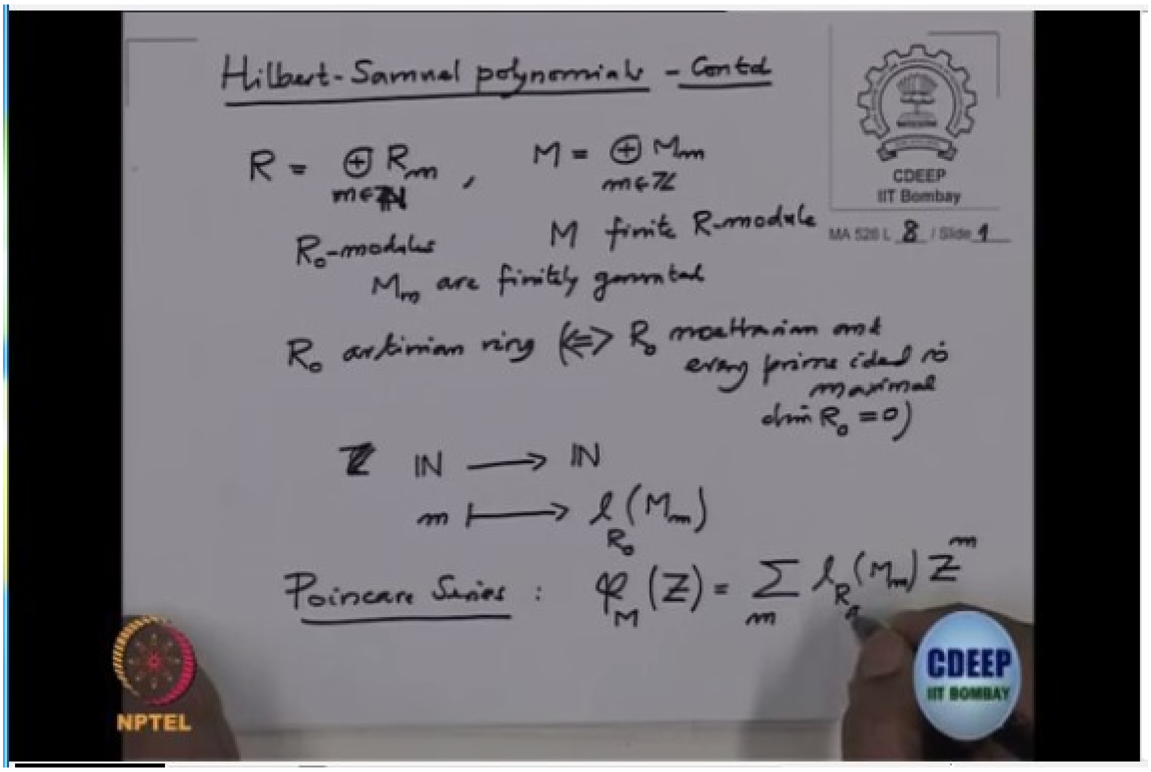
Lecture – 25

Numerical Function of polynomial type

Gnanam Paramam Dhyeyam. Knowledge is supreme.

DILIP: So, we will continue our study with Hilbert-Samuel Polynomials and its connection with the dimension. So, let us recall what we did in the last two lectures. So, we had a graded ring Rm and a graded module over R. So, when you do it for the ring, you have a positive, non-negative grading, -grading, and for module, it is $\mathbb{Z}$ -grading. And we are assuming that M is finitely generated, M is finite R-module, which is equivalent to saying that, it can have only finitely, meaning negative components, and all modules, all homogeneous components are modules over R_0 . R_0 -modules, Mm are finitely generated. These are our assumptions. And also, we assume that this R_0 is artinian ring. The slimiest way to think is, that is equivalent to saying R is, R_0 is noetherian, and every prime ideal is maximum. This simply means that the dimension of R_0 , Krull dimension of R_0 is zero. So, in this situation we have considered this function, so that is the function from $\mathbb{Z}$ or $\mathbb{N}$ to $\mathbb{N}$, namely, any m goes to length of M_m as a R_0 -module. This length is finite because we are assuming M_m s are artinian R_0 -modules. And, to this, we have associated series, which we call it Poincare Series, that is P, it depends on M and I want to also write that variable Z. So this is, by definition, this direct sum is running over M. It is length of M_m , this is the coefficient of Z^m . Remember that, when I did in the one more earlier lecture that instead of length, I've used a dimension. That is because, I was assuming that time this R_0 is actually a field.

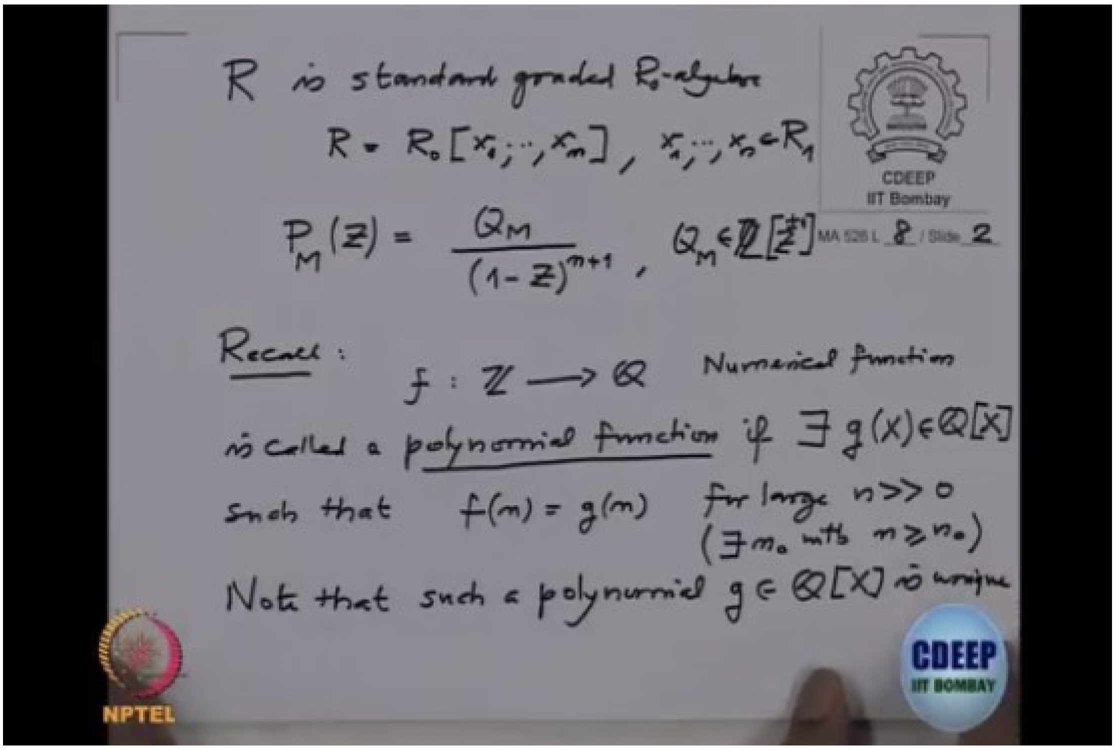
(Refer Slide Time: 04:43)



And what we proved is, this as a special form if we assume R is, now assume R is a standard graded R_0 algebra. This simply means R as a R_0 algebra generated by finitely many elements, and each element, all these x_i 's are homogeneous of component homogeneous degree one, so that is

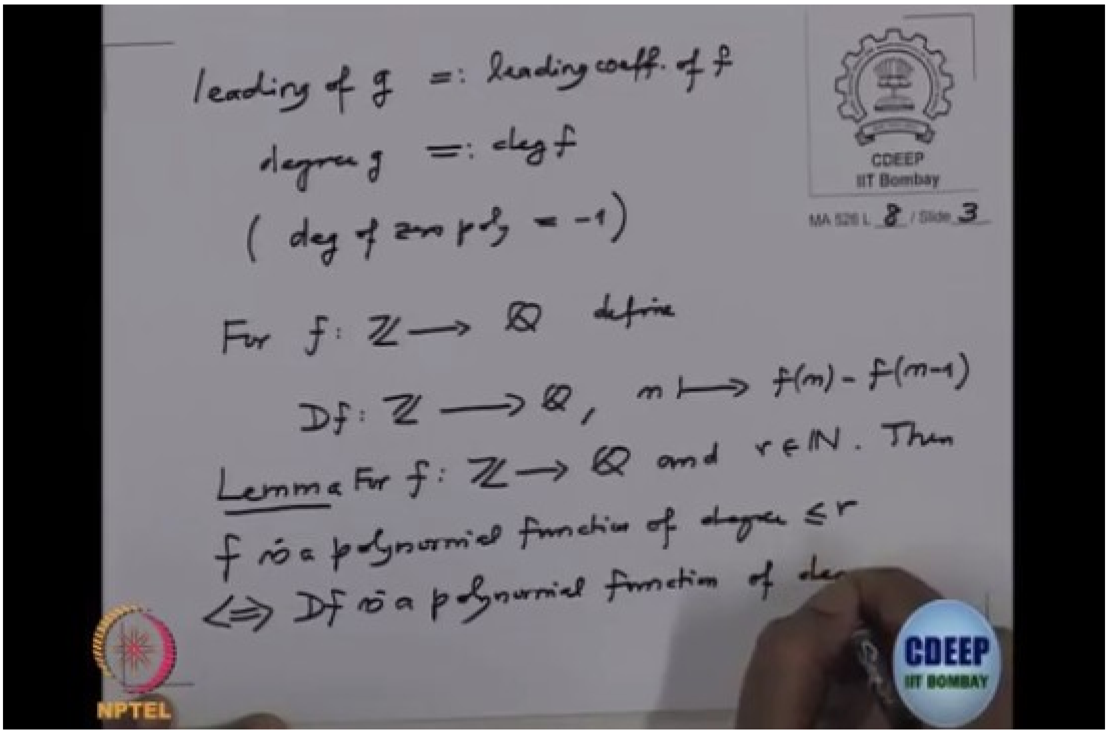
R_1 . And, in this case, what we proved is, this Poincare Series of M is of the for-- First, let us write it for R. So, it's of the form, some Laurent polynomial $\frac{Q_M}{(1-Z)^{n+1}}$, where Q_M is Laurent polynomial, so that is $Q_M \in \mathbb{Z}[Z^{\pm 1}]$. And we are discussing about the degree-- So, first, let me now make it clearer. So when says, somebody is a function, so recall, generally, if you have a function, numerical function f from $\mathbb{Z} \to \mathbb{Q}$, such a function is called a numerical function. We will say that, this is a polynomial function, is called a polynomial function if there exists a polynomial $g(X)$ over, with rational coefficients such that $f(n)$ equal to $g(n)$ for large n. Large n means, this means, there exist n_0 with this equality hold for all n bigger equal to n1. So, first note that such a polynomial g must be unique, because, if there are two polynomials which satisfy the same equality, then those two polynomials agree on many rational numbers, infinitely many rational numbers, and therefore, they have to be equal. So, note that such a g, such a polynomial g with rational coefficients is unique

(Refer Slide Time: 08:34)



So, therefore, in this case, the coefficient, leading coefficient of g, I will keep calling this is to with the leading coefficient of the function f. So, this is leading coefficient of f. This is a definition of leading coefficient of f. Similarly, the degree of g, we say degree of f. So, for a numerical function, which are polynomial functions, we have defined this leading coefficient and the degree of the-- And, also we will use a convention here, degree of a zero polynomial, degree of zero polynomial to be equal to -1 . This is a convention. Degree of a zero polynomial is not defined, but-- Okay. Now, how do we decide some numerical function is a polynomial function or not? So, for that, you define a new function, this is something like a derivative. So, for the new, for a numerical function f, define Df. This is also new map from $\mathbb{Z} \to \mathbb{Q}$ which is defined by any n goes to $f(n)-f(n-1)$. So, this is a new function. And then now, in simple lemma we'll prove, so, f is a numerical function and r is a non-negative integer, then f is a polynomial function of degree less equal to r if and only if Df is a polynomial function of degree one less.

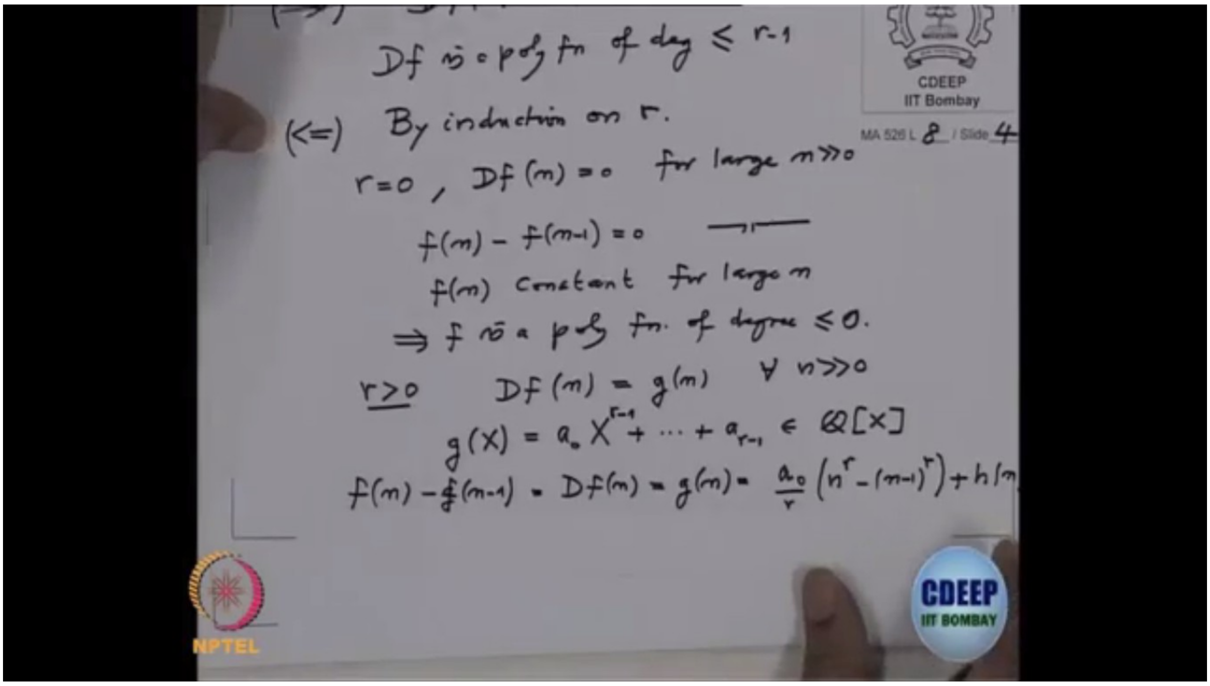
(Refer Slide Time: 11:34)



So, first, we are proving this. This is pretty clear because, see, look at Df, $Df(n)$ is by definition $f(n)-f(n-1)$. So, if f is a polynomial function, the leading coefficient of this right hand side is same, and the degree is same, so we'll get, the leading coefficient will get cancelled. So, for large n, this is the difference of two polynomials of the same degree and leading coefficient. So, it is clear that df is polynomial function of degree less equal to $r-1$, because we are assuming f is of polynomial function of degree r, less equal to r. Okay. So, conversely, this way we are doing the assertion by induction on r. So, we have now given that Df is a polynomial function of degree r minus one, and we want to prove f is polynomial function of degree less equal to r. Okay, so r is 0, r is o means this Df is the polynomial function of degree less equal to -1 . So that only mean that df the 0 polynomial, that means, Df, so this means, Df evaluated at n, if 0 for large n. So this means that $f(n)-f(n-1)$ is 0 for large n. But that means if n is actually constant for large n. that simply means that constant. So f, therefore, f is a polynomial function and because it is constant degree less equal to. So that implies f is a polynomial less equal to 1. So that proves the assertion for r equal to 0. Now assume that, we have assume r is bigger than 0 and Df is a polynomial function of degree less equal to $r-1$.

So Df will agree to some polynomial g for large n where g is a polynomial with rational coefficients $a_0 X^{r-1}+...+a_{r-1}$, these are rational numbers. Degree of g is less equal to r minus 1. Okay, now just from here, what is $f(n)-f(n-1)$ $g(n)$ which is, what I am going to do is just take $\frac{a_0}{r}(n^r-(n-1)^r)+h(n)$. So whatever remaining terms I club with $h(n)$.

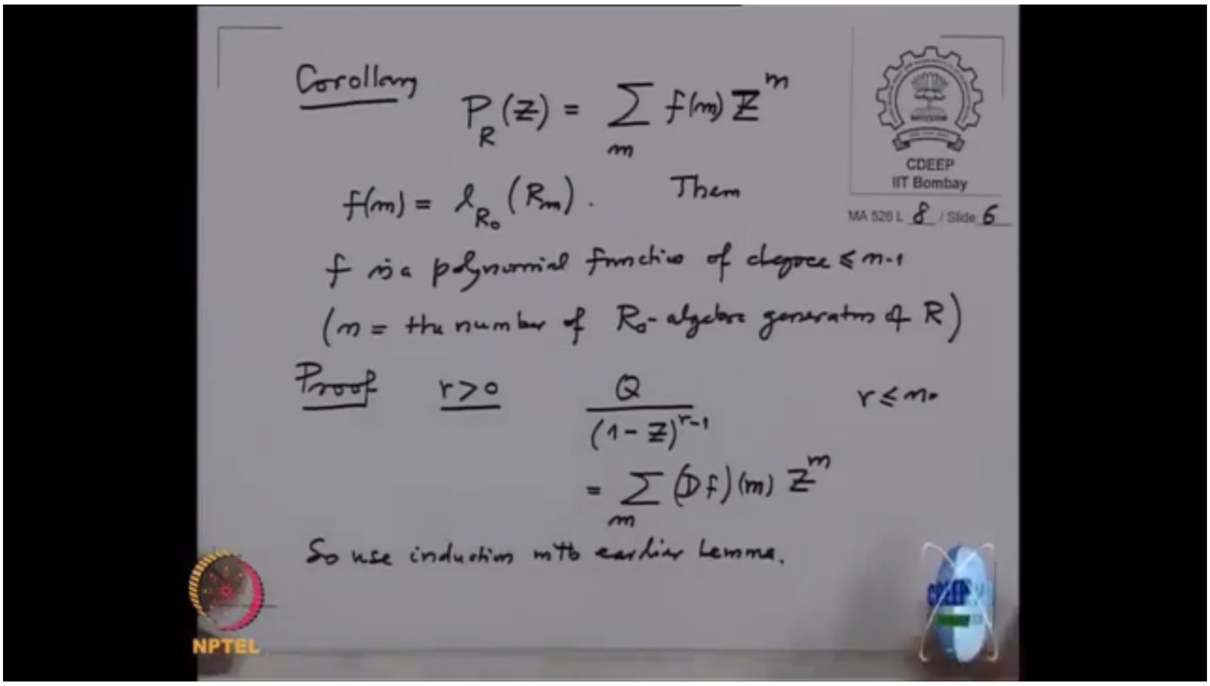
(Refer Slide Time: 15:53)



Where $h(X)$ .Where $h(X)$ is a polynomial with also in $\mathbb{Q}[X]$ and the degree of $h(X)$ is small or equal to $r-2$ because I have taken the top degree terms here. Okay, because these n power, when you expand this term by polynomial, n^r will get cancel and will r here and then that. So therefore $h(X)$ is a polynomial of degree less equal to $r-1$. Now if you define f' , this is a new function from $\mathbb{Z}\to\mathbb{Q}$, define by $f'(n)$ equal to $f(n)-\frac{a_0}{r}n^r$. so this precisely tells us that D of $f'(n)$ if that remaining residue with that is $h(n)$ for large n. And because degree h is small or equal to $r-2$. This Df' is a polynomial function of degree small or equal to $r-2$, therefore by induction hypothesis f' is a polynomial function of degree less or equal to $r-1$. And shifting this term to the other side is we get then f is a polynomial function of degree less or equal to r. So that proves the lemma. So recall that we have proved that if you take this Poincaré series its a rational function of the special type and the coefficients and the coefficients. So let me write it as a corollary. Let us $P_R(Z)$, this is like the definition was, this is $f(m)Z^m$ and $f(m)$ was the length of as R_0 module R. So this $f(m)$ then, $f(m)$ is a polynomial function of degree less equal to $n-1$. And remember this n was what, this n was the number of

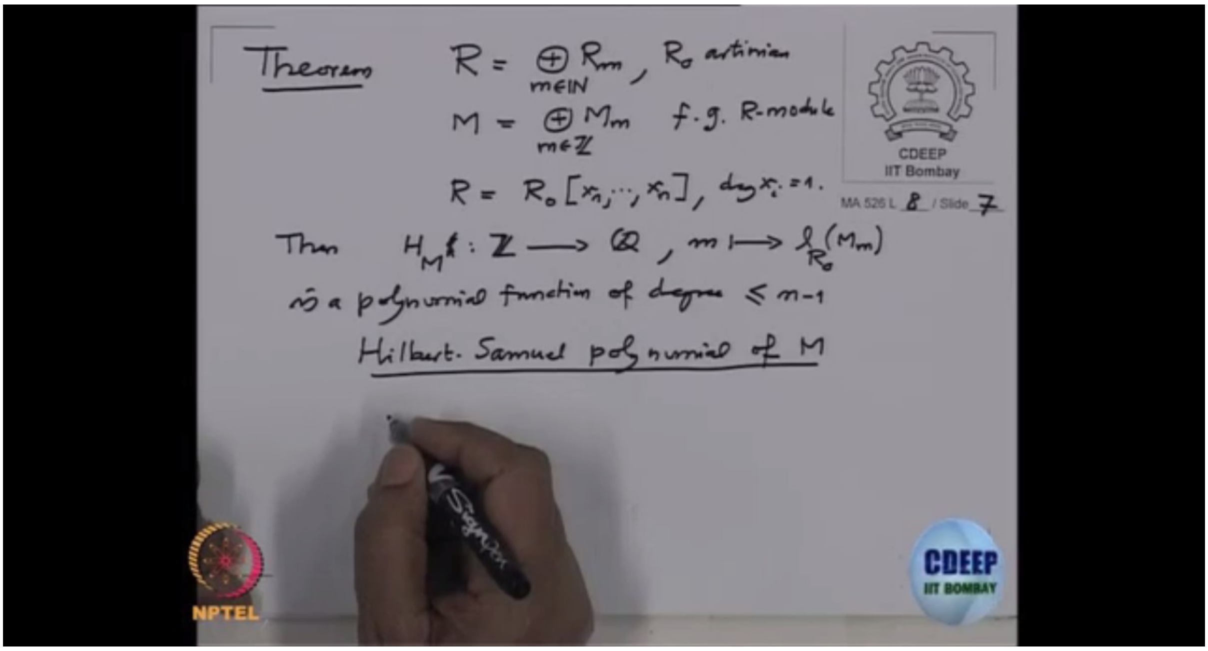
R_0 algebra generators of R. That is n minus 1. So proof, just note that the formula if you have R positive then that, remember this we have written in the form $\frac{Q}{(1-Z)^r}$. I forgot something, I forgot that when we have a rational function that if a numerator also has at 1 equal to $Q(1)$ is 0 then we cancel the numerator this $1-Z$ part there and below also. So we have this, now below is r and this r has to be less equal to n. So note that this is equal to submission $Df(m)Z^m$. So therefore by induction the assertion follows by induction by using the earlier observation. So use induction with earlier lemma.

(Refer Slide Time: 22:31)



So just let me note formally what we have proved. We have proved the following theorem. So R is a graded ring, $\mathbb{N}$ graded and with R_0 artinian and M is graded finitely generated R module and R is standard graded, means R is R_0 and generated by few elements. $x_1, \ldots, x_n$ where degree x_i are 1. Then, what we call it H_M , this H_M think of this is a function from $\mathbb{Z} \to \mathbb{Q}$, any m going to or length of either R_0 module of M_m , this is a polynomial function of degree less equal to $n-1$, where n is this number of R_0 algebra generators of this This polynomial so, unique polynomial associated to this numerical function and that polynomial is usually called Hilbert-Samuel polynomial of M. So that is a polynomial associated to this polynomialfunction. So this polynomialas a degree and leading coefficient.

(Refer Slide Time: 25:12)



So these two invariants, the degree we are preparing to prove to that the degree of this polynomial is actually the Krull dimension of the module. And when the module is R then it's Krull dimension of the ring. That is what we want to prove.