Lecture – 28

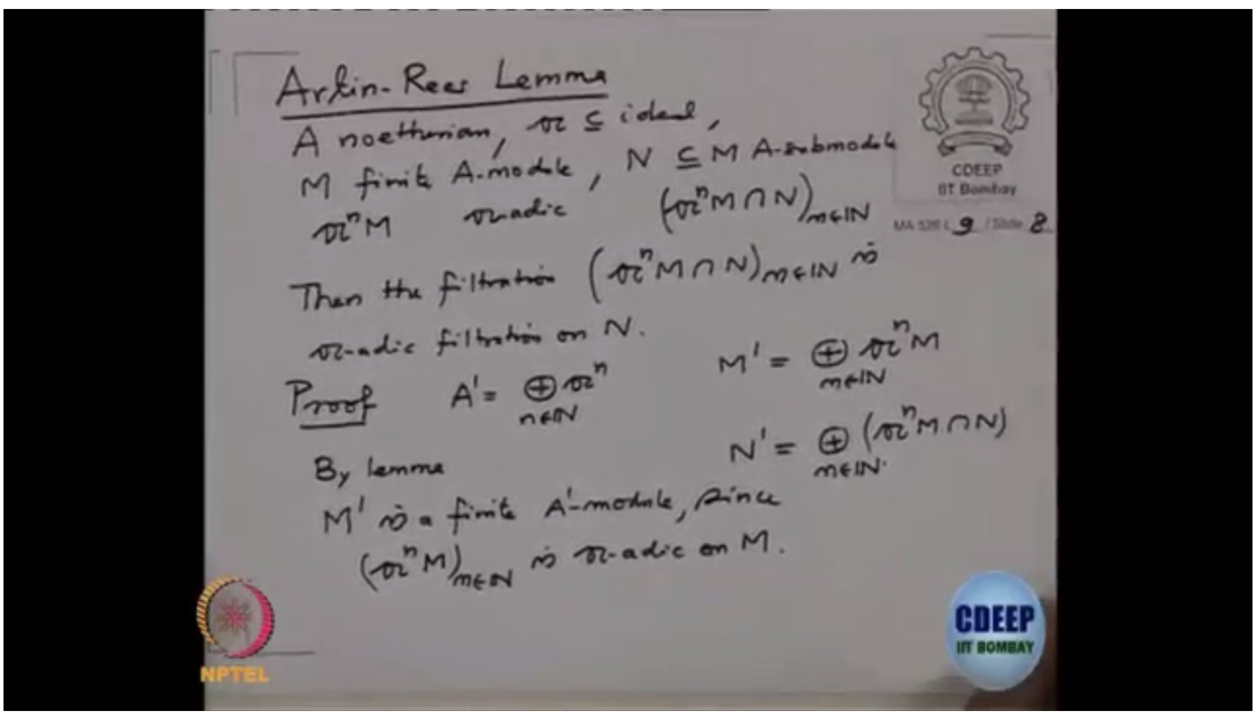
Artin-Rees Lemma

Gyanam Paramam Dhyeyam: Knowledge is supreme.

Okay, so now let us come to Artin-Rees lemma. So this is Artin-Rees lemma. So it's A Noetherian, A is an ideal A. M is finite A-module and N is a sub module of M, A-sub module. And we take this filtration on M which is given by the powers of A, so that is $A^n M$, this you know, this is A adic on M and we take this induced filtration, i.e., you take this $A^n M$ and intersect with N. so I can see that this filtration on the sub module. And I want to prove that this is A-adic.

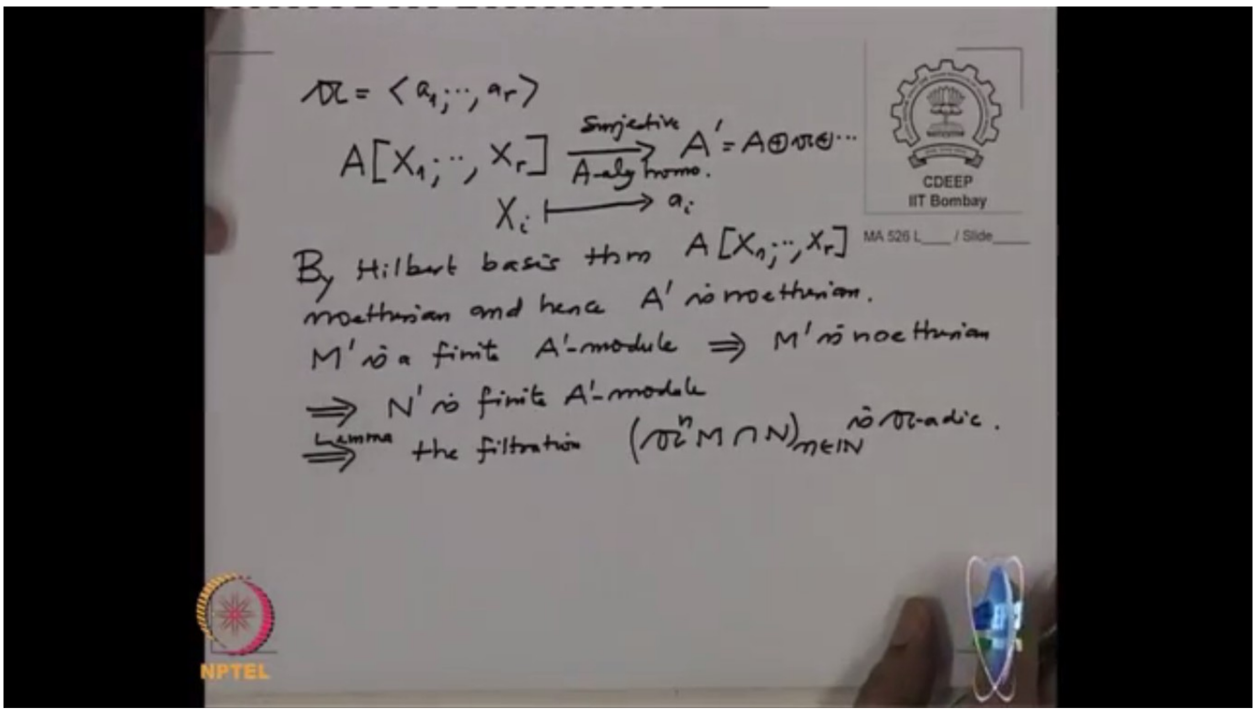
So then the filtration $A^n M$ intersection N, M in N is A-adic filtration of, on N, on the sub module. So, proof. So for about the filtration, to prove some filtration is A-adic, and the lemma says that we have to check that the module, the graded module is finite or the corresponding graded ring. So what do the corresponding graded ring is precisely $A^{'}$ is our ring direct some A^n, $M^{'}$ is $A^n M$ and $N^{'}$ is direct some $A^n M \cap N$. and we know that because this filtration $A^n M$ is a-adic on M, this $N^{'}$ is a finite $A^{'}$ module since $A^n M$, this filtration is A-adic on M and by lemma. So by lemma.

(Refer Slide Time: 04:00)



And note that this ring A prime, ideal A is in the noetherian ring A, so ideal a is finitely generated. So let's say it generated by a_1 to a_r and therefore we have a map from the polynomial being $A[X_1, \ldots, X_r]$ to $A^{'}$. Namely if I give you a map on a polynomial ring, so many variables with base ring A, then I just have to give values in a indeterminate. So let X_i map to a_i. This is also algebra, this is algebra and this map is clearly subjective. Because A' has an algebra over A generated by the first component. It's a standard graded algebra. See, this is A direct some A and so on. So it is generated by the degree of one elements. So anyway this map is A algebra homomorphism which is surjective. Therefore, by Hilbert basis theorem, because this is Noetherian, the homomorphic image of a Noetherian ring is Noetherian. So by Hilbert basis theorem, $A[X_1, \ldots, X_r]$ is Noetherian and hence A' is Noetherian. So A' is a Noetherian ring and M' is a finite module over A', therefore, M' is finitely generated. So M' is a finite A' module. And this M' is a sub module of that. It's a finite module in a Noetherian ring so M' is Noetherian. Therefore N' is finite A' module and that means by lemma, the filtration a power n M induced filtration on N is A-adic.

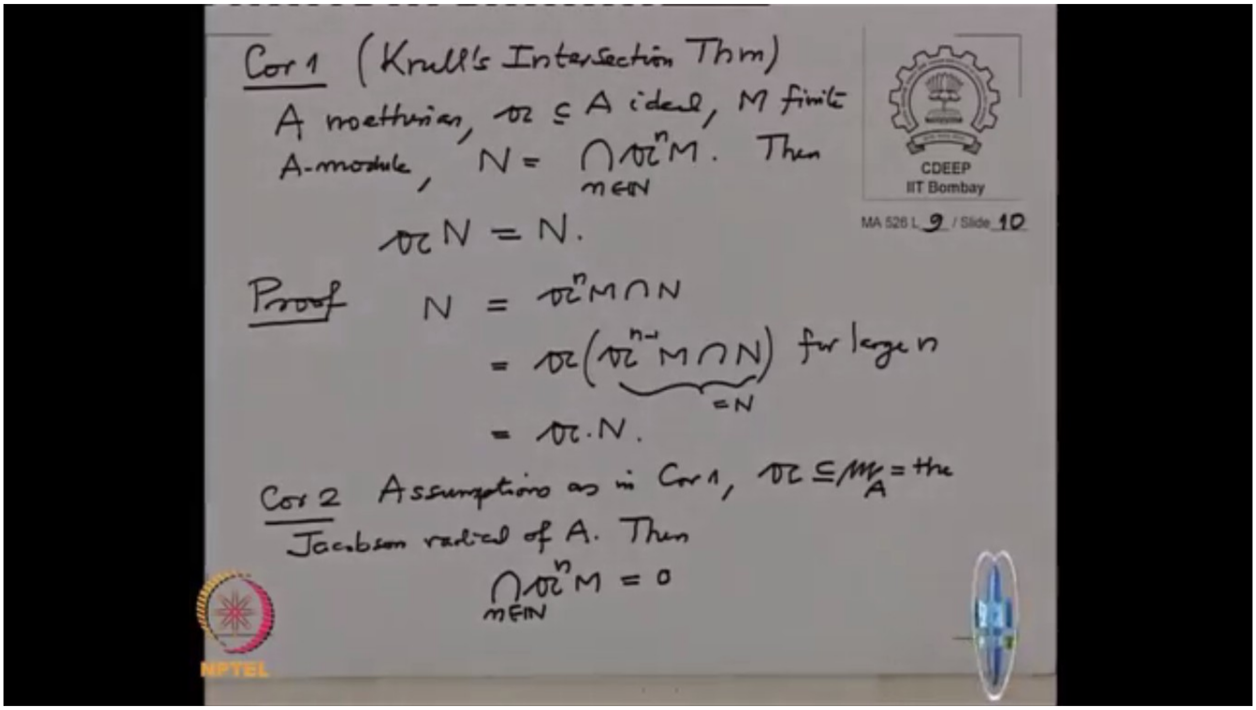
(Refer Slide Time 07:15)



Okay, now let me deduce some corollaries from here. So, corollary 1, that is usually called Krull's intersection theorem. Okay, so A noetherian , a even ideal in A and M is a finite A-module and N is the intersection of all $A^n M$, all the terms this filtration, A-adic filtration. A'. Then, $A N$ is equal to N. Proof: We got N, I want to prove, start with N, obviously if I take A' intersection with N, because only the intersection of all of them, this is same. This is a bigger set than N, so when you intersect, you get the same thing. But for large n, this filtration is A-adic means, this means it is $A A^{n-1}$ and intersection and for large n. But again, this is n. So this is same as $A N$. That is what we wanted to prove. Equal to N.

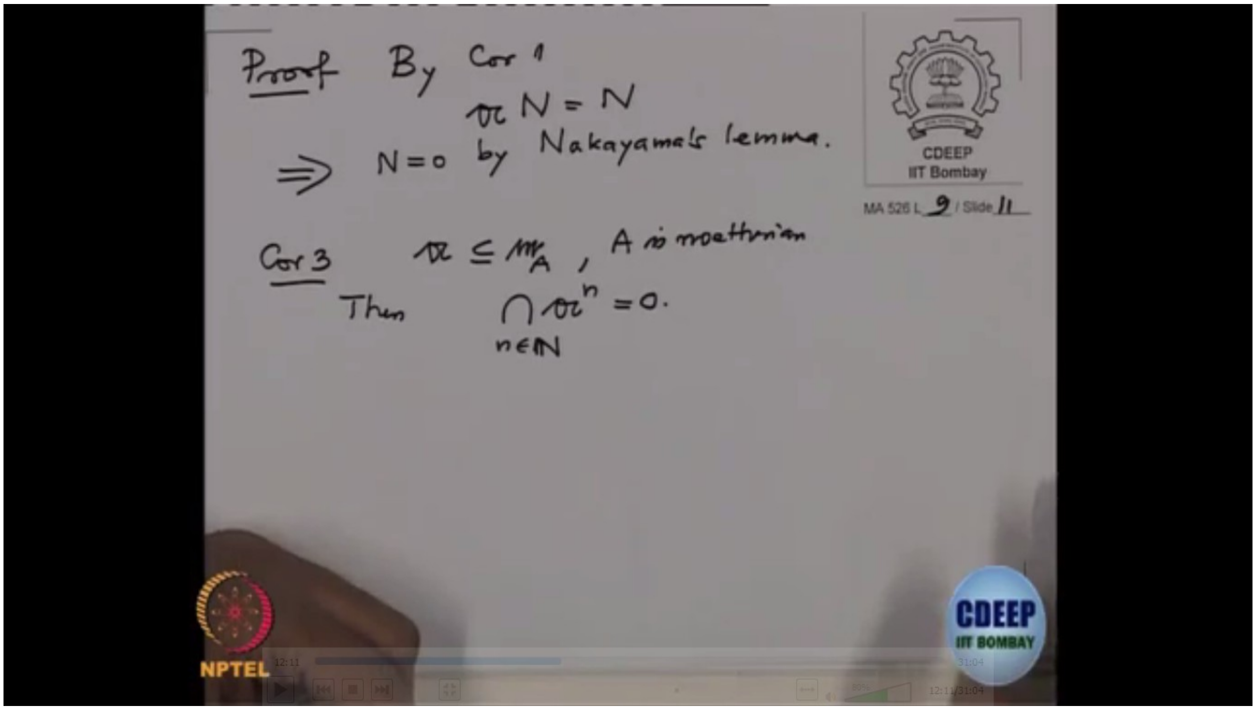
Okay, Corollary 2. Actually this is called Krull's intersection theorem. This is also, some people call it Krull's intersection theorem. So assumptions as in Corollary 1, and also I will assume that this ideal a is containing Jacobson radical of A. This is standard notation for the Jacobson radical of A. Then, intersection $A^n M$, this is 0. This is immediate form Corollary 1 because we know $A N$ equal to N. we know N is sub module of finite module over a noetherian ring. So it is noetherian. And Nakayama lemma, with Corollary 1, will tell then N is 0, then if we can say this is 0.

(Refer Slide Time 10:52)



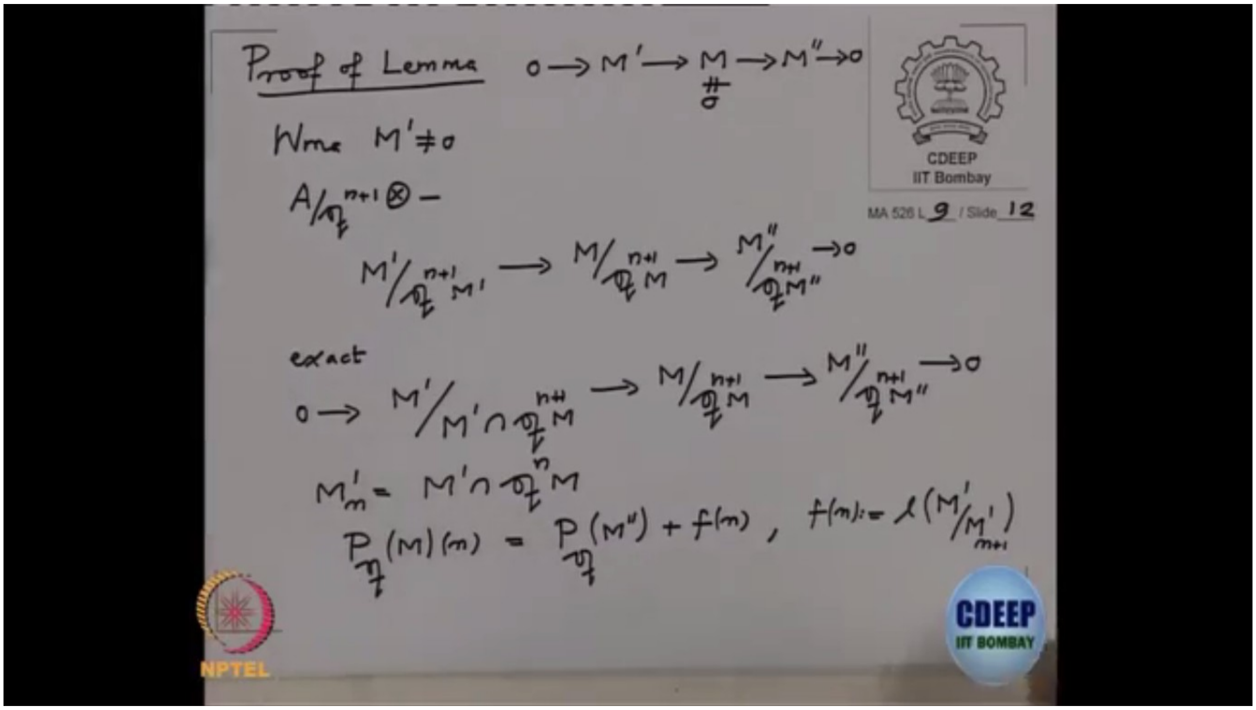
so let me just write, so proof, by Corollary 1, $A N$ equal to N and N is finitely generated, a contains of some articles, so on. So therefore, N is 0 by Nakayama lemma. Okay. So next Corollary, this is a special case of Corollary 2. Namely, suppose you have an ideal A contained in m_A, the Jacobson radical and of course we are assuming A is noetherian. Then, intersection A^n, this is 0. See, this Corollaries are very important when one studies, you see, if each filtration will give the topology on the ring A.

(Refer Slide Time 12:12)



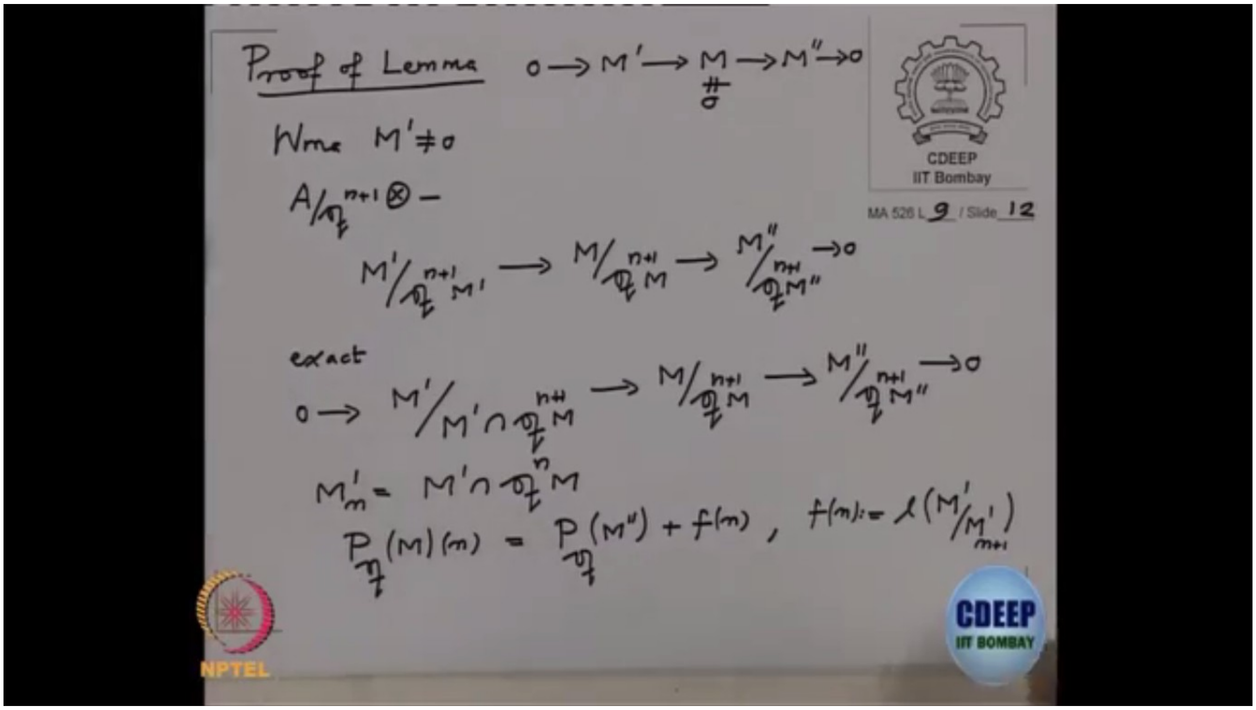
metric even and then we went on to study this metric space, so when completed the metric and then you, on the metric completion, there is a ring structure and then study this complete local ring, needs a complete ring, complete local ring. And the properties of this ring associated with the properties of A, it's an interesting study which will also give some analytic considerations, okay. So now let us get back to our lemma which we wanted to prove. So the lemma we wanted to prove was, we have an exact thing. So proof of the lemma I am writing. So we had given an exact sequence of, short exact sequence of A modules. A is local now and M is a maximal ideal and Q is M primary ideal. So such exact sequence. And we have given the middle one is non zero. So first is also we can, we may assume the M' is non zero, otherwise nothing to prove. If M' is 0, then this will be heteromorphic and then they are the same polynomials so the degree will definitely drop. So we can assume M' is also non zero. Now when we need to ensure the sequence with $\frac{A}{q^{n+1}}$, then you get tensor product is not a left exact so we will only get from the right side. So we'll get an exact sequence like this. $\frac{M'}{q^{n+1}M'}$ to $\frac{M}{q^{n+1}M}$ to $\frac{M''}{q^{n+1}M''}$ to 0. This is exact. But to make it exact, I will intersect, I will go further module, so to make it exact I will have to intersect this with M'. Yes, so we'll get exactly, let me write this, 0 to $\frac{M'}{M' \cap q^{n+1}M}$ to $\frac{M}{q^{n+1}M}$ to $\frac{M''}{q^{n+1}M''}$ to 0. When you compare the lengths and all, I think we came to this equation also last time. We put this is equal to, we put M'_n to be equal to $M' \cap q^n M$. so from here we will get equation like this. P_q , again, alternating. So $P_q(M)(n) = P_q(M'') + f_n$, where f_n , I will put f_n is length of $\frac{M^{'}}{M_{n+1}^{'}}$. And that also has a finite length because you see the support of this consist of only maximal ideal, therefore it will be finite length and so on. So we have put this,

(Refer Slide Time 16:47)



Then Q from here, we will get equation like this. $Q=P_q(M^{'})+P_q(M^{''})-P_q(M)$. this is what we are interested in finding the degree of this. And this is equal to $P_q(M^{'})-f$. And so we need to prove now that so the claim is, $P_q(M^{'})$ and f have the same degree and same leading coefficients. Because if you prove they have the same degree and same leading coefficient, the degree will get canceled and then the degree will drop. So once you prove the claim, the same will follow. So you'll have to prove the claim. Okay. So we have this, so just write down the definition. So that means you want to prove that $P_q(M^{'})(n)$, this is the length of $\frac{M^{'}}{q^{n+1}M^{'}}$ and f_n is the length of $\frac{M^{'}}{M_{n+1}^{'}}$. So now, this is where I have to use Artin lemma. Artin-Rees lemma says by Artin-Rees, this $q\,M_n^{'}$ will be equal to $M_{n+1}^{'}$ for large n. This is precisely the, so remember, did I? Yes. Where I put $M_n^{'}$, remember did we put? That was here.

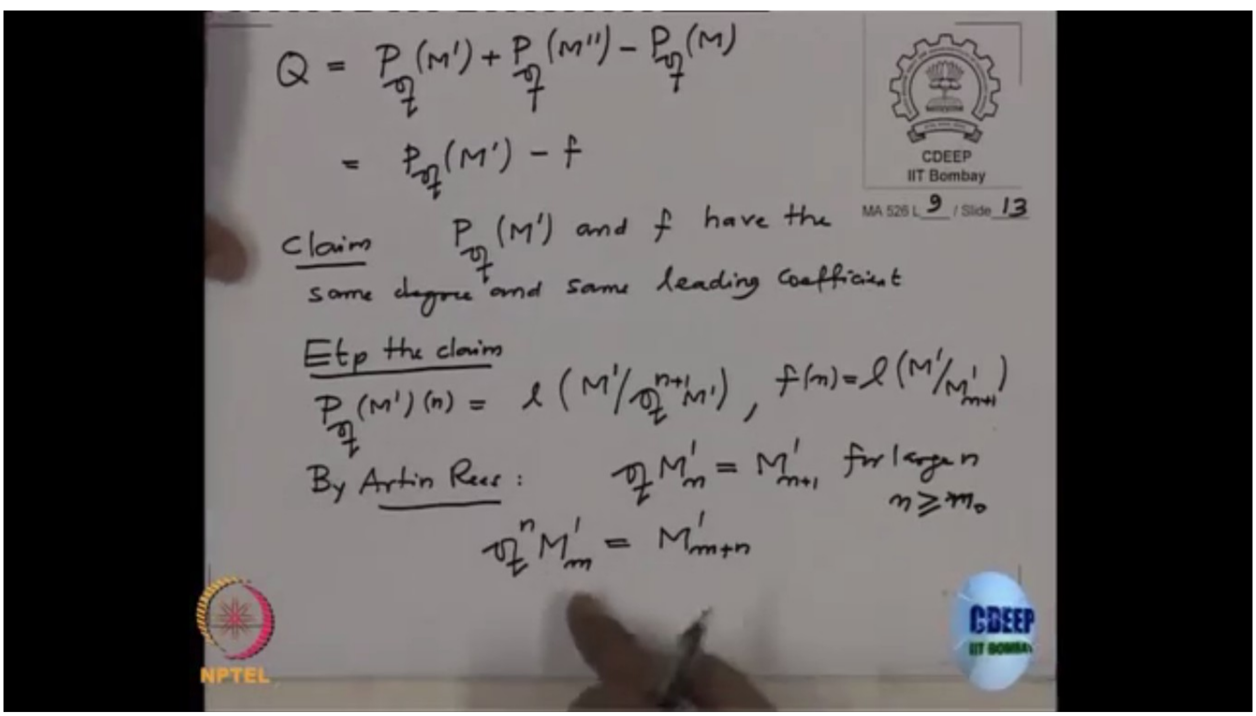
(Refer Slide Time 20:00)



M prime n is this the induced filtration, induced q-adic filtration on $M^{'}$.

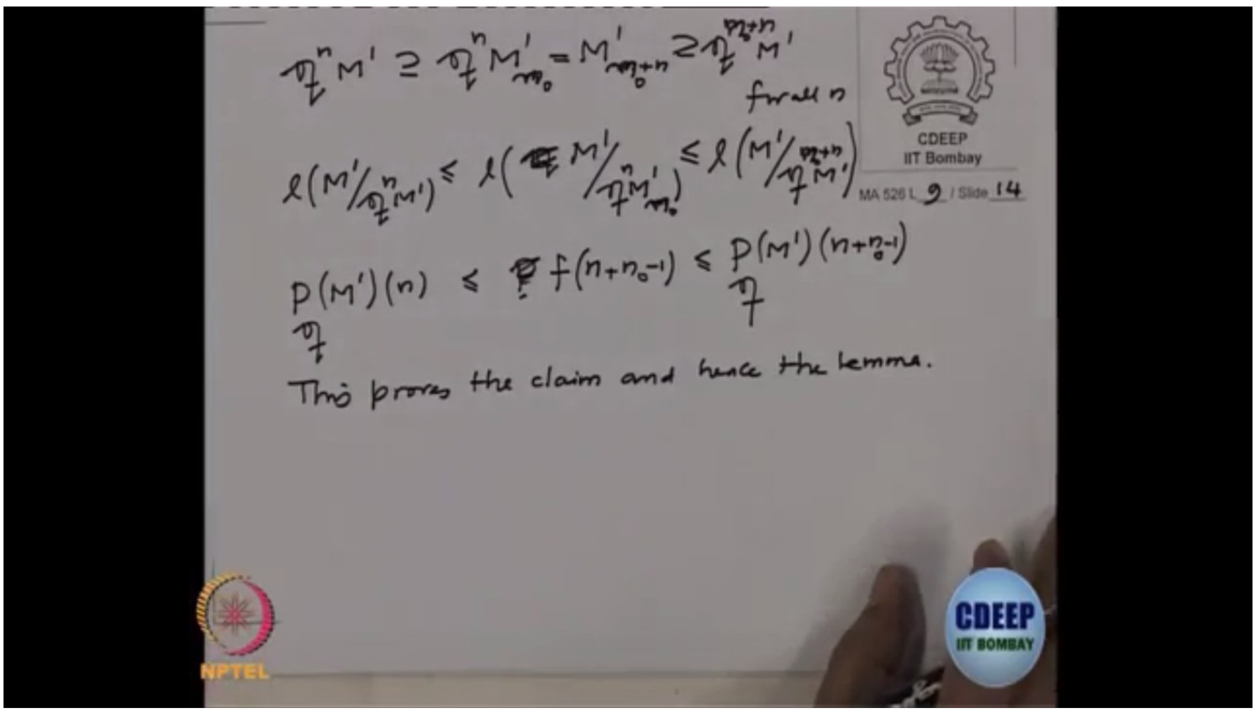
So because of Artin-Rees, this is M-adic. So that means this and therefore, so this is true for all n big or equal to some stage n_0 . So therefore if I take $q^n M_m^{'}$, this is same as $M_{m+n}^{'}$. Just keep applying this. So you get,

(Refer Slide Time 20:51)



so because of this, $q^n M^{'}$ this contains $q^n M^{'}_m$ which is equal to $M^{'}_{m+n}$, which contains $q^{m+n} M^{'}$, this is for all n. So the length when I go mod this and compare the lengths, you will get length of $\frac{M^{'}}{q^n M^{'}}$, this is will be less equal to length of $M^{'}$ mod this. This is less equal to length of $\frac{M^{'}}{q^{m+n} M^{'}}$. Actually that n, this m is n_0 . So therefore this is nothing but this one is $P_q(M^{'})$ evaluated at n, this is less equal to P, this is not P, this is f, $f(n+n_0-1)$ and this one is $P_q(M^{'})$ evaluated at $n+n_0-1$. So this is a polynomial in n and this is a polynomial, n_0 is fixed, neither polynomial, neither translated polynomial. So therefore so this is caught in between that. So that proves the claim. So this proves the claim. And hence the lemma.

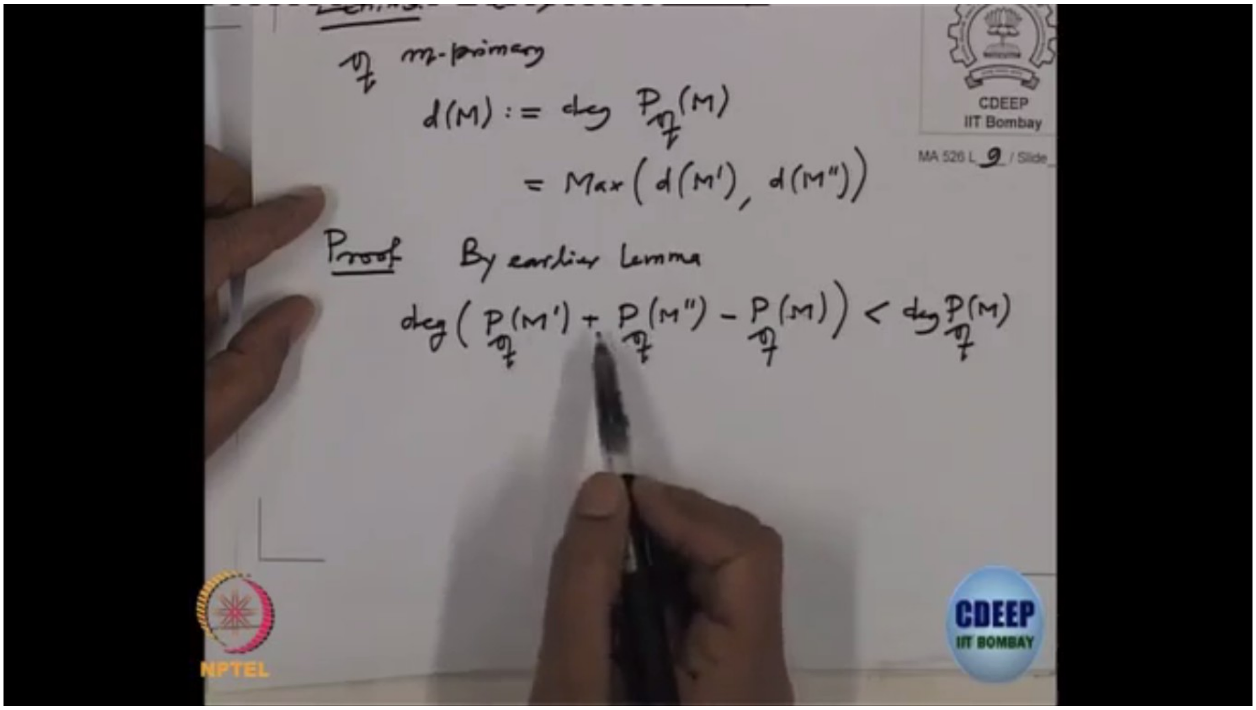
(Refer Slide Time 23:33)



Okay, the next one, you want to compare the degree. So the next one is, so the next lemma is when we have an exact sequence, short exact sequence like this. And q is m primary. And remember our assumption that (A, m) is local and all these are finite modules and $d(M)$ we have put, $d(M)$, this is the degree of this polynomial $P_q(M)$. It's a polynomial to the actual coefficient and the degree we are calling it $d(M)$. So what we want to prove is, if you have a short exact sequence, this degree is the maximum of, degrees which will come from $M^{'}$ and which will come from $M^{''}$. So proof, remember in the earlier lemma, we have proved that if I take the degree of the alternating sums of the polynomials. So that is degree of $P_q(M^{'})+P_q(M^{''})-P_q(M)$. this degree is strictly smaller than degree of $P_q(M)$. this is what we proved in earlier lemma. By earlier lemma. So from this, this assumption is clear. We got the degree, the maximum of, one of them, where this will not get canceled.

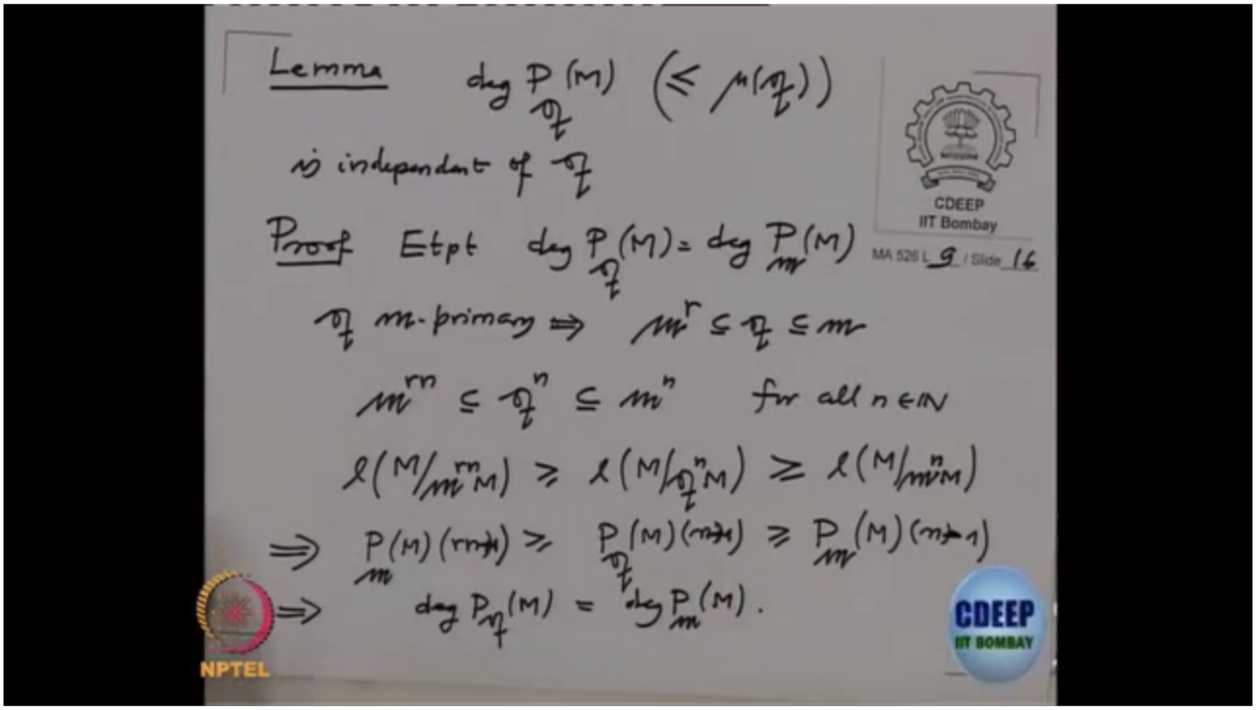
So the degree, if the degree is not maximum, so this leading term is getting canceled from this one. So either one or both will contribute. In either case it's maximum.

(Refer Slide Time 26:07)



Okay, so the next one. The next one is, the next lemma says that this degree is, we have seen that the degree of $P_q(M)$, this is bounded by minimal number of generators for q. But, so this lemma says that this degree is independent of q. That means if I take a different M primary ideal and do the same, form the numerical function and it's a polynomial function corresponding to that, then the degree will not change. So the degree will variant and this degree we will connect it to the Krull dimension. That is what we call the Dimension theorem. Okay, so to check that this degree is independent of q, you'll have to prove that this degree is same as degree of the Max, degree corresponding to polynomial corresponding to the maximal ideal. So it doesn't depend on the ideal q. okay. So it's enough to show this, but q is m primary. This is m-primary and m is maximal ideal in the noetherian ring. So that will imply that some power of the maximal ideal is containing q. so therefore when I raise the powers, m^{rn} is containing q^n is containing m^n , for all n. Now we may take the lengths. So length of $\frac{M}{m^{rn} M}$, this length and length of $\frac{M}{q^n M}$ and the length of $\frac{M}{m^n M}$. These lengths , they are, because smaller the ideal, bigger the length, mod. So we have these inequalities. So that will mean that the polynomials corresponding to these for large n, the polynomials are $P_m(M)$, this is evaluated at rn , this is big or equal to $P_q(M)$ evaluated at n and this is $P_m(M)$ evaluated at n. And this is true for all n. Not n, $n-1$. So this is also -1. All these are -1, sorry, this is - 1, this is - 1, this is - 1 because our definition of $P_q(M)$ evaluated at n is $\frac{M}{q^{n-1}M}$. So this and this are same polynomial. So that will imply the degrees are equal. See this, so from here it follows that the degree of the middle one, $P_q(M)$ is same as degree of $P_m(M)$. So that proves the lemma.

(Refer Slide Time 30:24)



Okay, so the next time I will do Dimension Theorem, that will in particular show that how does an complete Krull dimension of a noetherian local ring and it is finite.