Lecture – 26

Hilbert-Samuel Polynomial of a Local ring

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Okay. So, remember, for arbitrary ring A, the Krull dimension of A, we have seen it is supremum of dimensions of A localize at P, the supremum is running over prime ideals. So, in order to compute this or understand this, one want to concentrate on each localization. So, therefore, sometime onward, now, we will assume, we'll concentrate on local rings. So, that is ring, commutative ring with only one maximal ideal, that is also denoted by m, and m is the Jacobson radical of A. So, Jacobson radicals are usually denoted by m_A, that is intersection of all maximal ideals. And also, now, there is only one residue field. So, that residue field is also sometimes denoted by k_A. So, we have such a local ring. This is the maximal ideal, this is the residue field of m. Now from this, I want to get a graded ring, standard graded ring and apply this Hilbert, the Poincare Series. So, before I forget, this integer d we got, we want to prove that is the dimension, that I've not yet proved. So, that is, I will prove this d is actually the dimension, and then, how am I going to go further. So, for this local ring, from a given local ring, I will define, I will attach a graded ring. And for that graded ring, we have this theory of Poincare Series and so on, and then we get the integer out of that. And, that integer is nothing but the Krull dimension. That is what we will prove. So, in particular, what we would have got as a consequence in particular Krull dimension of a local ring is actually an integer. So, finite. But that doesn't show that Krull dimension of a arbitrary ring, arbitrary commutative ring, even assume now a theory in, that it is finite. In fact, that is not true because it's a supremum over this integers and this maybe increasing. So, also, I will show you examples of neotherian ring with infinite Krull dimension. Construct examples of neotherian ring with infinite Krull dimension. For this kind of treatment, you can see book by Nagata Local rings. In fact, only reference I know is this, this book. Only these books deals with such examples. And, actually, this book is one of the early book and this book is not so easy to read, but on the other end, the material this book contains, you can't find in any other book. Okay, so what is the graded ring which we define out of the local ring? So, that is called Associated graded ring of a local ring. That is the following. So, we have A and m, m is the maximal ideal, so you put K equal to $\frac{A}{m}$. And then, so, put $\frac{m^n}{m^{n+1}}$. Now, these are annihilated by m, so, therefore, they are A by m modules, right? So, I can form there direct sum. These are modules, this is field, so, I can look at $\frac{A}{m} \oplus \frac{m}{m^2} \cdots$ and so on. Direct sum m power n by m power n plus one and so on. Now, we have to, right now, we have only a direct sum of abelian groups. We want to make it as a ring. We know also, each one of this component is actually vector space over $\frac{A}{m}$. So, the standard notation for this is $gr_m(A)$. This is called Associated graded ring of the local ring Am.

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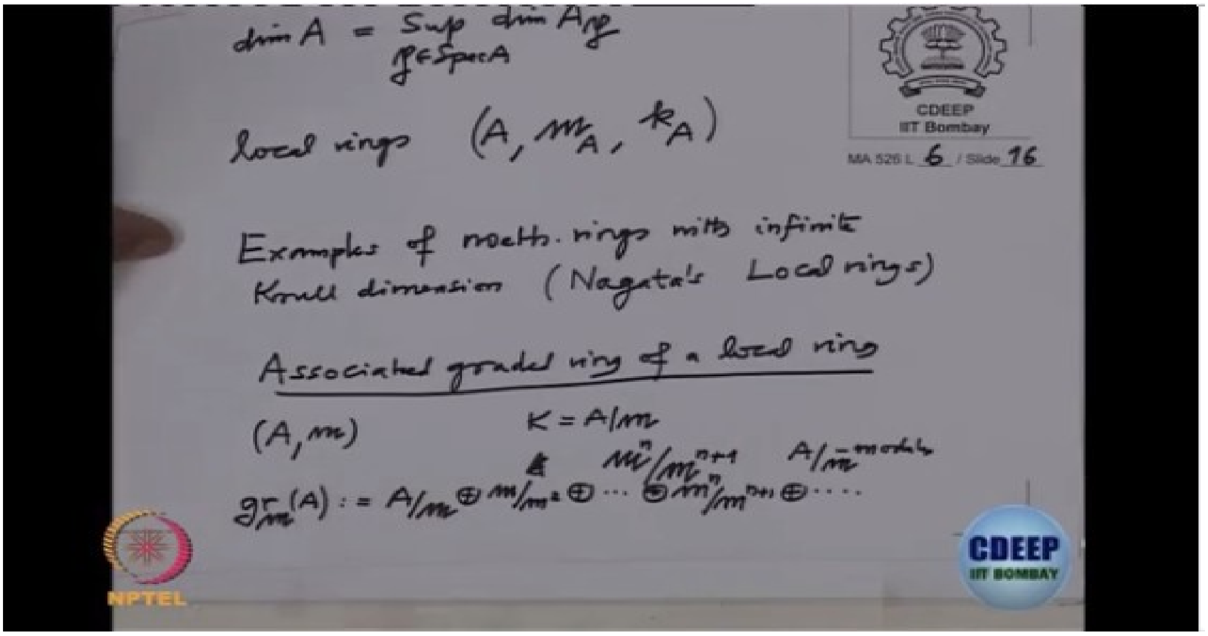

Now, what is the ring structure there? So, I only have to give you how you multiply the elements, right? So, if you have one element $\overline{x} \in \frac{m^r}{m^{r+1}}$, this is homogenous element of degree r. And also, other one, $\overline{y} \in \frac{m^s}{m^{s+1}}$. Then, I want multiply them. So, first you multiply as usual xy, no, what does this $\overline{x}$ is here means, that means x is an element in m^r, and this $\overline{x}$ is a image modulo m^{r+1}. Similarly, y. So, xy as usual, this will be an element in $m^r . m^s$, which is, contain in or equal to m^{r+s}. And, now I take its image in, when I write $\overline{x} . \overline{y}$, this should be $\overline{xy} \in \frac{m^{r+s}}{m^{r+s+1}}$. Take the image there. And, now, we will have to check this definition is well defined. So, check that with this definition, this $gr_m(A)$ is a graded ring. In fact, standard graded $\frac{A}{m}$ algebra. So, actually, really, the theory we'll use in this particular case is only for when R_0 is a field. But for more general applications, I will not only do this concept for the maximal ideal but the powers of maximal ideals, or not even powers of maximal ideal but so called m-primary ideals. So, more generally, define $gr_q(A)$ for arbitrary m-primary ideal q. And that will be now graded a by q algebra, standard $\frac{A}{q}$ algebra. Now, you see here, this $\frac{A}{q}$ is not a field in general, but it's an artinian ring because you have gone modulo primary ideal. So, there's only one maximal ideal, an artinian ring and is dimension zero, so therefore, it is an artinian ring, so, length makes sense and so on. So, then, one apply the theory for this graded algebra, and then, we will get the Hilbert polynomial, and then, we will get the degree, and we will get those integers and so on. So I will continue next time with this more general setup.

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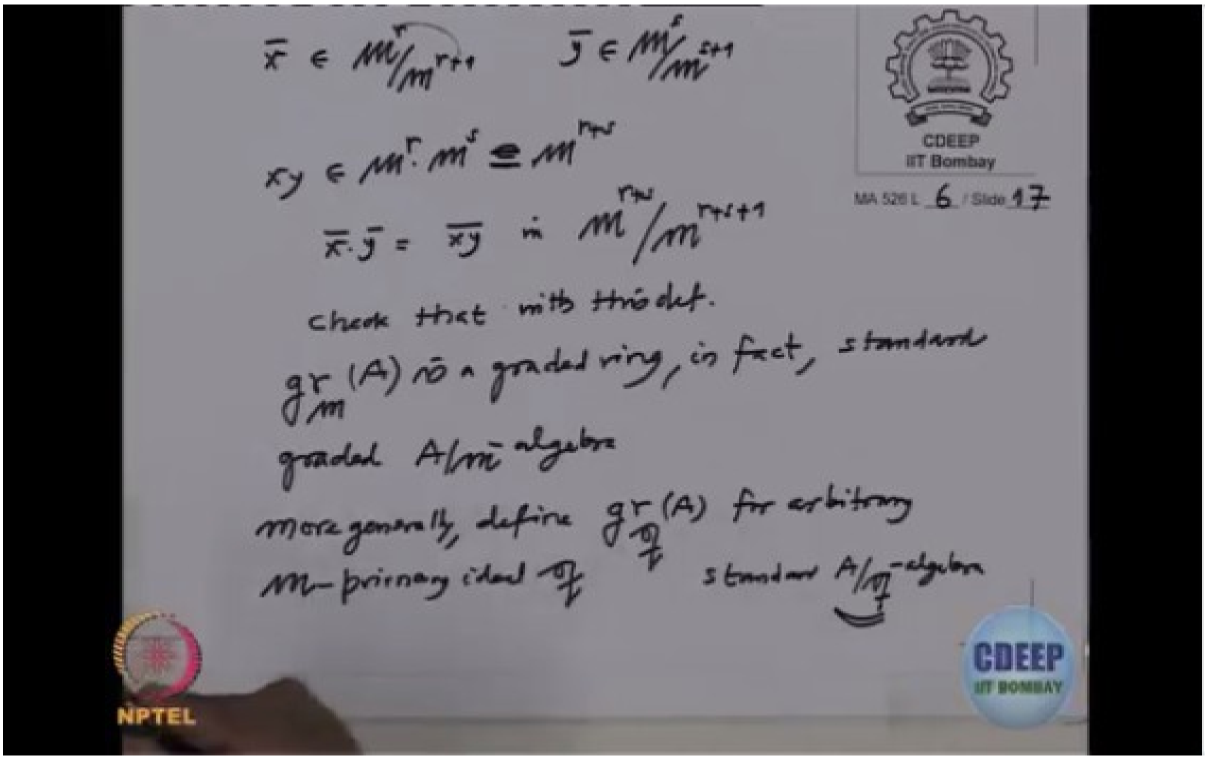

And use standard examples of the graded ring is only arise from a local ring. So, if you have a local ring (A, m) local ring, then, to this local ring, we associate this new ring which is $\frac{A}{m} \oplus \frac{m}{m^2} \cdots$. Note that this each component here, so this is a direct sum of abelian groups. But each component here is actually annihilated by m, therefore, they are also $\frac{A}{m}$ modules. So, therefore, this is indeed ring, and the ring structure is, the addition is clear because with direct sum of abelion groups and multiplication here. If you have an element $\overline{x} \in \frac{m^r}{m^{r+1}}$, and another one y-- So, to define a multiplication, I only have to define it on the homogeneous components and then, extend it by distributive law. So, this is, okay, I have two elements like this, you take a lift here, that means, you choose x so that x belong to m^r, and y so that y belong m^s, and you multiply them as in the ring. And take its residue class, so this is the residue class of xy in x, because x is here and y is here, the product will be in m^{r+s}. So, take its image modulo m^{r+s+1}. And here, check that this is actually a graded ring, it's a standard graded ring. So, this is called Associated graded ring A with respect to m. So, $gr_m(A)$ is a standard graded A by m algebra. And nothing special about m, but we could do it little bit more generally.

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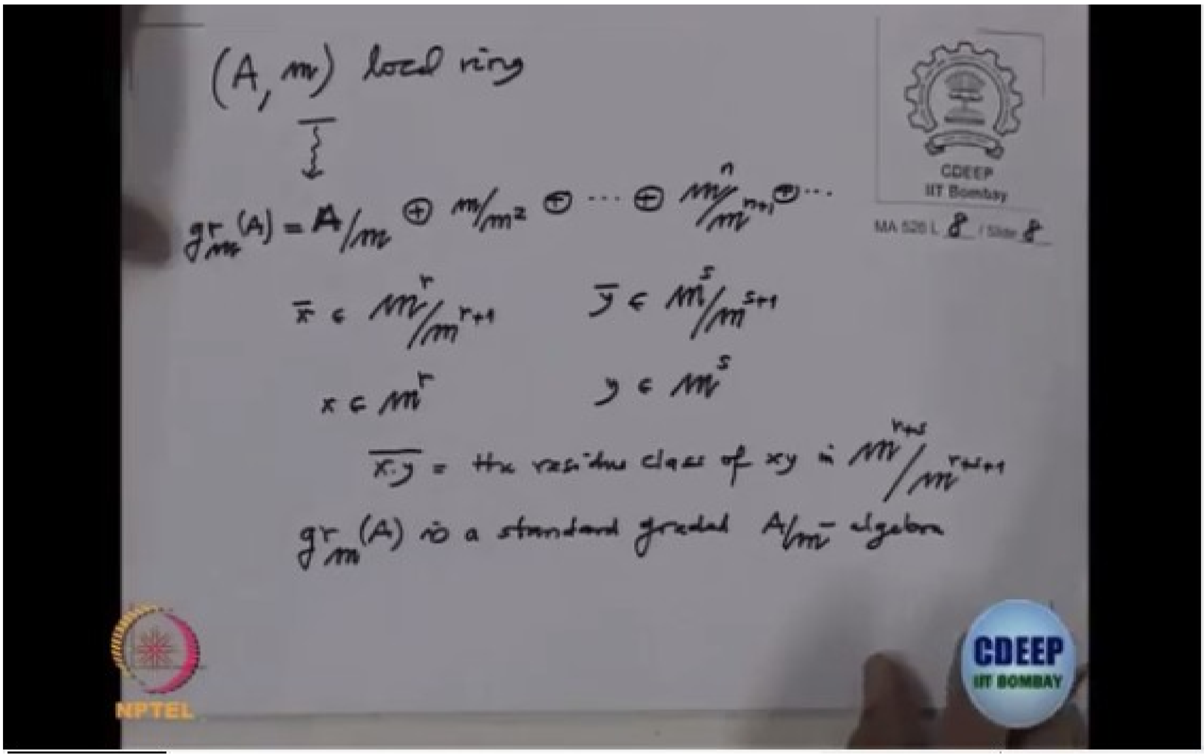

So, what we could do is, if we have a local ring (A, m) and we have a primary ideal q, q is m-primary ideal. So, that means, this ring $\frac{A}{q}$ is actually artinian ring. And then, we, instead of m, we can repeat the above process for q. So, associated graded ring of A with respect to q, it's denoted like this, $gr_q(A)$, this is $\frac{A}{q} \oplus \frac{q}{q^2} \cdots$. So, this is the standard graded $\frac{A}{q}$ algebra. And now, $\frac{A}{q}$ is artinian, so, we have the Poincare Series because $\frac{A}{q}$ is artinian and so on, and that is numerical function which is a polynomial function of degree. Now, this is standard graded algebra generated by this component. So, that will depend, that is, the number of generators for this as $\frac{A}{q}$ module is precisely the minimal number of generators for q. So, minimal number of generators for q, this is, if you have the generators, minimal generators for q, $x_1, \ldots, x_n \in q$, then this is n. So, you get a polynomial function of degree less equal to $n-1$. So that polynomial function arose from the Poicare field, that is the function m going to length of $\frac{A}{q}$ module $\frac{q^m}{q^{m+1}}$, the function. And then, when we want to do the integration of that, that will increase the degree by one. When I say integration, means, the operator f to Df and the other way. From Df we can get f.

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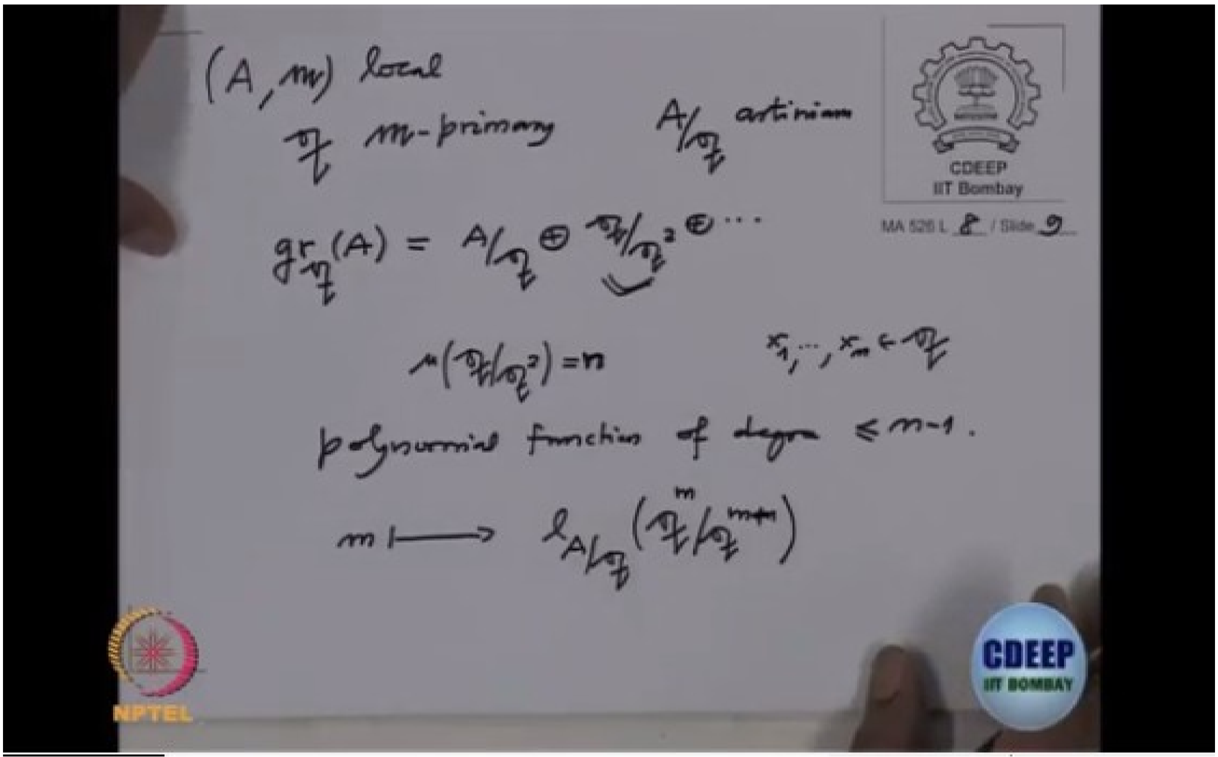

Okay. So, that polynomial usually denoted, so H_q, $H_q(M)$, more generally $H_q(M)$ for any finitely generated, M is finitely generate A-module. And q is primary ideal, then $H_q(M)$ is the function from $\mathbb{Z} \to \mathbb{Q}$ which is defined at every m, it goes to length of $\frac{q^m M}{q^{m+1} M}$. And then, this is a polynomial function, so for large m, it will agree with a polynomial with rational coefficients, and that polynomial usually will be denoted by $P_q(M)$. So, this is a polynomial with rational coefficients X so that they agree for large. Actually, there is a slide, a twist in the notation that, when I say P_q, I usually consider not this function but the integration of that function, that is, m going to length of, length as $\frac{A}{q}$ module of $\frac{M}{q^{n+1} M}$. So, when I add up this up to m, they will get this coefficient because the length gets added lie there. So, if I would have consider for this function, then the D of this will be, so, in this notation, this $H_q(M)$ at any m will be $DP_q(M)(m)$ for large n. So, when I differentiate it, I will get this formula for this. Okay. Alright. So, the theorem we have proved is this P_q, the same notation as above, $P_q(M)$ is a polynomial function of degree les equal to mue q, where mue q is the number of generators for q. Because, if I have minimal set of generators for q, then their residue class is mod q^2 will generate the associated graded ring as a standard graded algebra. Okay. So, we can write down a proof for this again. Proof, oaky, so we are putting R equal to $gr_q(A)$ and N equal to $gr_q(M)$. So, I have a graded ring, and I have a graded module, and we have the right assumption $\frac{A}{q}$ is artinian, this is standard graded generated by the residue classes of the minimal set of generators of $\frac{q}{q^2}$.

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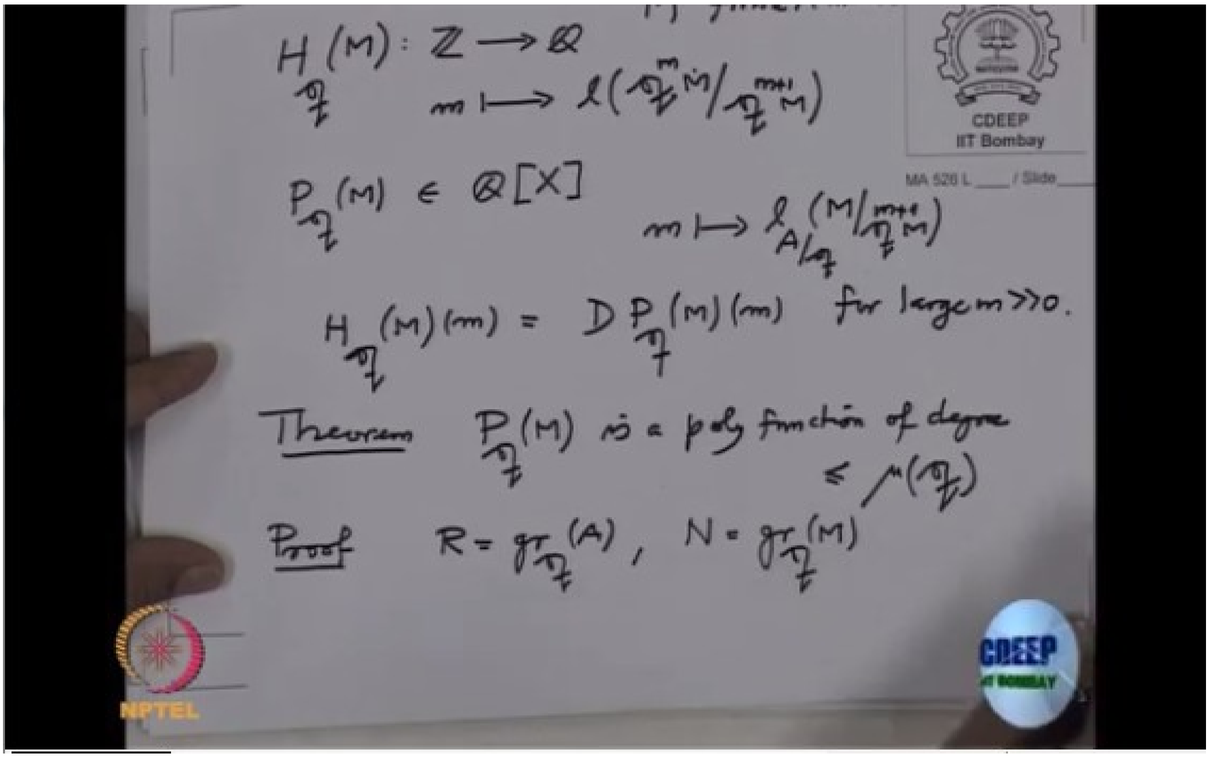

So, in that notation, this $H(N)$ is a polynomial function of degree less equal to $\mu(q)-1$, because we are considering, when I said $H(1)$, that is $H(0)$. You remember, $H(1)$ was a integration of $H(1)$. So, therefore, if you take this one, $DP_q(M)$, which is $H_q(M)$, which is $H_q(N)$. So, this is a polynomial function, so, is a polynomial function, because this H_q is a polynomial function of degree less equal to $\mu-1$. This one, P_q, Oh, I should write here, that implies $P_q(M)$ is a polynomial function of degree one more than this, less equal to $\mu(q)$. Okay. So, now, a couples of observations, like, now what happen to the leading coefficient. So, that is lemma. Lemma says, the degree, we know degree is less equal to the minimal number of generators for q. So, the degree of $P_q(M)$ does not depend on the m-primary ideal q. So that mean, if I take a different primary ideal, then the degrees will be the same. So, proof. Observer that q is m-primary, and m is maximal, so that should imply that some power if maximal ideal will contain in q. And, therefore, the m-th power of r^n, $(m^r)^n$, m^{rn} is contain in q^m, I should write m actually. M, this is contain in m^m. And, therefore, when I take $P_q(M)$, add this rm minus one, this is small, so that this length will be the biggest. And that will be bigger than this, $P_q(M)(m-1)$, and this is bigger equal to $P_m(M)(m-1)$. So, I have this polynomial functions. So, that shows the degrees. So this one and this one, so that will show that this $P_m(M)$ and $P_q(M)$ have the same degree. All these are polynomial functions, and because of this inclusion, now this bigger equal to thing, their degrees will be same. Yes, this should be P_m, this is coming from here know, this is P_m s. Thank you. So, these two polynomial functions are whatever degree is, when you substitute the variable to be equal to rm and it's bonded, this polynomial function is in between them, therefore, they have the same degree. Leading coefficient changes. So, actually, leading coefficient also, we can keep track. This lemma says that the degree of this $P_q(M)$ is independent of q.

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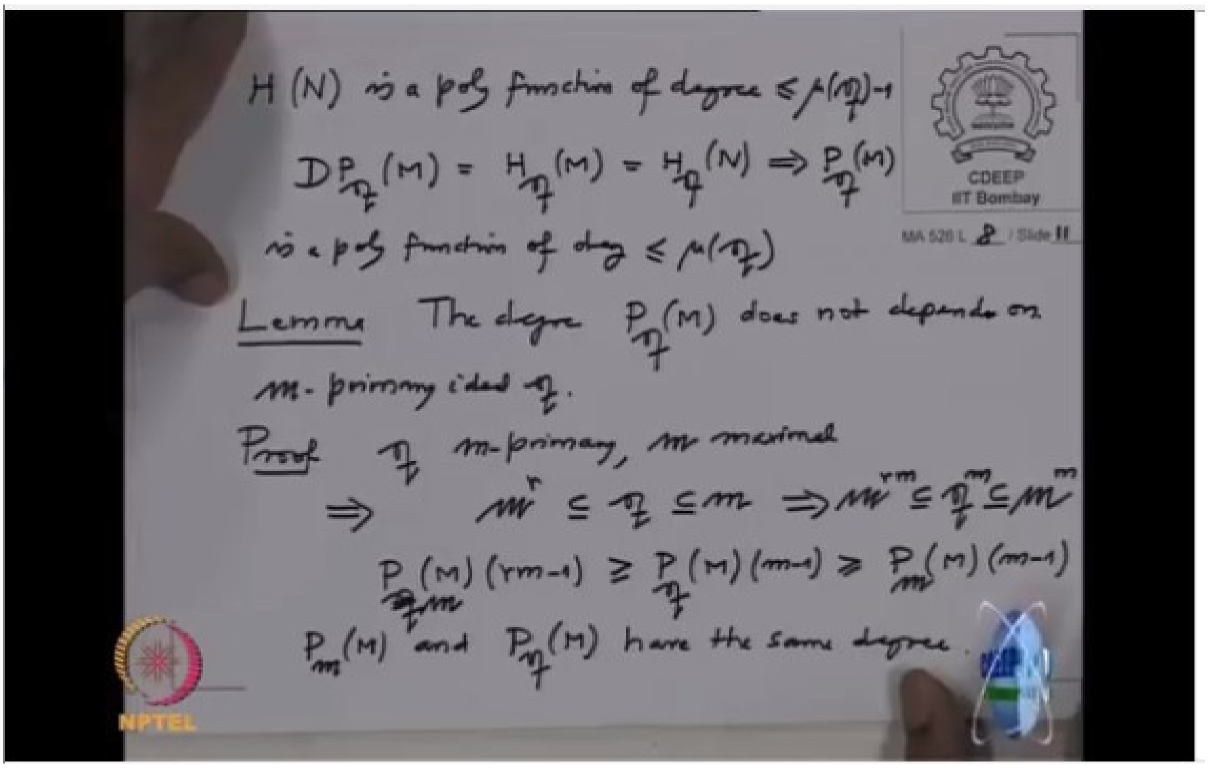

So, therefore, that is an invariant of M. So the degree of $P_q(M)$, this is usually denoted by $d(M)$. This is invariant of M. It doesn't depend on the ideal q. Okay. Also, simple observation like what we did earlier for a graded ring, if you have exact sequence of modules, short exact sequence of-- So, all this discussion goes under the assumption that (A, m) is a local ring. And all these M, M', M'' are finitely generated modules over A. This is short exact sequence of A modules, then the degree of P_q middle plus degree of this plus, so plus minus plus. Alright, so degree of this, P_q of, this is M', this is M'' minus $P_q(M)$. This degree is strictly smaller than degree of M''. Remember this in separate situation, when you take the Hilbert Series, it'll be, alternating sum is zero. So, from that you get—this is coming out of the alternating sum and the degree is strictly smaller than the degree of this. So, again, we have proved this, but again, we can write the proof. So, proof, we can assume, we may assume m prime is non zero. Otherwise, these are isomorphic and there's nothing. And q, we can put this inside thing. Put Q is this, $P_q(M')$ + $P_q(M'')$ - $P_q(M)$. So, now this sequence will induces exact sequence like this. When you tensor with $\frac{A}{q^{n+1}}$, the tensor product is right exact but not left exact. So, when you start writing the long exact sequence, this $\frac{M''}{q^{n+1}M''}$ to zero, then here it'll be, $\frac{M}{q^{n+1}M}$, and here will not be exact but I want to make it exact. So, to compare the lengths, so this is $\frac{M^{'}}{q^{n+1}M^{'}}$, and I want to put a zero here, where $M_n^{'}$, this will be actually $M^{'}$ and intersect with the kernel of that. So, intersect with $q^n M$, this is $M^{'}n$. So, when you intersect with the kernel and go mod the next one, this exact. So, this is exact. And then, you compare the lengths.

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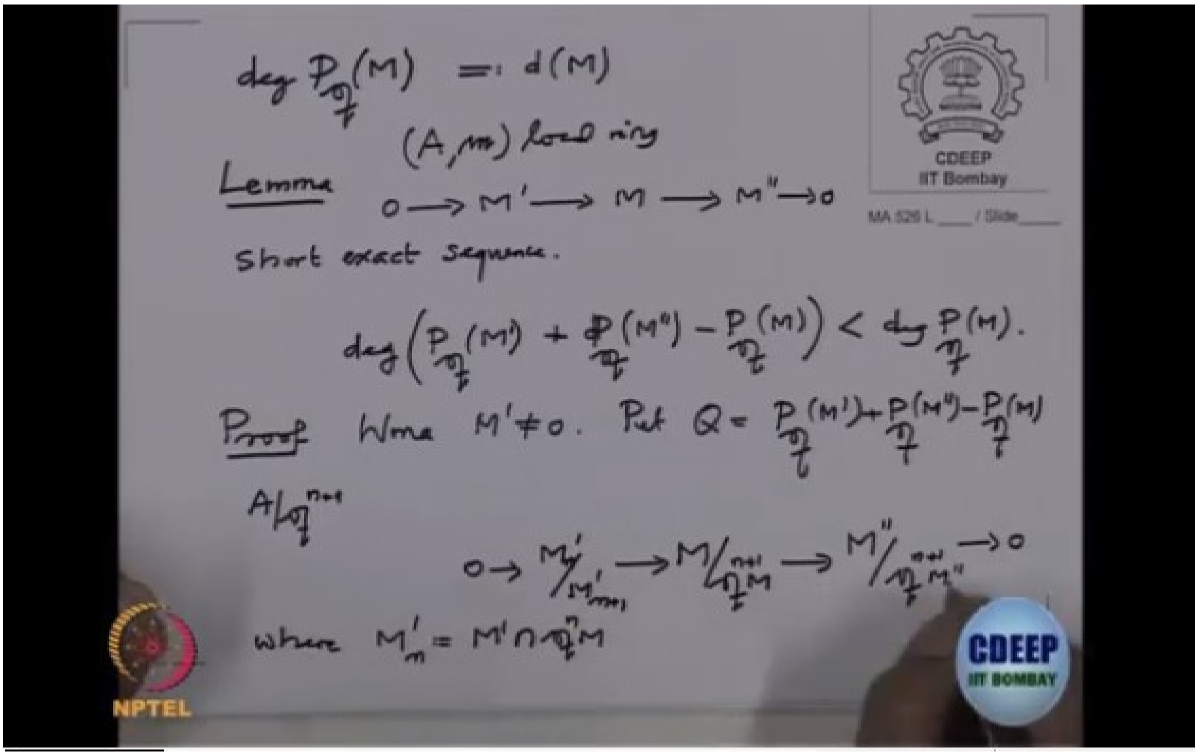

So that will give you Q, $P_q(M^{'})$ from here - f, where f is, will come from the length here. Where f at n will be equal to $\frac{M^{'}}{M^{'}_{n+1}}$. And now, here, to continue the proof, I need so called Artin-Rees Lemma.

Unknown: Sir?

Dilip: Yes.

Unknown: What is the length?

Dilip: Oh, so, this is the length. This is the length. Sorry. This length is same as summation lengths of $\frac{M_i^{'}}{M_{i+1}^{'}}$. This is i up to n. So, when I take the sums, from this exact sequence when I take the length and then add up, so this, here I will get $P_q(M)$, here I'll get $P_q(M^{''})$. But that difference, this is with the plus sign, this is with the minus sign, then that difference will be coming from here. That is what our q is. And that therefore, is Q will be equal to the difference of what I wrote, this function and at $P_q(M')$.