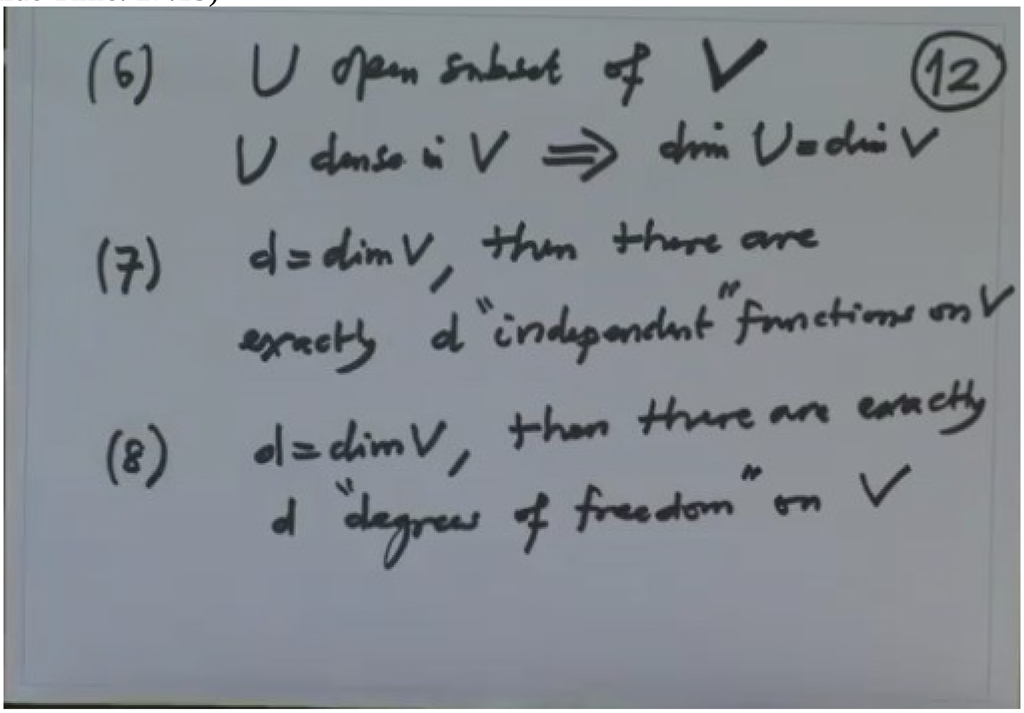

so now I would like to have a dimension so that it satisfy this 7 properties, but already we will show that the properties 2, 3 and 4 determines unique function from algebraic sets to natural numbers, where V is non-empty, for V non-empty, okay, so this we will prove it soon but we will use the fact that this polynomial ring is Noetherian, this is what we will use to check that already if I have the property 2, 3, 4 then it determine the unique function and that is called the dimension,

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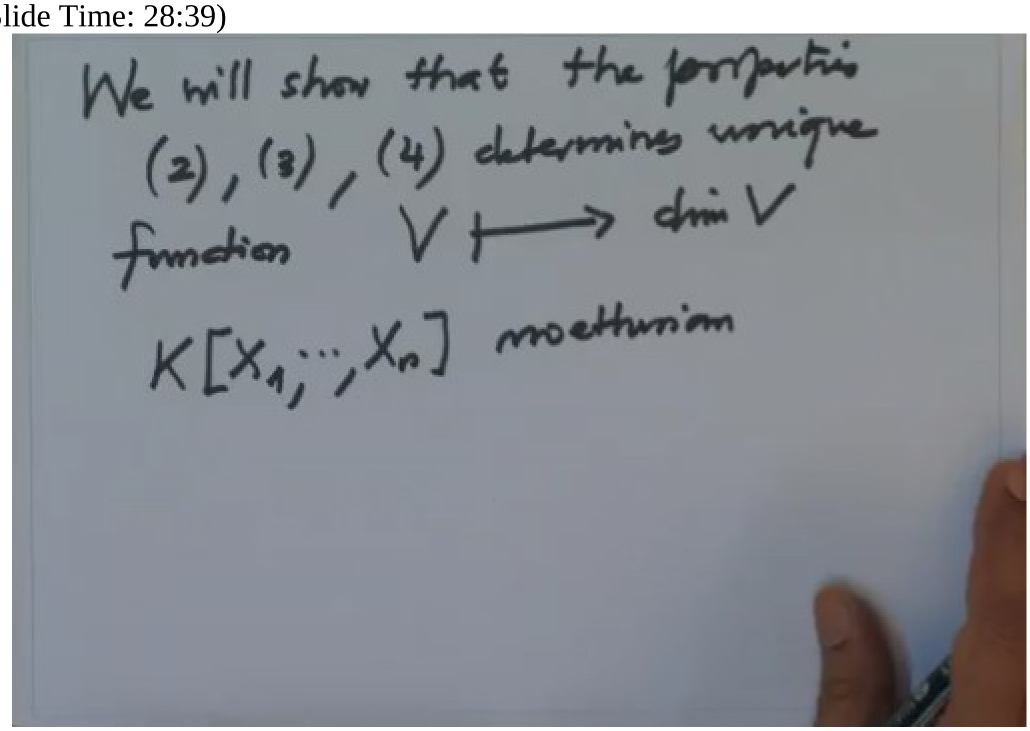

and this after the break I'll show you how it leads to the definition of Krull in terms of the prime ideals, supremum of the lengths of the chains of the prime ideals, this is what I'll show you after the break.

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