Lecture - 23

Hilbert-Samuel Polynomials

Gyanam Paramam Dhyeyam: Knowledge is supreme.

We will continue our study of dimension through Hilbert Samuel polynomials. So, last time I studied a theorem and we have to, we're not started the proof of it yet. Today we will finish the proof and continue with the study. So, let me recall from the last time that we have a graded ring R. That means, it is like this, it's decomposition into subgroups. This is $\mathbb{N}$ -graded. And the condition that R_n times R_m is contained R_{n+m} for all n and m. In particular, R_0 is the subring and all the subgroups, R_m are R_0 modules. And we are going to assume further that these R is a noetherian ring. R is noetherian. That means all ideals in this ring are finitely generated or equivalently arbitrary family of ideals have a maximal element. This one is equivalent to, this is equivalent to saying that R_0 is noetherian and this ideal for this $(R,+)$, if you take all non-zero direct sumands. R_m and these, positive, this is clearly an ideal. There's an ideal in this. If this ideal is finitely generated, and R_0 is noetherian then already that's equivalent to saying R is noetherian. This is not so difficult. This is because if this ideal is, so this way is obvious. This way is clear and for this way, R_0 in noetherian given and this ideal is finitely generated. This is the homogeneous idea. It's clear, it's homogeneous, and it is finitely generated. So, finitely mainly homogeneous elements will generate that. So, if you take those homogeneous elements. So, R_0 , $(R,+)$ is generated by homogeneous elements say, where our x_1 to x_r are homogeneous and positivity increase. Then the map from the polynomial ring $R_0[x_1,\dots,x_r]$. To R, R is generated as an algebra by this x_1 to x_r over R_0 .
So there is a natural subjective map from here to here. And because this is noetherian, this is noetherian by Hilbert. This is the law. So, by Hilbert, this is R is noetherian. That is usually the basic assumption we will make always.

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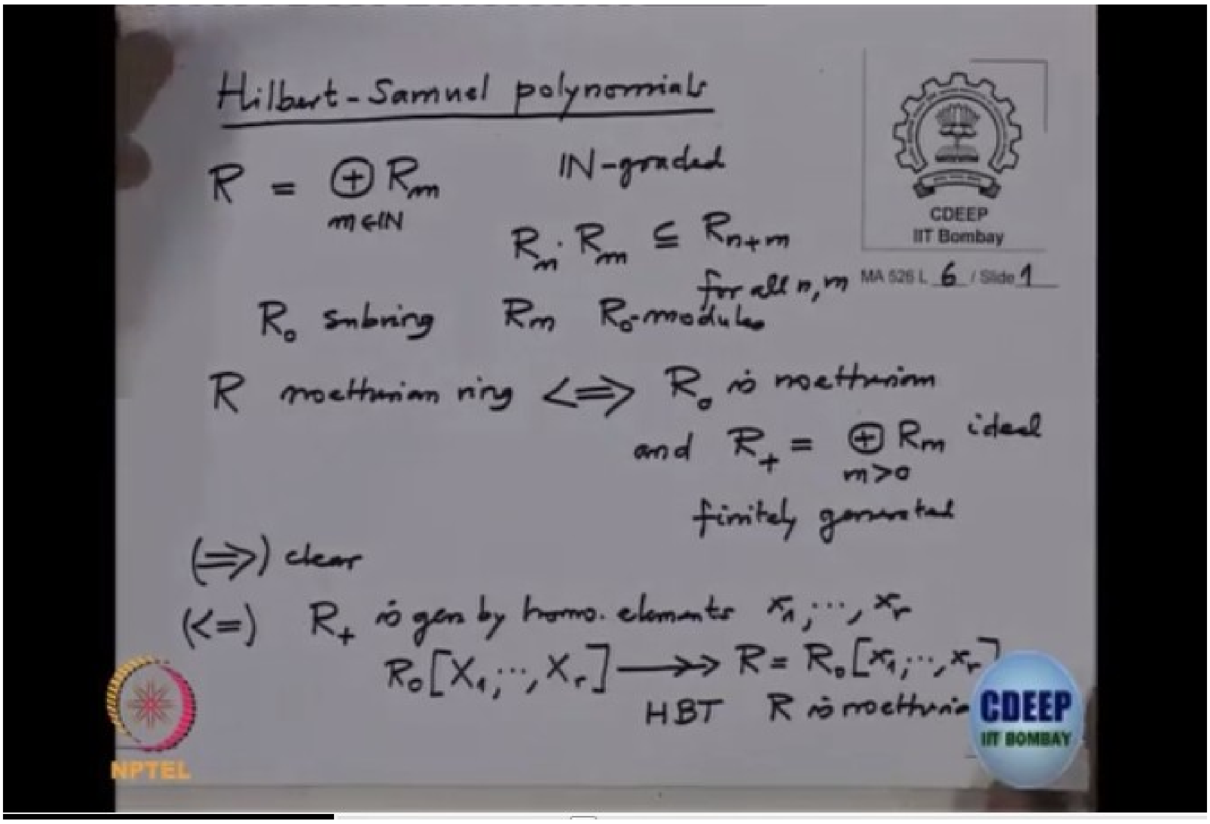

Alright. So, then, we consider the modals over this ring. Which are graded modules? So M is a graded module. So M is a as a decomposition into Abelian group like this. And now you allow some negative integers also. This is R module. And the graded means this $R_n . R_m$ is containing M_{n+m} . So also this, each, these are homogeneous components of degree M and they are R_0 modules. And now also, we make the standard assumption that M is finitely generated R module, which is equivalent to each M_m is finitely generated R_0 module for all M. And for large negative they are zero. M_m is zero, for all M, large negative. That is symbol for sufficiently in large negative. This is also very easy to see because. So, first of all, this way is clear and this way, if any finitely generated then, then, if you shift, if you look at M, less equal to or bigger equal to M, this is direct sum after M_m , n bigger equal to m. If M is finitely generated then all these are the submodules. Therefore, they're finitely generated and the successive quotient Mm is M bigger equal to m plus 1, module M bigger equal to M. Because when I go mod, when shift, we will get the homogeneous compound of degree M. So, this is R is noetherian. So these quotients are finitely generated that means Mm are finitely generated. And each Mm are finitely generated. So, unless, this M is, if it is not zero for large negative integers then you can produce a chain, ascending chain which will not have a, which will not become stationery.

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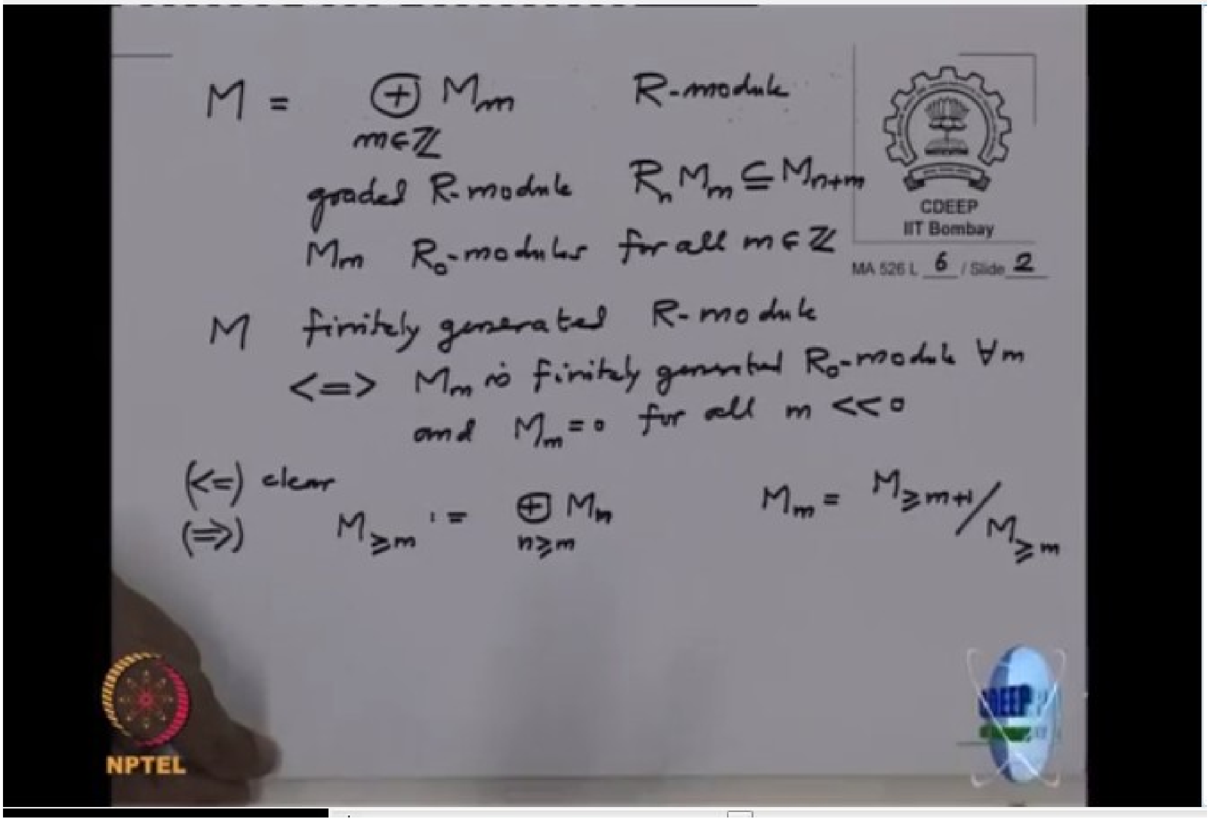

So therefore, all M_m are zero for large negative M. So these are the usual standard assumptions when makes and standard example one considers is the polynomial ring over a field $K[X_1,\ldots,X_n]$. And these are the standard grading. That simply means to each variable, you give grading to degree of variable X_i is 1 for all i. This is called a standard grading. With this, the homogeneous components are precisely the generated by homogeneous polynomials of that fixed degree. So those are the homogeneous components. And, the more examples we can create this standard graded ring by going module of the homogeneous ideals. So where we have enough number of examples for graded ring and so on. In fact, what I will do after I finish this module, I want to study in general dimension of Noetherian ring by using graded, by reducing their schedule to graded rings and use this theory for that. So to each graded module, to each graded module m, we've attached this series that is we called Poincare series, P_M. I know, how is with this, this defined Pm is defined as, this is a series, power series. So this is actually power series. It is a power series with coefficient in $\mathbb{Z}$, in the variable Z, and polynomial Z inverse in that. So that is dimension, look at the graded components here. Homogenous components M_m, some of this could have negative homogenous component. So this is the sum dimension of. Here I was also assumed at least for a while, I will assume that R_0 is, so R is graded ring. It looks like this. And R_0 is actually finite K algebra, where K is a field. K is a field. So just for the sake of understanding, take R_0 equal to K, because the general K is finite, K algebra, I want to soon even do it even more general than that. So I want to assume that R_0 is artinian ring. I will recall about artinian rings just before I start the general case. So for a field, if R_0 is a field, then all these M_m are K vector spaces. So take the dimensions, so this is some integer and take, this is a coefficient of Z^m. So it's power series in Z with integer coefficient. But you remember this power series as some negative terms. So therefore, and they're only finitely limit, because of our assumption that M is finitely generated. So therefore, it is a Laurent polynomial in Z, with coefficients in the power series. Okay. And what did we see? Last time we saw how does one compute, for example, when I have a twist, twisting means shifting the grading. So for each integer m, I have defined $M(-m)_n$. This is a new graded module, such that, the grading is shifted by this at n, equal to M_{m+n}. This is a new. This is a grading module, only shifting, only we have renumbered the components by the shifting. And then we saw, if you take the compare the Poincare series for m and Poincare series for the shifted graded module, then how does it behave we saw. This is equal to Z^m. In general, you could write for a k actually, that's better $M(k)$. For any integer k. This is just shifting by k. So how do you compute?

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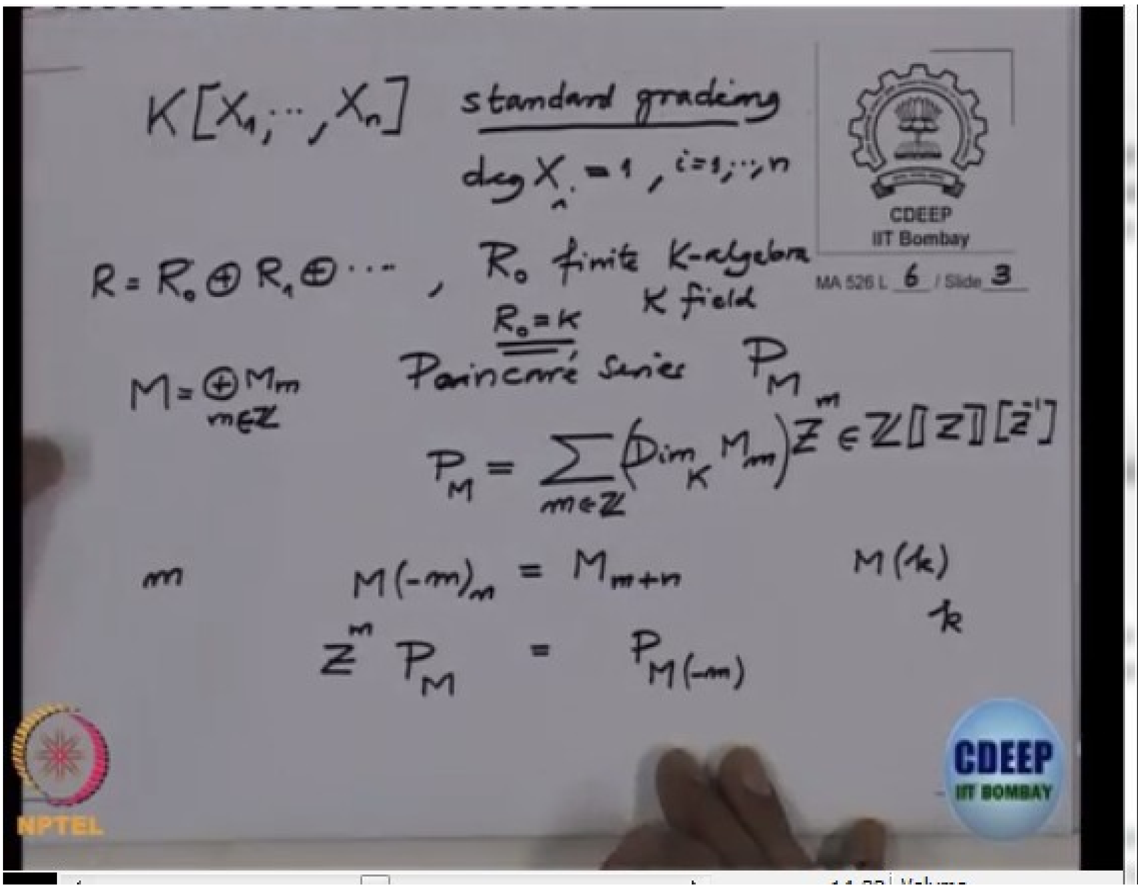

Also we saw when you go mod homogeneous element then how do you compute the Poincare series? Okay and then, with all these things, we have stated a theorem and we want to prove that theorem. So what is the theorem?

Okay, so we know now that our R is over R_0 generated by finitely homogeneously elements. So that is x_1 to x_r. And x_1 to x_r, each x_i, homogeneous and let us call the degree to be γ_i i is from 1 to r. They're non-negative integers, positive integers actually, so gamma Is are natural numbers, positive natural numbers. When all the gamma Is are 1 then we're in a standard case. And we will see some examples where allowing all gamma Is are not necessary one that also has helping calculation of some Hilbert series or some dimension to some, some more invariants. So, in this case, so R is this. Okay, and M, M is graded R module. What are the assumptions we have? Okay then, the theorem says, how, this theorem will tell how to calculate the Poincare series. R and M as above. The Poincare series P_M of M is a rational function of the type. See when I say rational function you see it's already we know it's a polynomial over $\mathbb{Z}$, a power series in Z and finitely meaningindicate terms. So, it's not really a rational function by definition but this part say that it is a function and of a particular type. So what type? P_M equal to Q, divided by $(1-Z^{\gamma_1})...(1-Z^{\gamma_n})$. Let us call this as n. So, Q where Q is Laurent polynomial with integer coefficients. We will see the prove is really simple. So first, so it's a rational function it will be because Q is also rational function it has only finitely negative. So it is really a rational function. So, it's really a rational function with integer coefficients.

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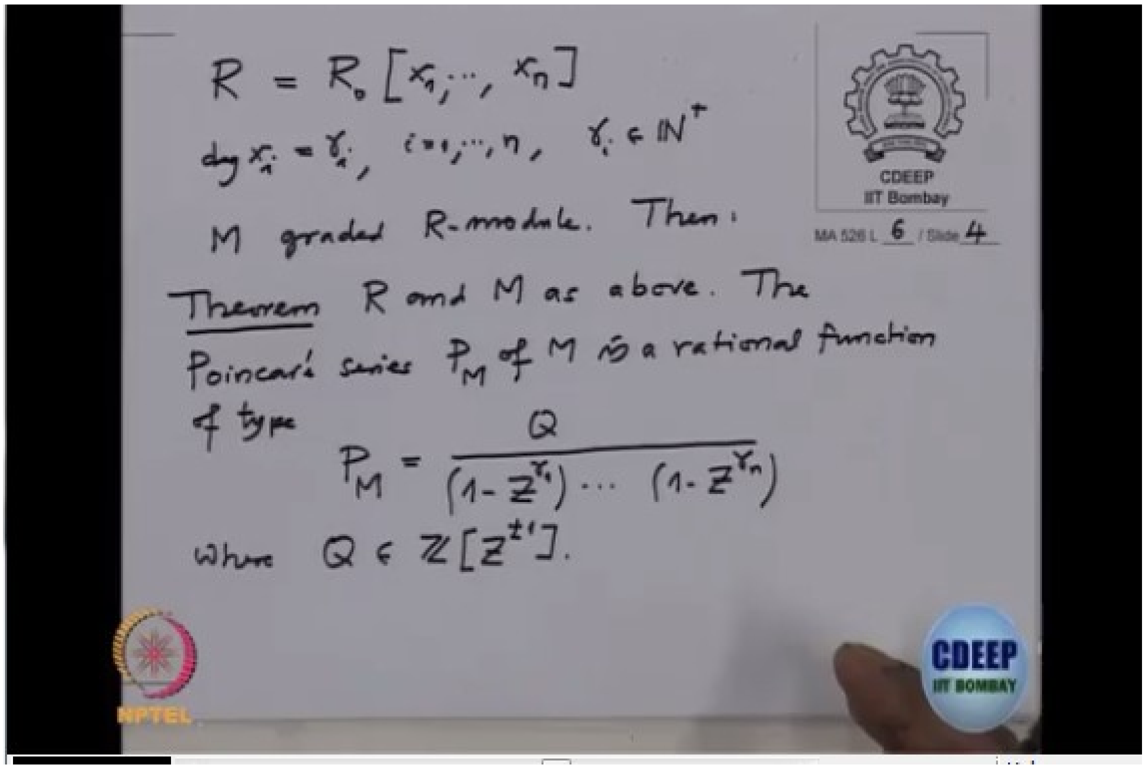

So proof. We will prove this statement by induction on n. Remember the n is the number of R_0 algebra generator of R. Right. The R_0 algebra, R is generated by this x_1 to x_n . x_1 to x_n are homogeneous of degree, γ_1 to γ_n and this n we are going to induct on.
So induction starting should be at 0. So n equal to 0, what happens? That means, i is R_0 and then all, all $M(m)$ are finitely generated R_0 modules. And, M is finite, R module, therefore finite R_0 module. Therefore, really only finite meaing components M are non-zero, because it's a finitely generated R_0 module. So, in this case, M is finite R_0 module. Just remember that when I say module is finite that means it's a finitely generated module. It's a finite R_0 module and in particular. So, okay. To let, we're assuming R equal to K. So it's a finite K module means, its finite dimensional k-victor space. So dimension k-vector space. So dimension K, M is finite. And in this case actually P_M is actually a Laurent polynomial. In $\mathbb{Z}[Z^{\pm 1}]$. R_0 equal to K, but I want to, with the next step means, right now I'm assuming only R_0 is K. That is because I'm not really sure whether you know module of finite length. Have you got exposure to modules of finite length? So, that's why I'm keeping it pending in general case, so that I will, after this, we will, I will recall some basics about modules of finite link then we will come back to this. So in this case, R_0,

Poincare series is a Laurent polynomial. Because after a while, all $M(m)$ are 0 and only finitely meaning negative terms are there. Therefore, this is really a Poincare series. So, and that matches with this because what we wanted to prove is, the Poincare series is a Laurent polynomial divided by this particular polynomial. And, in this case, all γ s are not there, so this part is not there because N is o. So, this proves the theorem in case when n is 0. Now assume n is bigger equal to 1. Okay, and now, look at this x_n . x_n is of degree γ_n . So, M, if I shift m, by gamma and the negative side. So γ_n , to M, and take a multiplication map by x_n . And why did I shift it? I shifted it because I wanted these to be homogeneous of degree zero. Which should be a graded homomorphism, That is the reason I shifted these by the degree of x_n . So this multiplication map has a kernel and cokernel. So kernel, let us call N to be kernel and P to be the cokernel. So we will get an exact sequence like that. Remember cokernel means, this modulo by image, so that this becomes exact sequence. Exact sequence, you know. That means that each page is exact. So this is an exact sequence and last time we called at whenever we have a exact sequence of homogenous graded modules, graded modules with homogenous homomorphisms then the alternating sum of the Poincare series will be 0. So, in this case, what will we get? So, first of all, P_N . P_N will look like, now this N and P, note that N and P are not only R modules, but $\frac{R}{x_1}$ module. See R is a graded ring, original graded ring, is x_1 , not $x_{1,}x_n$. x_n is a homogeneous element of degree n. So this generate homogeneous ideal. So module with that, it is a graded ring again. So this graded now as a R_0 module it is generated by one lesser limit. Namely x_n , so it is generated as algebra over R_0 by $x_1,\ldots,x_{n-1}$. And this N and P, both are and related by x_n . x_n times n is zero and also x_n times P is 0. Because there are related by x_n , both of them can be taught as a R by ideal generated by x_n module and which, now cut down the number of R_0 algebra generators. So by induction hypothesis, the Poincare series of N and Poincare series of P_r of the required form. So, Pn and Pq, how do they look like? P_N and P_P . This will look like some Laurent polynomial. So, I will denote it by Q_N . Because it will depend on N divided by this $(1-Z^{\gamma_1})\ldots(1-Z^{\gamma_{n-1}})$. And similarly this P_P will also be Laurent polynomial Q_P divided by $(1-Z^{\gamma_1})\ldots(1-Z^{\gamma_{n-1}})$.

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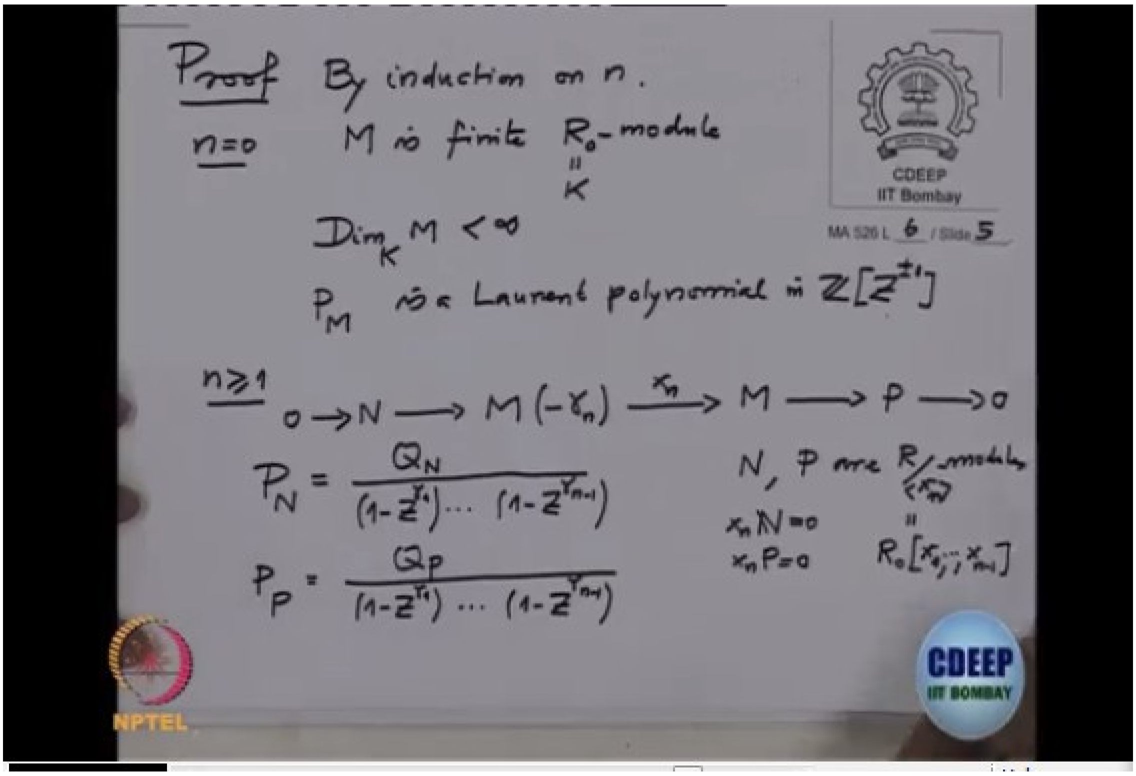

So we should write where this Q_N and Q_P are Laurent polynomials with integer coefficients. This is the induction hypothesis. And now, because of the short, this exact sequence above, we will take the alternative sum is 0. So let us write the sequence for that. So what do we get? We get starting with N. That is P_N, the next will be this shifted M by $-\gamma_n$. And that we know what is the Poincare series for this. So that is, with gamma, with the negative sign $-Z^{\gamma_n}$, P_M. The next is M, so that is P_M with the plus sign. Next will be the minus sign and Pp and this is zero. Now in this equation, we know what is P_P, we know what is P_N by induction and we want to know what is P_N but that is very easy now, because from this we get P_M times $(1-Z^{\gamma_n})$ times P_M, this I want to keep one side, the other side is shifted that is this goes P_P - P_N. And this is not Q_M, Q_M will look like the Laurent polynomials from the numerators from each one of them. So that is $Q_P - Q_N$ and divided by $(1-Z^{\gamma_1})\ldots(1-Z^{\gamma_{n-1}})$. And just shift this to the denominator down and you get Q_M is the difference of these two Laurent polynomials, so it is also Laurent polynomial and we get what we want. So this, this proves the theorem. Now before I go on, I want to spend some time to relieve this assumption that I'm not in the field. And therefore I will need more, I will concept of modules of finite length. So what I want to say that, if I, if I have a modules of finite length means length of a module should make sense. And this length concept should be more general than dimension concept. So over a finite K algebra, are the finitely generated modules will be of finite length. And therefore all these things should make sense. So again I will recapitulate after we recall this concept of modules of finite length.

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$$Q_N,\ Q_P \in \mathbb{Z}[z^{\pm 1}]$$

$$P_N - z^{\gamma_n} P_M + P_M - P_P = 0$$

$$(1-z^{\gamma_n}) P_M = P_P - P_N = \frac{Q_M = Q_P - Q_N}{(1-z^{\gamma_1})\cdots(1-z^{\gamma_{n-1}})}$$

$$Q_M \in \mathbb{Z}[z^{\pm 1}]$$

This proves the assertion.