Lecture – 27

on

Filtration on a Module

Gyanam Paramam Dhyeyam: Knowledge is supreme.

We will continue from the last lecture, the study of the degree than leading coefficients of the Hilbert-Samuel Polynomial. So let me recall few. So we have a local ring, noetherian local ring m, and using and q the m-primary ideal. So this implies that $\frac{A}{q}$ is artinian. And when we consider the graded ring, Associated graded ring, which is denoted by $gr_q(A)$ which is $\frac{A}{q} \oplus \frac{q}{q^2} \ldots$ and so on. And this is standard graded $\frac{A}{q}$ algebra. So this is generated as $\frac{A}{q}$ algebra. By the residual classes of the generators of $\overline{x_r}$, where r is-- where this bars. Where $x_{1,} x_2, \ldots, x_r$ are elements in q, and bar denotes image in $x_i \in \frac{q}{q^2}$. So it in noetherian, the finite type algebra or an artinian ring and then we consider the map in the numerical function.

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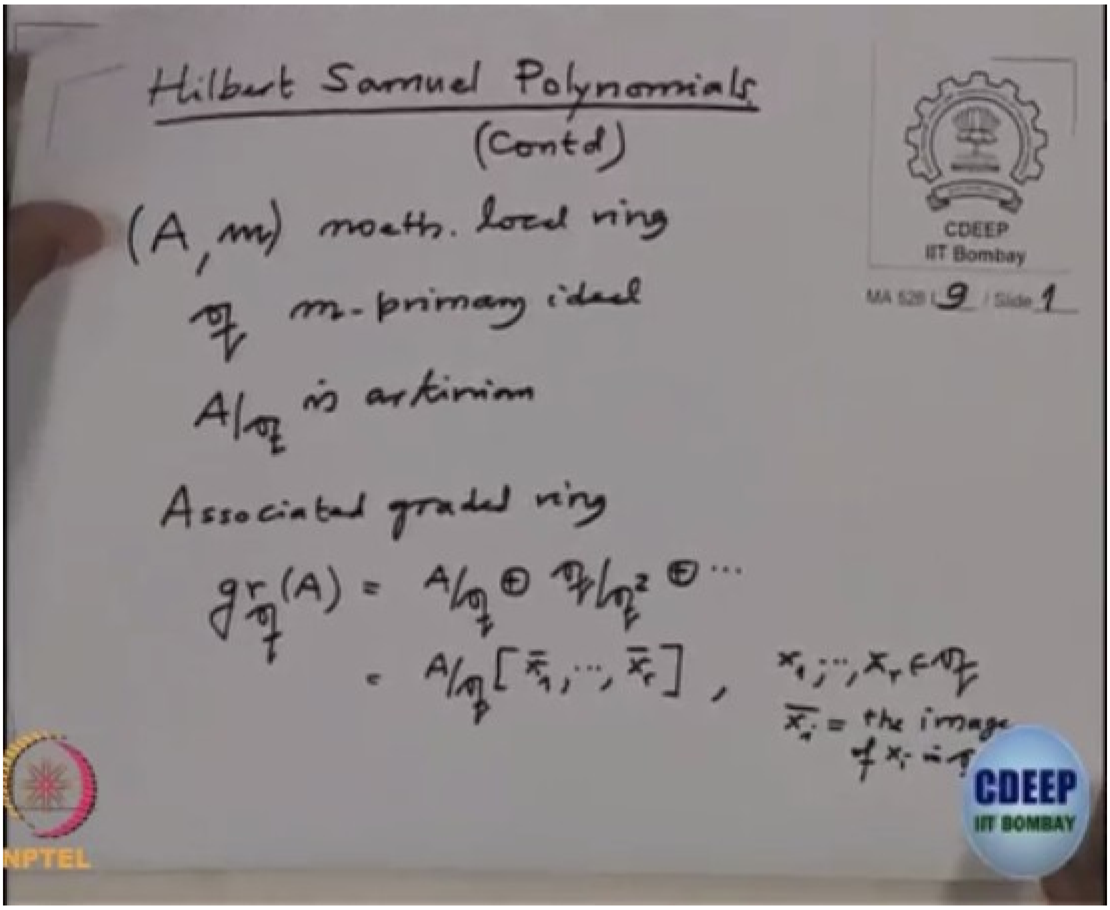

From $\mathbb{N}$ to $\mathbb{N}$, this was any m going to length over $\frac{A}{q}$. $\frac{q^m}{q^{m+1}}$. And we approved it for a large m. For large m, this is a polynomial function. This is a polynomial function with ration coefficients in Q. In that polynomial function denoting by H_q , so because the coefficients are in Q and Q is infinite, this polynomial is immediately determine and that polynomial and denoting by H_q . H_q and not only for the ring here, we have done these for though module. So we have done. So we have for module M, $H_q(M)$ and notation is at any m, this is length of $\frac{A}{q}$ as $\frac{A}{q}$ module, $\frac{q^m M}{q^{m+1} M}$. And for the large m, it's a polynomial of degree less equal to number of generators of q, minimal number of generator of q - 1. So I will denote H_q without large, I will generate this size of polynomial only, $H_q(M)$ this is a polynomial with the rational coefficients.

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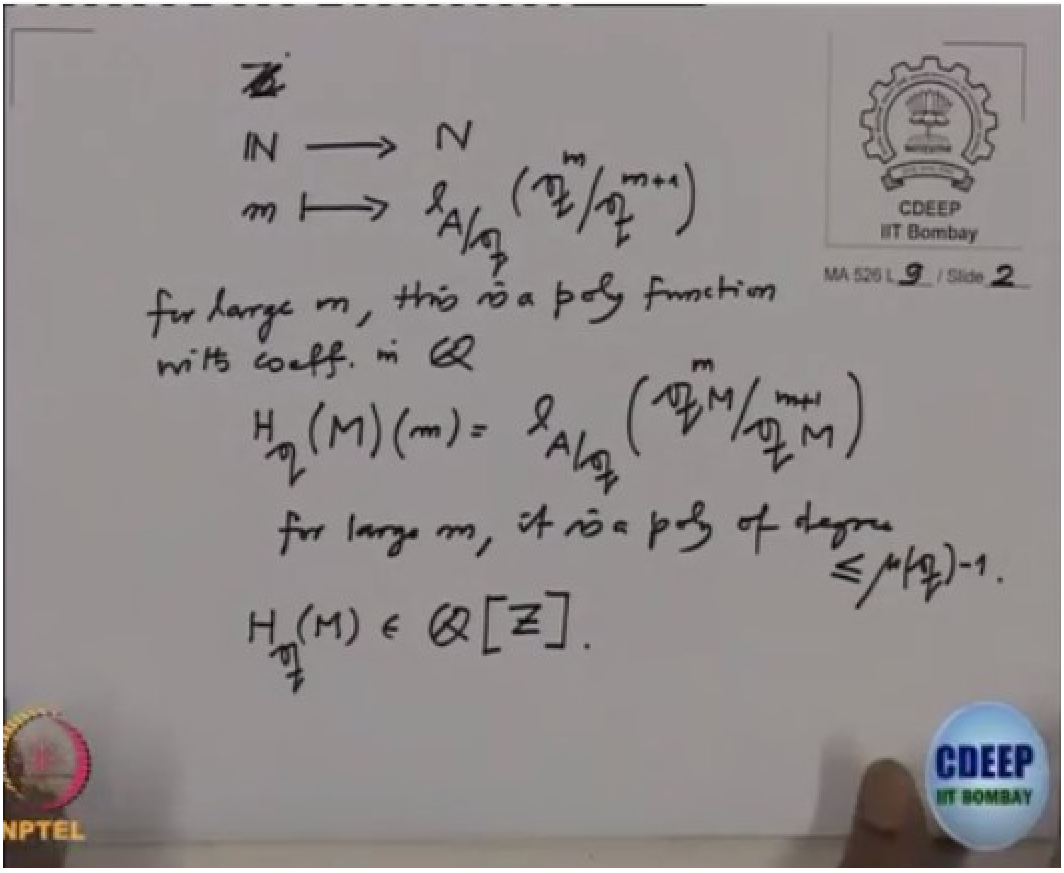

Okay, and then we have also noted that, if I take this function. For this M, we might have to do with the integers, because-- no, no not for this M. But in generally, we have a finite module or graded ring and if you allow some negative grading also then negative grading it doesn't go all the way to-- all the side, do negative side. So, only finitely mini negative components again occur. Okay, so what is P_q ? $P_q(M)(m)$ is defined by length of length over A of the module $\frac{M}{q^{m+1} M}$, this is-- this module even highlighted by power of a primary ideal, therefore the length is finite. It's a finite length module. And this is for large m, in polynomial function of degree. Now, degree link is theone one because this definition clearly, it say, that if I take the D of $P_q(M)$, this is $H_q(M)$. Where is a derivative that means, the difference of successive terms.

So what we proved is P_q the polynomial function of degree less equal to minimal number of the generators for a q. And what I was proving, I wanted to prove this Lemma and then I realize that we need Artin- Rees Lemma. So, the Lemma I was proving was-- so this Lemma we were proving, if I have exact sequence of 0, $M^{'}$ to $M^{''}$, this is exact sequence of A-modules. Then M is the middle one is non-zero. Then if I take the alternating sum of the functions, so that is-- if I take $P_q(M^{'})$ + $P_q(M^{''})$ - $P_q(M)$, these are the polynomials. And if you take the alternating some of the corresponding polynomials, then the degree of this is strictly less than the degree of $P_q(M)$. So that means that the leading term of the sum of these two will be equal to the leading term of this. So they get cancel and the degree drops. That is what we wanted to prove.

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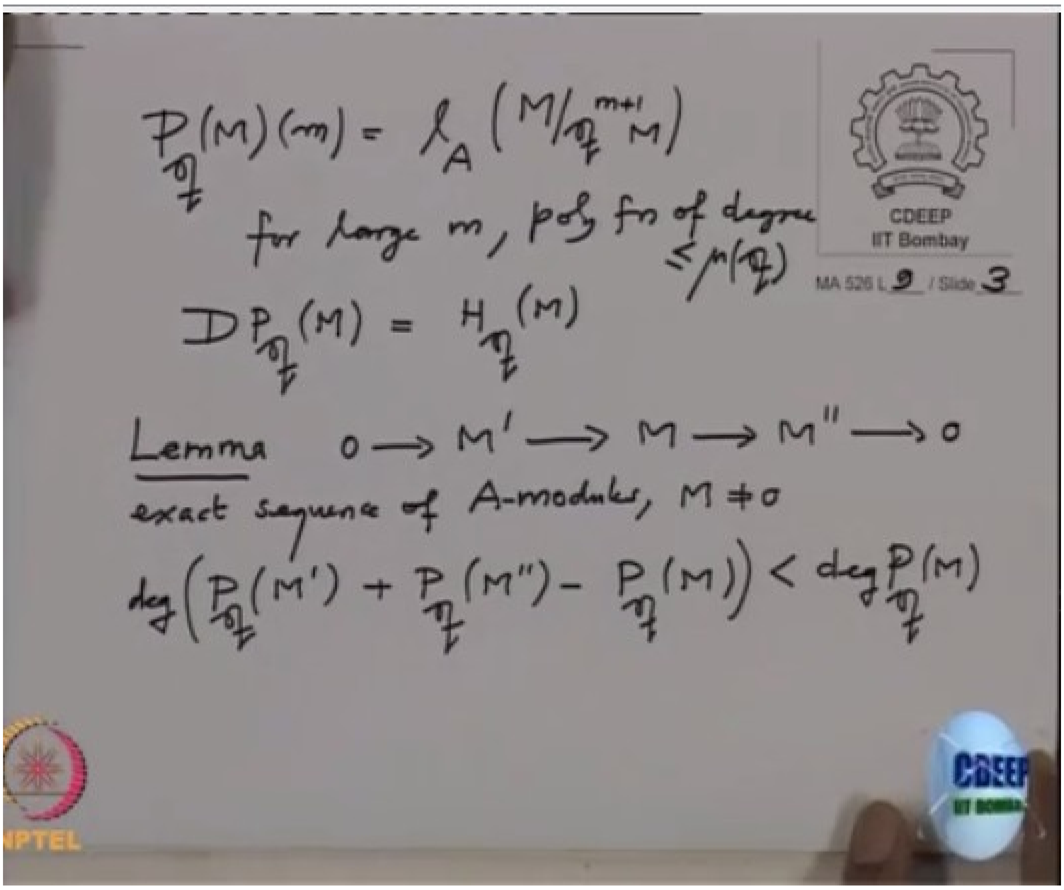

And for this, I wanted to use so-called Artin- Rees Lemma. So let me recall that Artin- Rees Lemma. So for that I will little bit more general I recall. So I'm recalling now. So we will complete the proof after Artin- Rees Lemma. So this is can be stated more generally, so I will state more generally. So when I say, filtered ring, so like a graded ring as a gradation. And so filtered ring also have a ring with the filtration, so it's a ring, is a ring A with a filtration on A. And what is the filtration? Filtration is simply a descending chain of ideals. So that is a descending chain, it starts with A moldering unit ideal A_0 containing A_1 , containing, containing, containing A_m and so on. Chain of ideals is called a filtration on A.

The typical example of a filtration is you just take any A, any ideal and take the powers of that ideals. So A_0 is A, A_1 is A^1 is A and so on. So this is a filtration, this filtration is called A-adic filtration or filtration given by the ideal. And usually filtration, so it's a either it you denote it like this. Or you denote it simply by F, this is usual notation for a filtration. And for a filtration, you can consider of the outer filtration, you can get a graded ring. So graded ring associated to a filtration, F is-- you just take the direct sums. These are ideals, so there are billions of groups. So you take the direct sum. So that is it denoted by $A^{'}$. This is $A_0 \oplus A_1 \oplus \ldots$ and so on. So we get a graded ring out of that filtration.

So that is called graded ring associated that filtration F. So there is another graded ring associated to that, that is coil that is usually denoted by $gr_F(A)$, that is the successive quotients. So that is $\frac{A_0}{A_1} \oplus \frac{A_1}{A_2} \oplus \ldots \frac{A_n}{A_{n+1}} \ldots$. Actually this is called associated graded ring of the filtration F.

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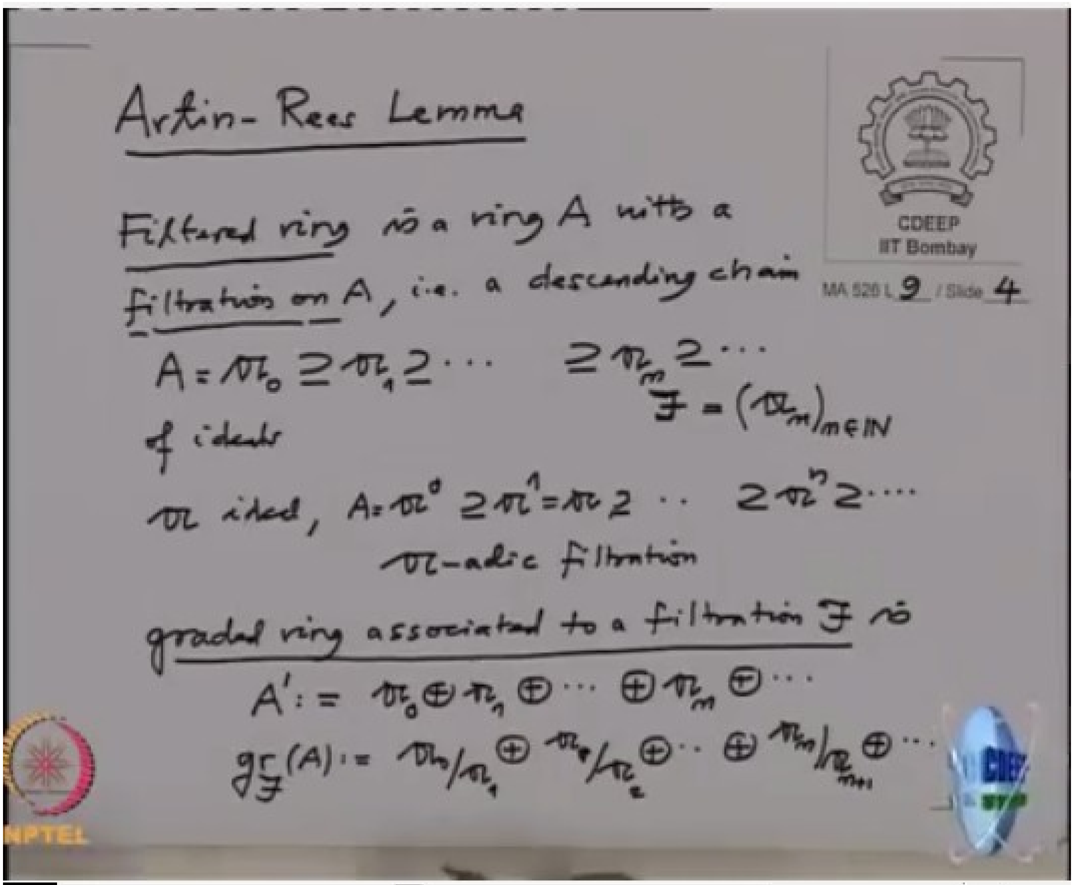

So in particular for this adic filtration, you get really the associated graded ring with respect to the ideal A. So $gr(A)$, this is $\frac{A^n}{A^{n+1}}$. And this is a standard graded algebra over the A by ideal A generated by homogenous elements of degree one. Okay, similarly we can do it for the module. So if you have a module over A, so if M is a A module, filtration on M is descending. Filtration on M, this is a descending chain of submodules, so that M equals to M_0 containing, containing, containing the M_n containing so on. So A-submodules. Or in general, I forgot to mention here when you say, filtration, here, filtration should also mean that, when I take A_n, $A_n . A_m$ that should be containing A_{n+m}. So that the multiplication makes sense.

So, you see, sorry I forgot to mention this. A filtration means, the descending chain of ideals with this property. $A_n . A_m$ is containing A_{n+m}. This is-- this will make, this ring A' as a graded ring because in order to multiply the elements you get in the next component. So such a filtration-- no and so the filtration on anywhere descending chain of submodels and you call it given an ideal A. We say that the filtration is A-compatible or A- stable. Or A-compatible, if you $A . M_n$ is containing M_{n+1} for all $n \in \mathbb{N}$, then we call it A-stable. The filtration on the module is a descending gain of submodules. As for a given ideal A equally to A-stable or A-compatible. If $A . M_n$ is contained in the next one. And you call it A-adic. If this becomes equality for large n. For large n. That means this becomes equality for all n bigger equal to after some stage n_0. So there exist a n_0. So therefore all n bigger equal to n_0, this equality happens for all. This is called A-adic filtration and of course compatible. So A-adic means this condition and A is stable. So they do A-adic.

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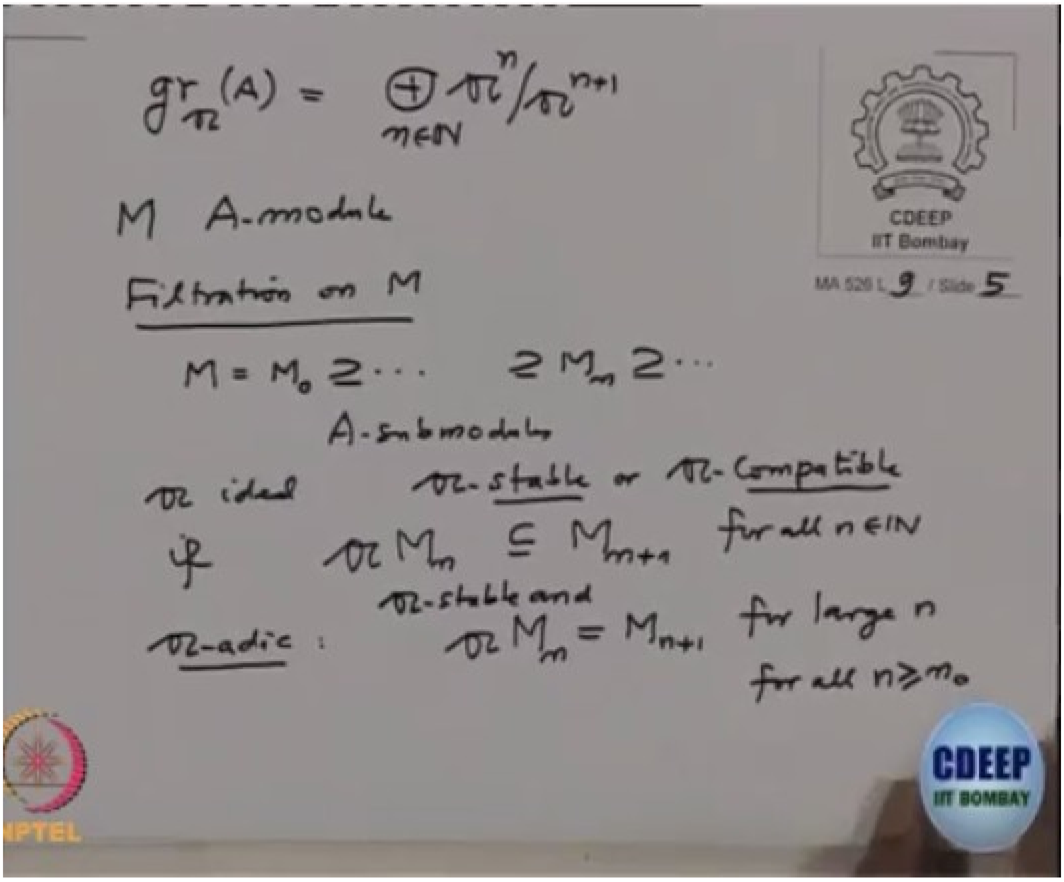

So the typical example is if you take the ideal A, and the powers of that ideal. So, typical example is if you have an ideal A, ideal A, and M is the module then $A^n M$, this is A-adic filtration on M. Okay, now at first, let me write it as a Lemma. So, Lemma, this is about when the module is finitely generated. When the filter module. So we have-- remember we have A' so let me write state clearly. So A then ideal in A, M is the new module. And (M_n) is A-compatible filtration or A-stable filtration. And let assume all these of finite A-submodules. M_n s, all M_n s are finitely, finite modules. And then we form this graded ring A', which is direct sum A^n, and we found the module M' which is direct sum of M_n s. So this is a graded ring, and this is a graded module over A'. So M' is a graded A' -module. That is because the filtration is compatible, okay. Then the following are equivalent. Then the following are equivalent. Number one, the filtration is A-adic. And two, M' is a finite A' -module. So, it's-- it gives a criterion in to check when the-- this M' is a finite module over A'.

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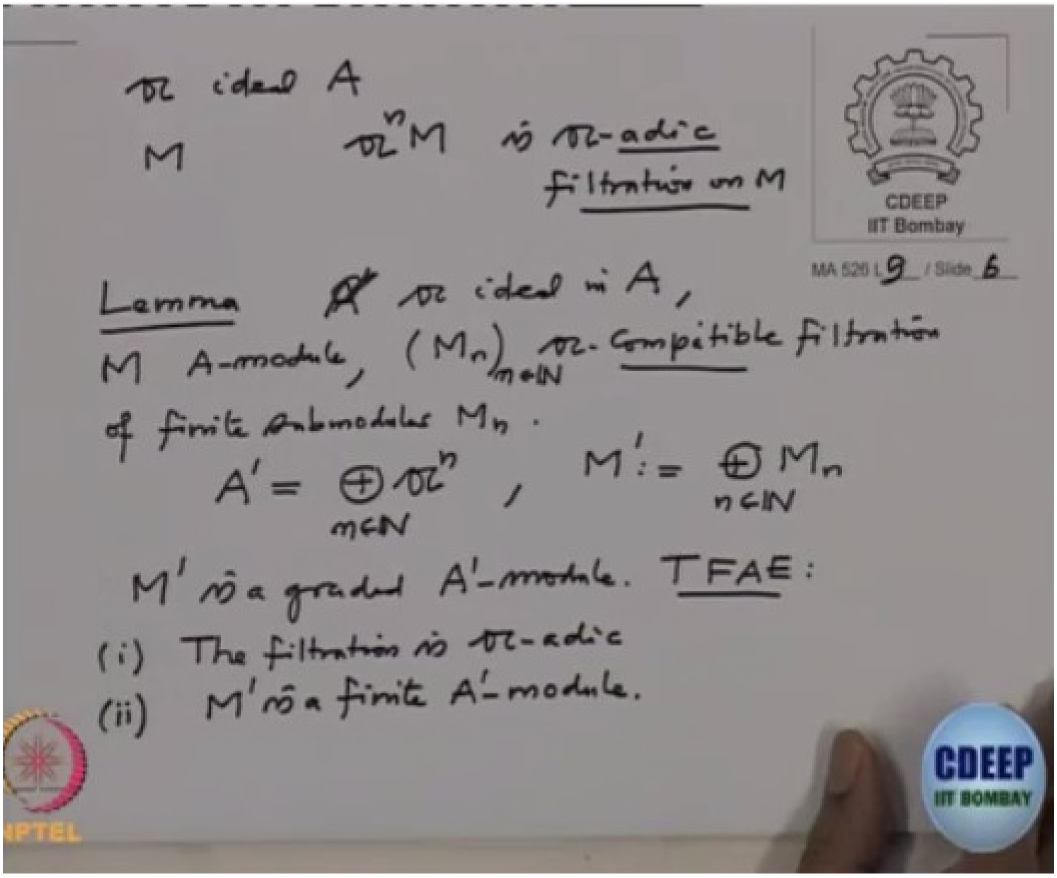

Okay, so proof. This is very simple. So, you already noted that M' is a graded A' -module. So let us indicate the proof of one implies two. So, we are given that the filtration is A-adic and we want to change that, it is a finite module. So, filtration is A-adic, so for large n, M_{n+1} equal to $A\,M_n$ for all n bigger equal to some stage n_0. That's given, that is a meaning of let's say, A-adic flirtation. So obviously if I look at $M_0 \oplus M_1 \oplus ... \oplus M_n$. This, this M either finite ideal modules over here, so these are-- this finite-- so this set the generators here, generators et cetera. This is finitely, finite over-- so this set we'll generate M' over A'. Because the later will-- later all of these elements are will come from here and coefficients are in ideal A. So, therefore later part will be generated by this part. So in particular it is finite A' module. What I'm saying is, you take the one of these components of degree zero, degree one, degree n, all together-- put it all together, they generating sets. And it's about that all later components will be generated by that later one. So, two implies one. Suppose it is finite module then, it's finitely generated over $A^{'}$. Because it is a graded module, again assume the generating set is homogenous elements. So suppose that x_1 to x_r are homogenous elements, elements of degrees n_1 to n_r. And generate-- we generate $M^{'}$ as $A^{'}$ -module. And let us take n_0 to be maximum of all these guys. Maximum all the degrees, n_1 to n_r Then we claim, we should take that, that stage onward, this equality. The equality, so we should check now, $M_{n+1}=A.M_n$. That is the meaning of-- that it is A-adic. For all n bigger equal to n_0. So let's check this. So, note that if I take n bigger equal to n_0. Look at this set ot look at this $M^{'}_{n+1}$. This one is precisely all-- because this x_1 to x_r generate, therefore this is precisely the finite sums from i equal to 1 to r, combination of A linear combination of this x_1 to x_r, there so. $a_i x_i$, where a_i is-- actually I can assume by comparing the homogeneous components of degree n plus 1, I will assume that, that the a that actually in $A^{'}_{n+1-n_i}$, which is, which is A^{n+1-n_i}. This is obviously equal to $A.M^{'}_n$, since A-- this $A^{'}_{n+1-n_i}$ which is equal to A^{n+1-n_i}, which is A_{n-n_i}.

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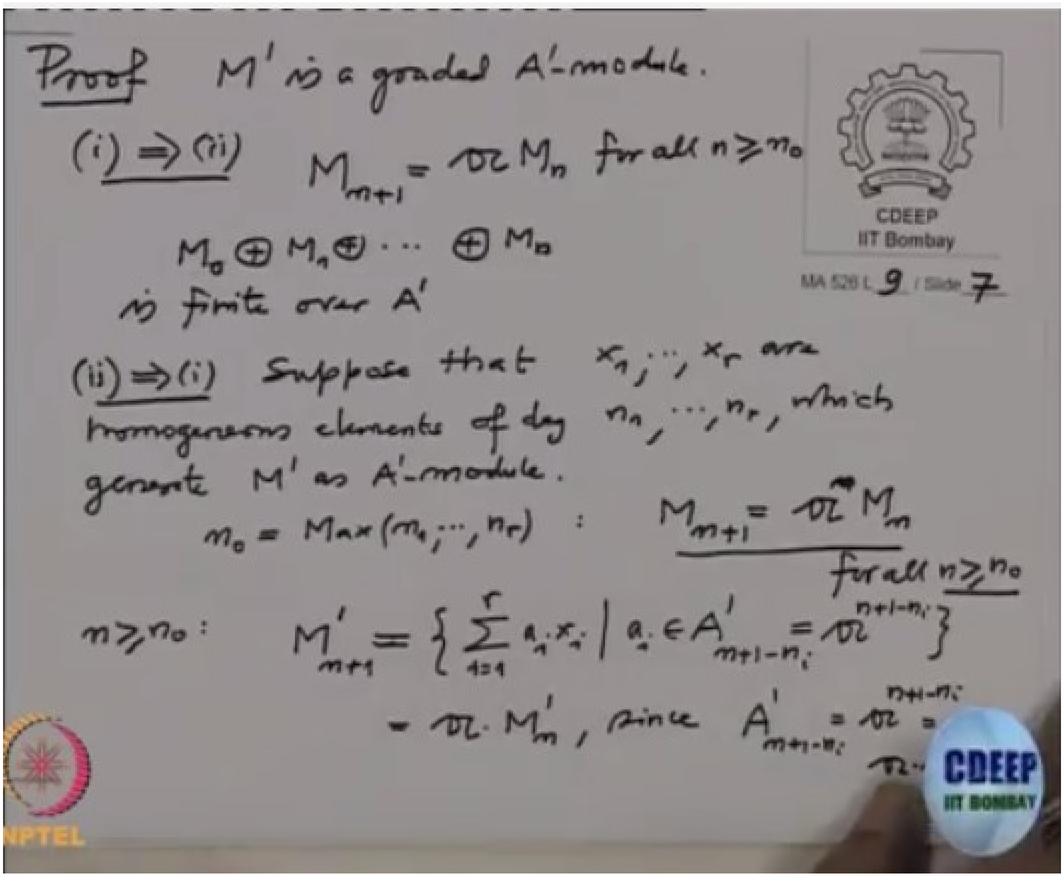

That proves that the filtration is A-adic. Now let us come to the-- so when you want to take some filtration in the A-adic, we have to take that the graded module which is finitely generated.