Lecture – 24

Hilbert-Samuel Polynomials(Contd)

Gyanam Paramam Dhyeyam: Knowledge is supreme.

So now, I'm going to recall modules of finite length. It's just a review. I will not prove the statements but they'll make the statements so clear that you can prove on your own. So R is our ring. Let me call A as a ring. A is our base ring, always commutative and M is an A-module. We have studied Noetherian modules and also Artinian modules I suppose. So let me recall. So what do you do for a fix module M, I consider all submodules of that. That is $s_A(M)$, this is the submodules. N is a submodule. A submodule. And this is said, and the inclusion only said, $s_A(M)$ with this inclusion. This becomes an ordered set. Ordered set means, a set with a relation and the relation should satisfy three properties that is reflexive, antisymmetric and transitive. So just it is called an ordered set. In many books you'll find that such are called partial ordered set but partial word is unnecessary there, so I dropped it and more you go to older books, you'll not find a word partial. I don't know why it came from. So orders set for us is a set with an order on that. Okay, now in a ordered set ascending, descending chains make sense. So a module is called noetherian, if every ascending chain become stationary. And now when do you call a module to be artinian when a descending chain is stationary. But now you see, to each ordered set there is concept of dual. Dual is opposite order, so whatever theorem you want to prove for noetherian, if I change this set to the dual set the statement will prove. So the noetherian will become artinian and artinian will become noetherian. So you only really have to prove one theorem. So if you prove a theorem for noetherian then the same theorem will be true for artinian by just proving, by just changing this set. Changing the order to be the opposite. So for example, if you want to show that, if you have short exact sequence. So 0, N, M, N' . If this is short exact and if the middle module is noetherian then other two modules are noetherian. So same statement, if M is artirian then these two if and only if these two are artirian, how will you prove? We just change the order and this is really neat.

So now it may happen that you would've seen examples of modules which are noetherian but not artirian and artirian but not noetherian. So if it is both those modules have nice properties. So whatever, so module M with the property ACC as well as DCC. So noetherian and artirian, these modules are worth noting. So what does that mean? That means… This means, when we have such a module then we have a finite sequence M, which is M_0 contains M_1 , contains and this cannot go on forever. M_k and then it becomes 0. And each stage not equal. And also this containment, of course, not equal but I want to also assume that there nobody in between because if somebody in between I'll insert it. So that means, I will make the chain more and more finer. So there exist a chain such that the successive quotients $\frac{M_i}{M_{i+1}}$ are simple modules, simple A modules for all i. Simple you know, simple means, there is no submodule other than 0 and the whole module.

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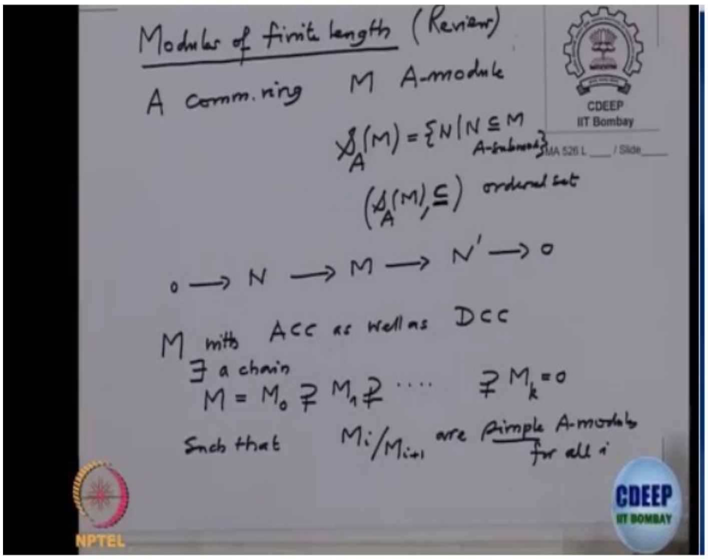

So such a series is called Jordan-Holder series. Such a chain is called a Jordan-Holder Series for M. Some people also called it a Composition series. Okay, now if you have two such chains. Two such Jordan-Holder series then this theorem that they have the same length. A need two Jordan-Holder series for M are equivalent. Let me write it are equivalent. So what does equivalent means, so that is if I have two chains like this. M equal to M_0 containing M_l ,.., M_k which is 0 and other one is N equal to N_0 , N_l ,... this is 0 and here the successive coefficients are $\frac{M_i}{M_{i+1}}$ are simple. Here $\frac{N_j}{N_{j+1}}$ simple and we call them equivalent. If K equal to l, and these exists, these guys are permutations of this. That is there exist a permutation σ on k letters which is l letters so, S_k these are the permutations and k symbols such that $\frac{M_i}{M_{i+1}}$, this is isomorphic to $\frac{N_{\sigma_i}}{N_{\sigma_{i+1}}}$. So up to permutation the successive coefficients are same, isomorphic. Two such series are called equivalent. So Jordan-Holder theorem say that any two Jordan-Holder series or a module are equivalent. So in particular, they have the same length and this is called the length of a module. So I will not prove this theorem. Either do it yourself, I see some standard book. See, may be a McDonald.

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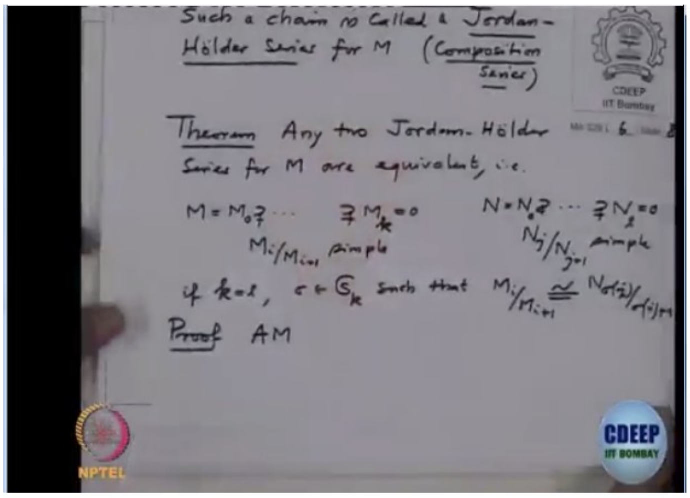

So that is called a length of a module. So I will write that symbol by length l, just to remember which ring we are working on and this is the length of M. This is called a length of M. These are called modules of finite length, because length should make sense. In order that it make sense, we have assume the modules are of both these should satisfy DCC and ACC. So in particular when your base ring is a field. Suppose, your base ring is field then when will module M. Module M means, K is a vector space. When will it satisfy ACC and, when will it satisfy DCC, when it is both finite dimension. So length of A by M, in that case this is nothing but the dimension of that module, dimension of vector space M. So this make sense when it is a finite dimensional vector space. So now going back to our assumption on R_0 . See R was a graded ring and as an algebra over R0, it is generated by finitely mini elements. This one came because R is noetherian. That was our assumption original. So these four say, that R_0 should be noetherian but I want assume in above exposition we have R_0 is actually a field. But I could also assume that R_0 is finite k-algebra. Remember, finite means, finitely generated as a module that means, as a vector space it is finite dimensional. And in that case, the dimension, see in a Poincare Series we have these dimension over k of this modules $M(m)$ s. So now this $M(m)$ s are of finitely generated modules over R_0 and if I assume R_0 is actually finite dimensional vector space then this dimension I will have to replace by length over R_0 .

This will make sense, because this is a finitely generated module and this R_0 is artirian ring and therefore it is artirian and this length will make sense because it will have, it's a noetherian and also artirian. Therefore this length will make sense and in the above theorem everywhere only definition will have to replace dimension by this length. And all other induction et cetera everything will go through in the calculation. You have a graded module R, the graded module we started with. So we have a graded ring and graded module over that and we assuming the R noetherian and M is noetherian and then we define P_M and in the definition of P_M, this were the coefficient of Z^m. So in the general case now you will have to replace this dimension by the length because if you see the proof of that if will depend on the Jordan-Holder series. So every time successive coefficient. Just to make that change and check whether a same proof will go through, if you make a change replace dimension by this length.

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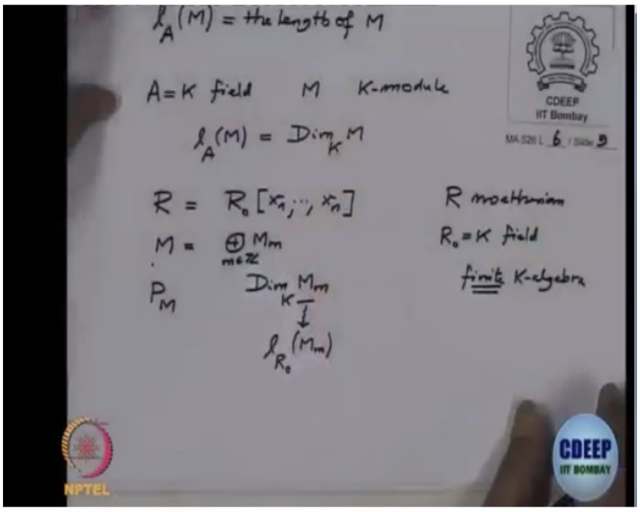

R_0 is a field but I'm saying now you can even assume now R_0 is a finite k-algebra. That R_0 is a field. But we have used for a definition, no? Because I have use the word dimension there. That's what I said, for a definition of a Poincaré series, the way I've written, we have use that they are finite dimensional k vector spaces, right? So R_0 was our K, otherwise Poincaré series M, what we have to define it has summation running over M. Now, length of R_0 over $M(m)$ and then Z^m. See, earlier when R_0 was a field here it was dimension. Now, I'm saying, you take this as a new definition and this definition. So what we did there was the properties of the dimension, and here also we'll have to use the properties of the length and what properties when we have a exact sequence shot exact sequence. Length of middle module is length of the sum from the other two modules. And if we have a long sequence then the length and if you take alternating signs they need to be 0. The proof will be the same. So I will not go in to that details. So first of all, if you assume R is standard graded that means, R, now I will use R_0. R is R_0 algebra generated by x_1 to x_n on all x_i s, all x_1 to x_n. They are actually degree 1 element, R_1. That is why the standard 1.

So sometimes it is also return that this R is as a R_0 algebra generated by R_1. And R_1 is finitely generated R_0 module therefore it is. You can choose finitely many elements. So in this case, the Poincaré series were any grade module M, P_M then looks like, some Laurent polynomial Q and all now will be $(1-Z)^n$ because this will be $(1-Z)^{\gamma_1}...(1-Z)^{\gamma_n}$ and so on then n now. So now but see, this Q is a Laurent polynomial, maybe this 1 - Z maybe factor of this and so on. So I want to reduce right in the middle. First of all, I want to define a new function. This is P_M, and this $P_M^{(1)}$, we'll see, why I want to do this. This is just P_M, one factor I want to add more, $1-Z$. This is same as $\frac{Q}{(1-Z)^{n+1}}$. And this will be power series Z^m. This is a power series, right? Think of it's a power series and this is also power series. This is power series 1 + Z. Remember this formula, $\frac{1}{1-Z}$ is $1+Z+Z^2+...$ and so on. So the coefficients here, they are, I will denote them by $H_M^{(1)}(m)$ and what $H_M^{(1)}(m)$? See, here what was it originally, it was only the length or dimension and here now it will get added up because when I'm multiply by this power series, this one over this, when I multiplied by this power series they will get added up. So this $H_M^{(1)}(m)$ will be equal to $Dim_k M_i$ and i will be up to m. Now, I'm still writing dimension over k, that is because we are assuming these R_0 is actually finite K-algebra, if R_0 is a finite k-algebra with dimension, what is the relation between the dimension and length? Then length over R_0 will be this dimension. Length over R_0, so if you take now. First length of R_0 over R_0 is just dimension of R_0. All right, so now, with this simplification now, what we have to do? We have seen this P_M is a rational function of a particular type and this is further rational function. Now, I want to use a partial fractions. So partial fractions. Do you remember what is Partial fraction? Partial fraction decomposition of a rational function. See, I will just remind you that we have done this many times.

For example. Let me give you a simple example. You remember when you wanted to solve integration of rational functions. So you wanted to solve integration of $\frac{f(t)}{g(t)}$. What did you do? You wrote the denominator polynomial you make split and then so simple. In a quadratic case you did this, no? This t minus a, t minus b and 1 over this. First, you divide this polynomial f by g and then assume that the degree is smaller. Degree of denominator is smaller than the degree of g. And then look at this, this you wanted to write, somebody here, somebody here, t minus b, right? And you made a computation and see. If it was a power here, then you wanted to do this and also you wanted to add square term and so on, right? And why did you do that because the integration of this individual became is here. But very important assumption what you have made is the denominator polynomial in that rational function splits into linear factors, right? Which is not a proof or for example, bl polynomials but real polynomial it is not too bad because either a linear factor or a quadratic effect but rational polynomial over other fields and that is actually, that is why [inaudible] stated the fundamental theorem of algebra. You wanted to prove fundamental theorem of algebra precisely for this reason not because of any algebraic reason. You want to come to an integration and therefore your looking for such a partial fractions, decomposition with that integration calculation becomes easier. Therefore he stated that time, fundamental theorem of algebra and he was using only

V_L 's but if you proof our complex numbers for V_L s also it follows. Fundamental theorem for algebra V_L s is every polynomial with real coefficients splits into linear quadratic factor. So that was, so this we are going to imitate for the this rational function generator P_M . So what will happen then? So we have this P_M^1 and we have written it like, $\frac{Q}{(1-Z^{n+1})}$. And remember n is the number of algebra generators for R. And this is also we have written a series $H_M^{(1)}(m)$ Z^m .

All right, in this I know, with the-- so the first is I'm going to divide the Q by this, and write this as some polynomial P by the remaining one. The reminder divided by this, the denominator. But in that I'm also going to use, So this P is what? P is the unique polynomial, Laurent polynomial which I got after dividing by this, right? And in some more terms will come, in those terms I want to use this formula .This is the decomposition of this partial fraction. So $\frac{1}{(1-Z)^{\nu+1}}$. What is the formula for this? This is a sum over m. $\binom{m+\nu}{\nu}$ Z^m . This is more general than 1 - Z, no? So this formula, I'm going to plug it in here and then re-write this term. So what will I get? And I want to compare and after that I will compare the coefficients of Z^m . Because then I will get a power series when I write this formula and then I have this power series then I'll compare the coefficients. So what will be coefficients? So the coefficients will be they will look like this, $H_M^{(1)}(m)$. This one is like this and this one is also like this. $H_M^{(1)}(m)$ at m this will look like, when i use this formula then compare.

So some integers and then this coefficient $m+\nu$ and this sum will run from 0 to somebody because you see it's a power series. This way, what did I do on the left side. I first, divide it by this, $1-Z^{n+1}$ to this Laurent polynomial and got this Laurent polynomial. So now n minus Z power that divides and then I use or directly if you want Q is Laurent polynomial and take this $1-Z^{n+1}$, and then use this formula. And then compare the coefficient of Z^m on both sides. So here this binomial coefficients will come and they will come with, when I made a division so this d is a unique integer and this is you wrote to e_d , they are integers and ed is non 0.

Yes, they all depend on Q and also… whether $1-Z$ divides or… But this also formula holds only for large M. You see, because this Q as negative terms, no? So when you compare, I want to compare only for the large power so that the negative coefficient are not playing any role. And you see, this left side is a length or dimension? So which is a non-negative integer. So from this formula it is clear that this $H_M^{(1)}(m)$. or we will see example for calculation. For calculation, $H_M^{(1)}(m)$ this is actually, asymptotically equivalent to e_d and when you expand the binomial it is $\frac{m^d}{d!}$. Asymptotic. Asymptotic means for large. So because when the m is smaller in this binomial expansion the terms will not contribute to the large one. So they will be smaller and smaller. So this in particular implies that this e_d is positive because this side is sign is determined by the leading term so that is e_d positive.

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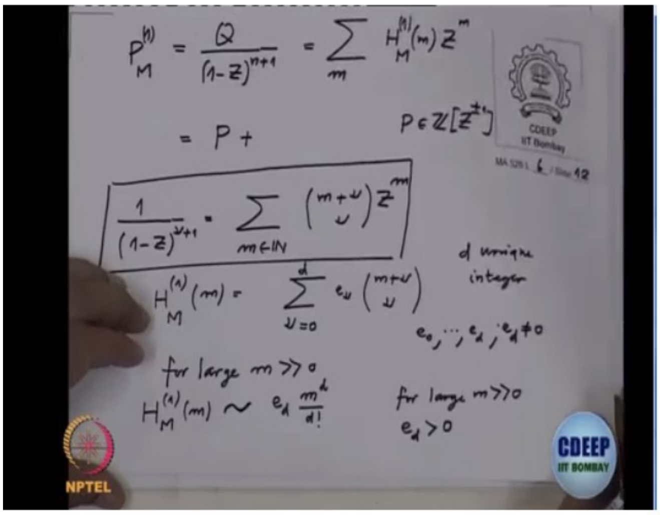

Now this integer d which I got out of this that I will prove it is a dimension so that will give us a nice theorem that how do you compute the dimension? So let me write this formula normally a corollary. So that what corollary we are proved is the following. So R is the standard graded, $R_0[x_1,\dots,x_n]$, degree of x_i is one and M finite graded R-module, then there exist unique integer d, actually natural number d, which is $d(M)$, it depends on M, so natural number and integers $e_1,\dots,e_d \in \mathbb{Z}$ with e_d positive, this is if d is positive. Such that $H_M(m)$ which is the dimension over K of M_m this is equal to summation form ν equal to 1 to d, $e_\nu \binom{m+\nu-1}{\nu-1}$ for larger m. Now here the only difference I made is, we have compu $H_M^{(1)}(m)$ ted . $H_M^{(1)}(m)$ is sum of the dimensions. From $H_M^{(1)}(m)$ if you want to compute a dimension you just take the derivative. It's called derivative. So that means, you take $H_M^{(1)}(m+1)-H_M^{(1)}(m)$ and we will get. See, how are going to get, remember this $H_M^{(1)}(m)$, this is the dimension minus equal to n. So if I take m + 1 and subtract $H_M^{(1)}(m)$ from that, I will get this $H_M^{(1)}(m)$, so we get, we have this formula, $H_M^{(1)}(m+1)-H_M^{(1)}(m)$, this is $H_M(m)$ m here and $m-1$ here. You see, here, it's the top degree term will get cancelled that's why it is here the $\nu-1$. See, the binomial coefficients. These polynomial, these , th $H_M(m)$ is a polynomial so, also the polynomial , this is $H_M(m)$ he polynomial in m, you see, it's visible here, because here it is this formula. It's a polynomial in m or degree d. This polynomial is called the Hilbert Samuel polynomial of M. Now let us see, one example at least. So we will get a custom to the calculation. So example, let us take K is a field. And let us take our graded ring to be the standard ring, which I will denote X, not one variable but many variable. X_1 to X_n . And let us take module also that M is also the same. So M is a graded module. And it is standard, R is standard graded K-algebra. In this case R_0 is K. Okay, so what is $P_{K[X]}$ that is what we want to compute, right? What is the definition of $P_{K[X]}$, so this is a series. So this series is what are the coefficients of Z^m, the coefficients here are precisely what dimension of over k of M_m. So you need to compute what is dimension of M_m. So what is M_m ? M_m is homogeneous components of degree m and what is that? What is the coefficient? This is the binomial equation, right? This is $\binom{m+n-1}{n-1}$. You see, you can test it, take one variable so what is one variable case? So n is 1 then it should be what? One only,right? So it is m plus 0, choose 0 which is 1 always. It's actually the series $1+Z+Z^2+\ldots$ and so on. This one is nothing but $\frac{1}{(1-Z)^n}$. This we approved it. The Laurent polynomial is 1 in this case. So it is this. So what is d in this case? d is n and which is of course, this anything nothing but the krull dimension of our ring. This we have proved it in the last lecture we approved. If we take a polynomial ring over a field then the Krull dimension is n. And what is e_a, now it as coefficients up to e_n. So en is 1 and all other coefficients are 0 for all $i \leqslant n$, in the corollary which you have stated.

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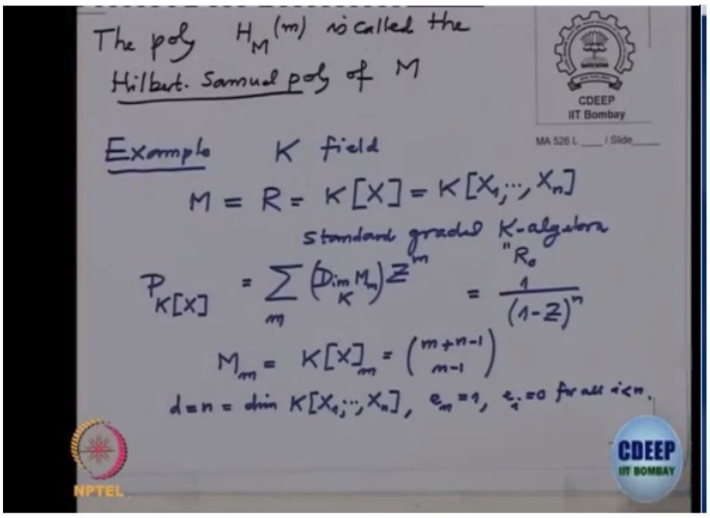

Because it's only the this and this and the remaining coefficients are 0. And now you could also compute an example where you take polynomial ring and choose one homogeneous polynomial of sum degree and then calculate. We take F homogeneous polynomial of degree r and take R to be equal to polynomial ring in n variables, module F, ideal generated by F. This is your graded and now do calculate P_R first. And how will you calculate P_R ? P_R you know, to calculate P_R , you calculate you know the P of the polynomial ring and this F degree r used at multiplication by kind of argument and you will calculate. Then the rest is the numerical calculation with the polynomial.