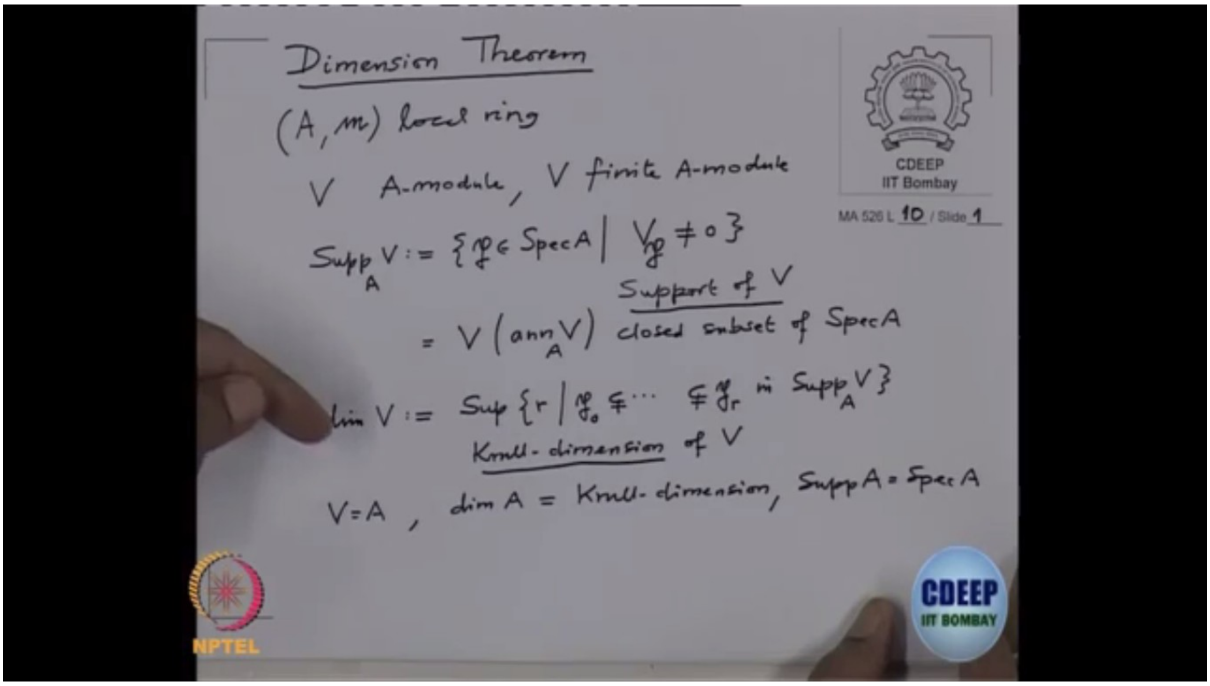
Lecture – 29

Dimension Theorem

Gyanam Paramam Dhyeyam: Knowledge is supreme.

Today, I will prove what is called Dimension Theorem. And I want to prove it for modules actually. So let us set up the notation. So we have a ring A, A is local ring. And V is an A-module and we will assume V is finite A-module. To the module we have the support. So support of V as a A-module this by definition all those prime ideals P, such that V localized P is non-zero. This is called support of the module V. Support of V. And you would have seen in earlier course that this is a closed set. This is V of an annihilator of V, this is a closed subset of Spec of A, remember when you say Spec of A, we considered the Zariski topology on that then this is a close subset. So and the dimension of, Krull-dimension of V is by definition the supremum of r, so that we have a chain of length r in the support of V. This supremum is called the Krull-dimension of V. So V is equal to A then this the definition of a Krull-dimension of the ring A. V equal to A, this dim A this is what we have been taking in earlier lecture, this is the Krull-dimension of A. Because in this case of support is the whole spectrum. Support of A, the Spec of A. And I have been saying that earlier we did not clear that this supremum is finite. So we will prove today it is finite if this ring is local and in that case we will have another two definitions of integers which are attached to V, so also I want to remind the convention.

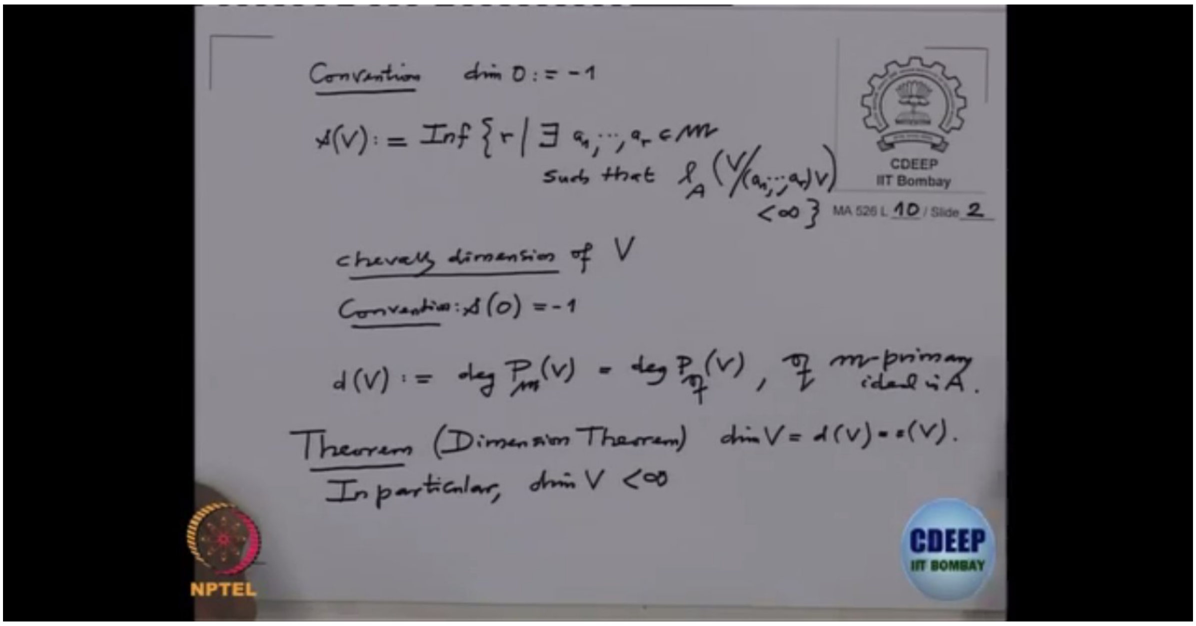
(Refer Slide Time: 03:52)



The dimension of the 0 module is -1. Okay, the another definition is, and also I am assuming V is non-zero. So, in this above V is non-zero. So that's why we have to make a convention and dimension of a 0 module is -1. So, also we defined now and you look at the another number which is called a Chevalley dimension. That is $s(V)$, this is by definition Inf of integer r such that their exist r elements in the maximal ideal of A, such that, when you got Mod ideal generated by a_1 to a_r in the module V that is length of $\frac{V}{(a_1,\ldots,a_r)V}$, this is finite. So that means this residue class module $\frac{V}{(a_1,\ldots,a_r)V}$ this is a finite length. This is called Chevalley dimension of V. this make sense because certainly if I take $a_1,\ldots,a_r$ to be the generating set for the maximal ideal, then this module is $\frac{V}{MV}$ and obviously, this is the finite length module because, in this case, this residue class module as a support only M. And one tests that module is a finite length when the support consist of maximal ideals.

So also., we will put by convention that s of zero-module is -1. This is convention. And remember we have already defined $d(V)$. $d(V)$ is by definition the degree of the Hilbert-Samuel Polynomial $P_m(V)$ or for that matter $P_q(V)$, where q is any primary ideal, m-primary ideal. And we achieved that the this degree doesn't depend on the primary ideal we choose. So we can take which ever primary ideal we need to take. And that theorem says now, these three numbers are same. This is called dimension theorem. We say that the dimension of V is equal to $d(V)$ is equal to $s(V)$. In particular, dim V is finite. That is because it's a degree of a polynomial, r also this also it is clear that this $s(V)$ is bounded by for example, the minimal numbers of numerators for m.

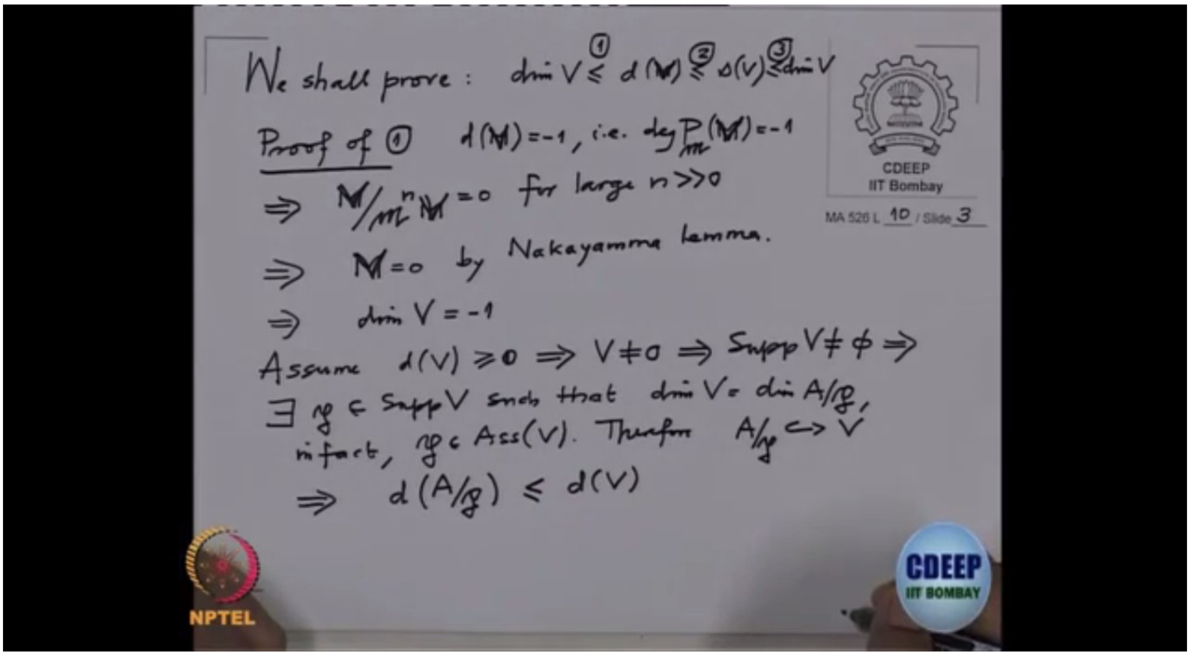
(Refer Slide Time: 08:19)



And because we should also assume, when I say locally by definition that it has only one maximal ideal and it is noetherian. Noetherian is also the part of the definition of locally. So we will prove this theorem so proof of the plan of the proof is, we shell prove the inequalities from dimension of V is less equal to $d(M)$ is less equal to $s(V)$ is less equal to dimension of V. So this prove their equalities. This is, let's call this as one, this is two, so we have to prove this three equalities. Okay. So the first one, proof of one. So, first let us dismiss the case, suppose $d(M)$ is -1, that means this module M is zero. No not zero but $d(M)=-1$ means, so that is the degree of the Hilbert-Samuel Polynomial is -1. But that means m by, that means in zero polynomial for large M so that means $\frac{M}{m^n M}$ is 0 for large n. But that means $M=0$ by Nakayamma Lemma because we are in the local ring, M is finite or why would I change it M now V.

So this V, and this is also V, everything is V here. V is 0 by Nakayamma Lemma. But V zero then we have put by definition dim of V is -1 by convention. So that proves this is equality. If this is -1, this is -1. That is what we have to prove. Now, assume $d(V)$ is bigger equal to 0. That will imply V is non-zero and that means support of V is non-empty. So their exist a prime ideal p that implies, their exist p in the support, such that dimension of V is equal to dimension of $\frac{A}{p}$. Remember we have put this dimension to be, dimension of V to be the length of the chains of the prime ideals in the support. So, I take the, at the left end I will take the more so that we cannot go on for that. So, That is dimension is, dimension of $\frac{A}{p}$ because if the lower end of the chain is some prime ideal p, I choose so that this is a maximum attained there. So it is dimension of $\frac{A}{p}$. In fact this p we can choose in the associated primes of V. Because do you remember the minimal prime, minimal elements in the support and minimal elements in the associated primes of same. So, therefore p is in the associated primes so that is, that means $\frac{A}{p}$ is a sub-module of V because associated prime means it is annihilator of some element. And so it is $\frac{A}{p}$. $\frac{A}{p}$ is a sub-module of V. So that will mean that the degree, you remember when somebody is a sub-module in the last lecture we have seen the Lemma's that the degree of the Hilbert-Samuel polynomial for $\frac{A}{p}$ and degree of Hilbert-Samuel polynomial for V is less equal to this. So with this it is enough to prove.

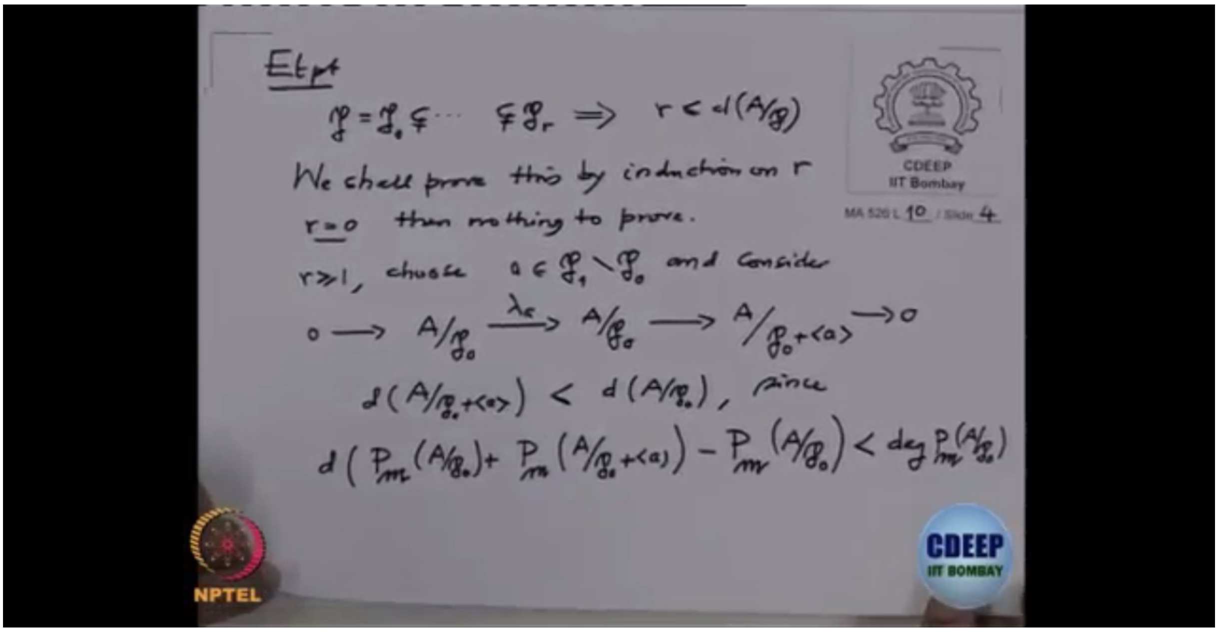
(Refer Slide Time: 14:25)



Now to prove that in locality one it is enough to prove that, if I have a chain like this, p is equal to p_0 containing proper length of chain r then this r should be smaller equal to d of $\frac{A}{p}$.

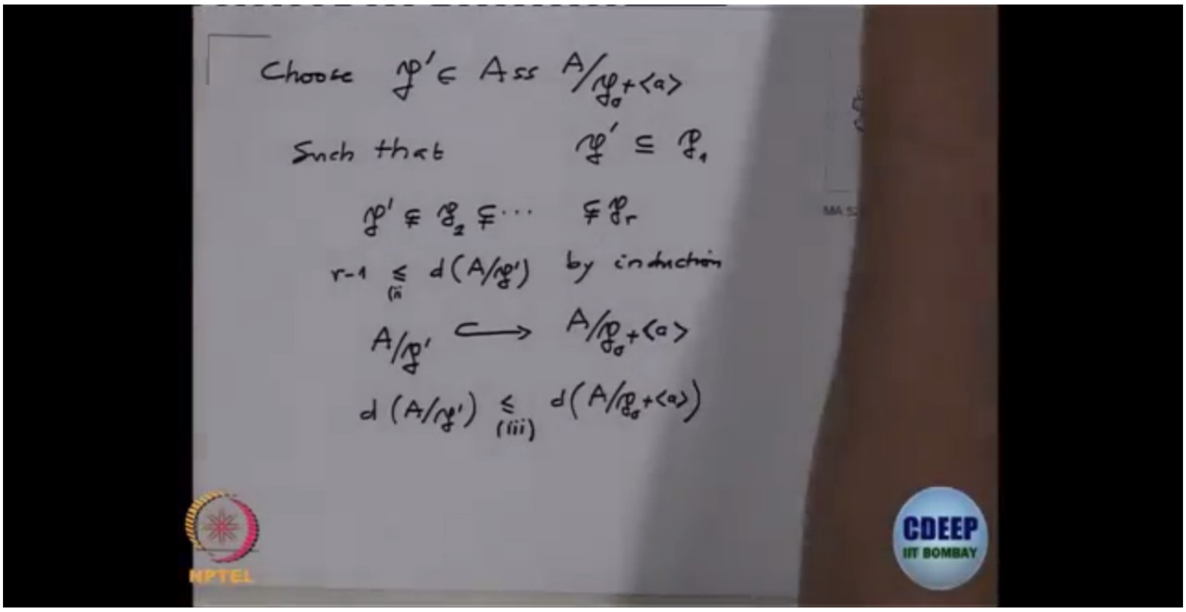
So this we shell prove, we shell prove this by induction on r. Okay. So, if r is 0 then there is nothing to prove. If r is bugger equal to 1. Then chose an element a in p_1 which is not p_0 . That we can choose because this chain is proper and consider the exact sequence. 0 to $\frac{A}{p_0}$ and multiplication by $\frac{A}{p_0}$ and multiplication by a, one $\frac{A}{p_0}$, so this make sense because this is a multiplication by an element a, and this element a is not in p_0 , $\frac{A}{p}$ is a integral domain. So a is non-zero element there, so multiplication map is injective. And the co-kernel is $\frac{A}{p_0+\langle a\rangle}$. So we have this exact sequence. And then one to have an exact sequence, so then we know that d of this $p_0+\langle a\rangle$ is strictly smaller then d of $\frac{A}{p_0}$, since we have proved that if the take the alternating some of the Hilbert polynomials and take the degree that degree is smaller than the degree of the middle one. So since d of P_m middle one, that is $\frac{A}{p_0}$ + this one $P_m(\frac{A}{p_0+\langle a\rangle})-P_m(\frac{A}{p_0})$ this degree is strictly smaller then degree of the middle, that is $P_m(\frac{A}{p_0})$. What do you see this is same polynomial, so the leading coefficient will get cancelled. So therefore this degree is smaller equal to this degree. So that is what it means that d of that is strictly less than this. Okay. Now, once you have proved this.

(Refer Slide Time: 18:34)



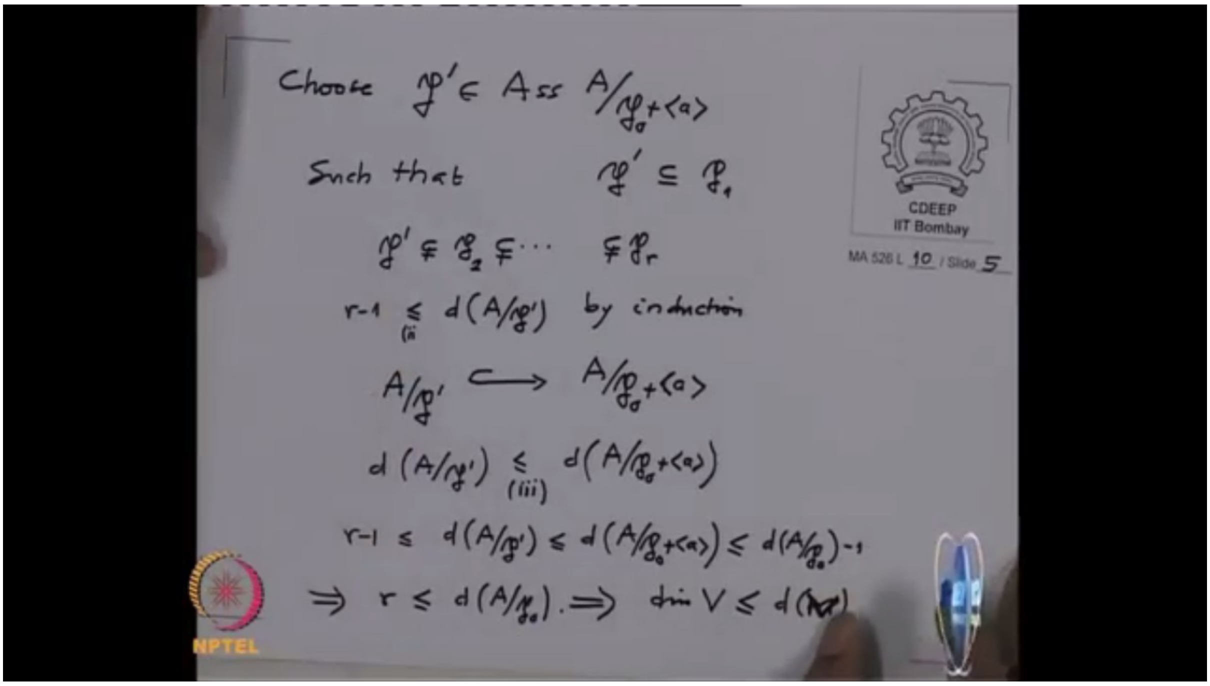
Now, choose prime p' in the associated primes of $\frac{A}{p_0+\langle a\rangle}$. remember I am trying to choose sequence of elements which we generate so that the module V by those elements we'll have a finite length. So this, choose associated prime, p' of $\frac{A}{p_0+\langle a\rangle}$, such that, p' is containing p_1 . Remember this ideal is in between p_0 and p_1 because a was in p_1 . So p_1 may not be associated if I choose a prime which is containing p_1 and which is associated with prime. And look at this sequence. Now I removed p_0 and p_1 from the chain and look at p prime which is containing p_2 , this is not equal because this is containing p_1 and so on. Look at the chain, and now induction that $r-1$ is, this length is $r-1$ because it starts with, which is starting with the some-- So it's length is $r-1$. So this is less equal to d of $\frac{A}{p'}$ by induction. And now, you look that this is associated prime so there is an injective map, this is $\frac{A}{p'}$ is a sub-module of $\frac{A}{p_0+\langle a\rangle}$ and therefore d of this, is smaller equal to d of this. So now, we have inequality, so this is, this is one, let us call this as two, this as three and also we have proved, this is one.

(Refer Slide Time: 21:20)



So r is less equal to d of $\frac{A}{p}$. And this I want to combine to get $r-1$ less equal to d of $\frac{A}{p}$ prime and this one less equal to this one, d of $\frac{A}{p_0+\langle a\rangle}$ and this is, now the last one that the first one, that is d of $\frac{A}{p_0}$ - 1, because that one I should put the other side. So therefore all together that implies r is small equal to d of $\frac{A}{p_0}$, that is what we wanted to prove. That is the first inequality. So I proved that if I am doing any chain on prime ideals of length r in the support then that is bounded by this degree , so therefore that proves that dim of V is less equal to d of M or d of V. Because we have chosen p_0 with that property.

(Refer Slide Time: 22:49)



Okay. That proves the first inequality.