Lecture – 22

Graded Rings, Modules and Poincare Series.

Gyanam Paramam Dhyeyam: Knowledge is supreme.

Okay. So easy to see some first, first easy remarks, so suppose I have a module M, this is a graded module. Suppose, I have integer k from this module I want to get a new module that I want to write it in $M(k)$. this is a twist. So that means to define this graded module I have to give you what are this homogenous components. So homogenous components are this, if I write n here that is by definition, M_{k+n}. So I have shifted this only, if this module was like this, M_{-1} direct sum M_0 direct sum M_2, and if I have K is 4, then this component are the new components will be now, I shift to this side. So that will be new $M(k)$ will be, so $k+n$, so this will become minus 3, $M(k)$, only the position is changing, right? This is only just shifted. So I don't write that it's unnecessary to think about it. So it just positioning. So obvious the question is what is the relation between the Poincare Series here and Poincare Series there?

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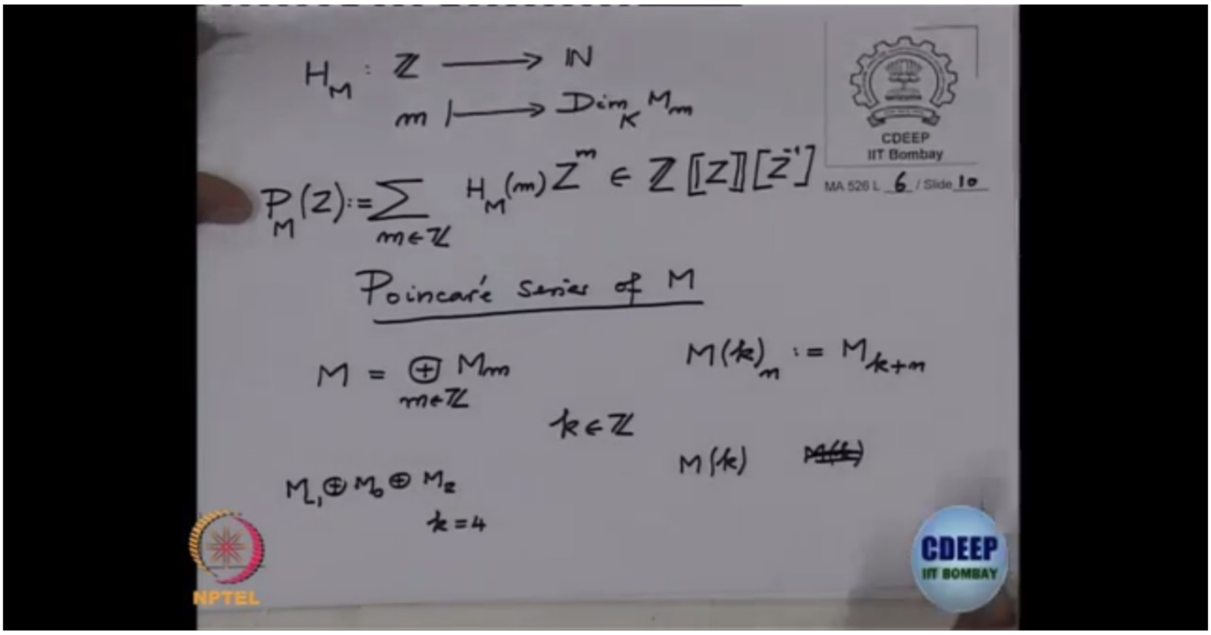

So obviously the Poincare Series of $M(k)$, this is $Z^{-k}P_M$ is obvious. We just have to shift the dimension to the other side, left side. Okay. And also so this is, I am collecting easy properties of this Poincare Series. So suppose we have, now, before I state the next one and suppose I have two graded module M and N, there are graded module, R modules. Then what is the homomorphism between them that should respect the gradation that means, homogenous component of degree n should go inside, homogenous component of degree n. So it's called, f is called graded homomorphism. If f of M_m is containing N_m then you call it graded homomorphism of degree 0, is called graded homomorphism. If you want degree k then it need not go in N_m, but it should go in N_{m+k}. So graded of degree k means not this but this should go in. This is for all m. $m+k$ for all m. And if I don't say degree then that is assumed at degree 0.

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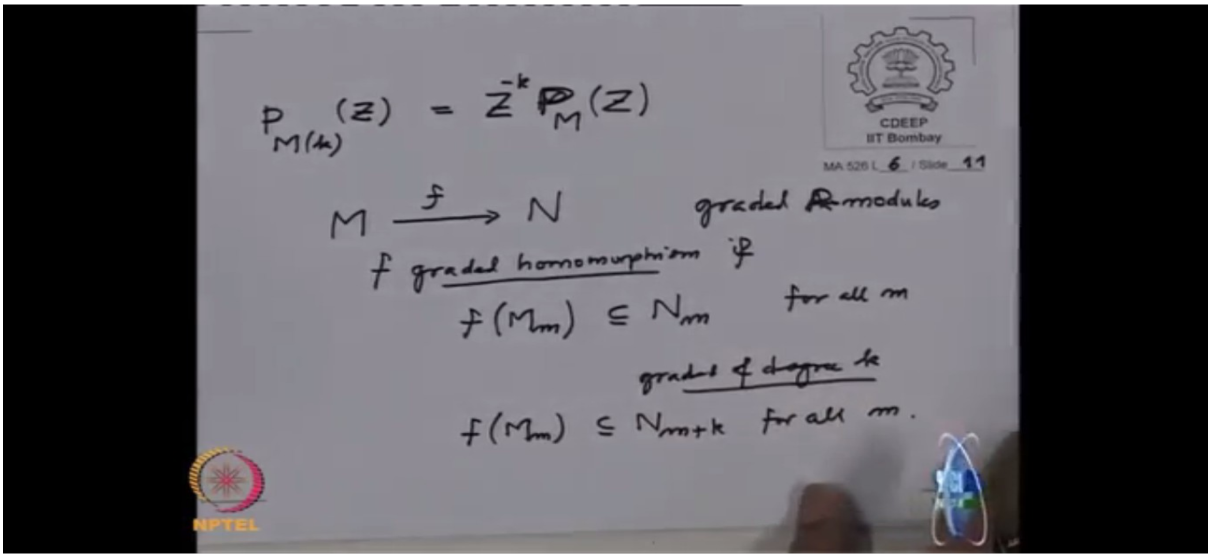

So now suppose I have a sequence like this graded modules and I will assume, I will also assume M finitely generated graded R module. That means as a module it is finitely generated over R. And because it is graded then I can assume that there is the generating side consisting of homogeneous elements of whatever degree. So now if I have a sequence like this 0, M, now I cannot put a suffix because I want a modules to be numbered so I will write it here. So this is a graded module, another graded module, and this is a graded homomorphism and so on. And do you know, when do you say a sequence is exact? So this is a long sequence of graded modules of graded homomorphism and we say it is exact means the-- so exact means, image here equal to cardinality here. And at this stage it is injective at this stage it is surjective. If sequence of graded module is exact, then at each homomorphism stage it will also be exact, because we know it's a graded homomorphism. So this will mean that this M_m component here will go to M_m component here, and so on. So then this sequence will also exact for all m. So if this is the exact sequence of graded R-modules, then this sequence is an exact sequence of finite dimensional K-vector spaces. But then when you have the exact sequence of vector spaces, then you can compute the dimension. And the dimension is alternating some of the dimensions is 0. So that means, so then summation. Summation is on 0 i equal to 0 to r, alternating sum, dimension over K as $M_m^{(i)}$ this is 0 for all $m \in \mathbb{Z}$. But this means what? These precisely means that, so that is, so this means.

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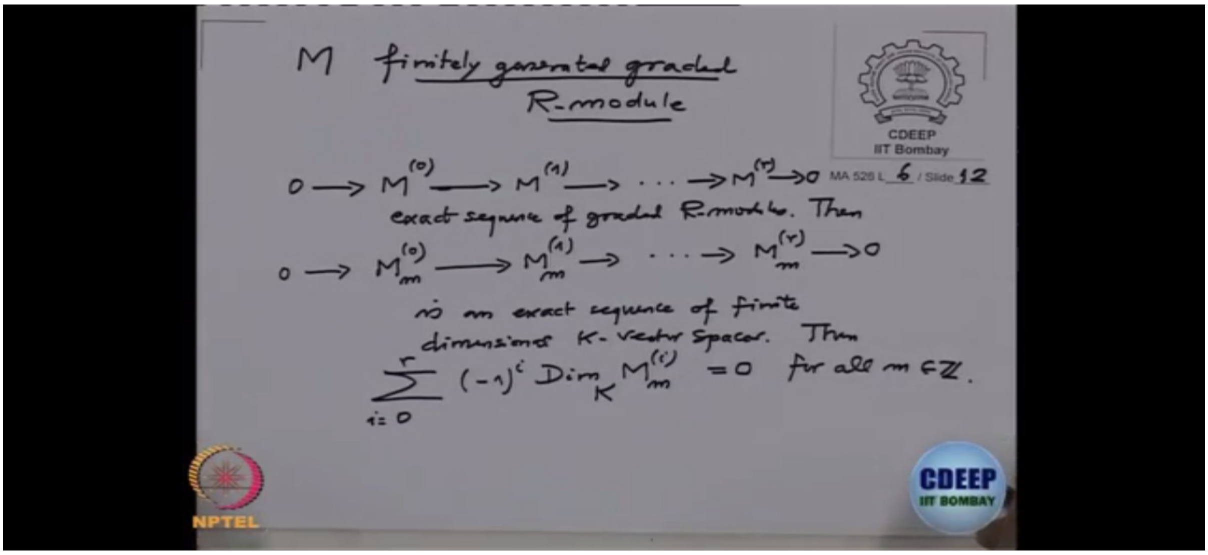

If I take this series, Poincare Series, summation alternating sum, of $P_{M^{(i)}}$, this are all 0, because P M is by definition this. So all these coefficients will add up sum to 0 that means as a power series it is 0. So let us see a concrete example of such a situation. So example, so let's we have these graded ring, with our usual assumptions it is standard graded. And suppose f is a homogeneous element of degree m. This is ℕ. Suppose f is a homogeneous of degree m. Homogeneous of degree m which is strictly positive. And suppose f is a non-zero divisor in R. And let us, this means non-zero divisor means your map is multiplication by f is injective. This is injective. This is multiplication by f. this map is a going to f a, non-zero to other means this map is injective. But this is not a graded homomorphism. So if I want to get vector spaces out of this I should put the correct indices. So this means, this is I should think instead of R_n we will go where, if I multiple by f we will go to R_{m+n}. So if I take homogeneous element of degree n and multiple by f I get homogeneous of degree m + n. And then if I steal one to write this R here, then I better shift this two the other side, then it will become homogeneous. So if I still want to think this λ_f as a graded homomorphism of graded modules, then to put the correct degrees, I should shift this R to the left side, so that it becomes homogeneous of degree 0.

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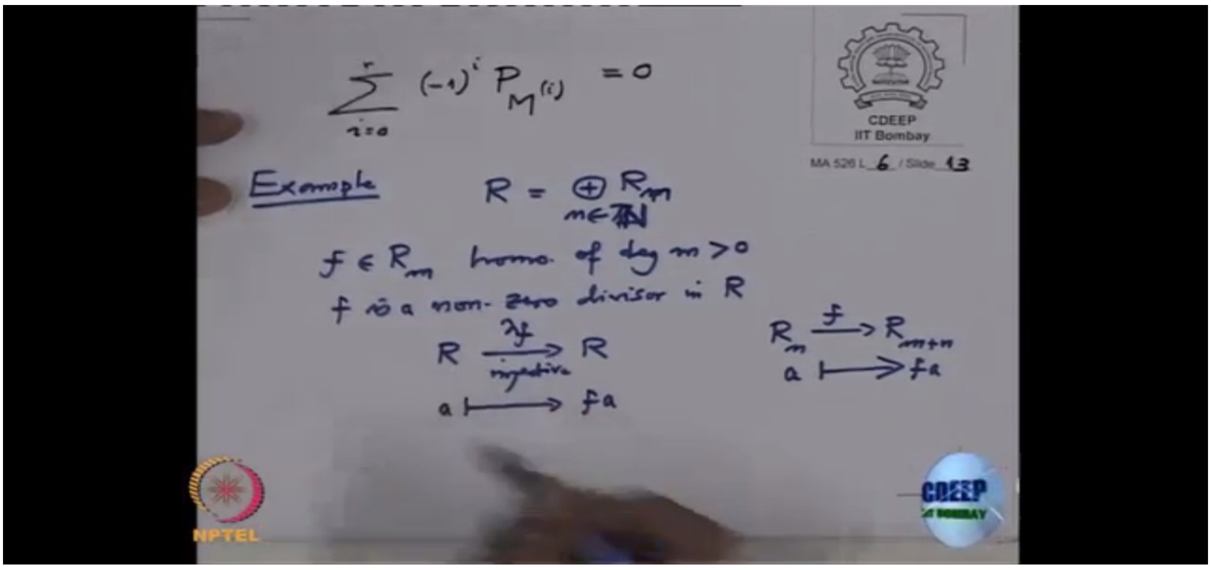

So then that means I have to consider $R(-m)$ to R. So $-m$ the homogeneous components are shifted to the minus side, so that is left side. So if I have homogeneous of degree some element here then, now this becomes correct. This is a graded homomorphism. And then what is the co-kernel, so this is injective. So I have injective homomorphism. And what will be the co-kernel. Co-kernel is R module ideal generated by f. And this is I don't have to do anything here, because this is homogeneous. This is also have a grading already, because it's a quotient of a graded so here you can put this is same thing as R_n mod. This is we say the same grading R_n mod R_n intersection with this. This is already so this, I have this exact sequence of graded modules. And just now what we have seen that the Poincare Series is here, alternating sum is 0. So that means the middle one, so the P_R is equal to P of this plus P of $\frac{R}{\langle f\rangle}$. Because alternate sum is you start with plus here, plus, minus, plus, so this minus I am shifting to the other side, so it is P_R equal to the P of both this and this one again, just know you have seen that this is that power will come out so this is same as $Z^m P_R$ plus P_R by f. So, and this is nothing special about the ring I could have done this for a module. Only thing I need to assume that this homogeneous element f of R should be non-zero to other for the module m. So let me write now the statement. Let us simplify this little bit. So we should write in these terms of the other P_R. So P_R by ideal generated by f this is same thing as, I am shifting this to the other side, this to the other side that means $1-Z^m P_R$. So if you want to compute the Poincare Series modulo f. Then we have this inductive formula, right?

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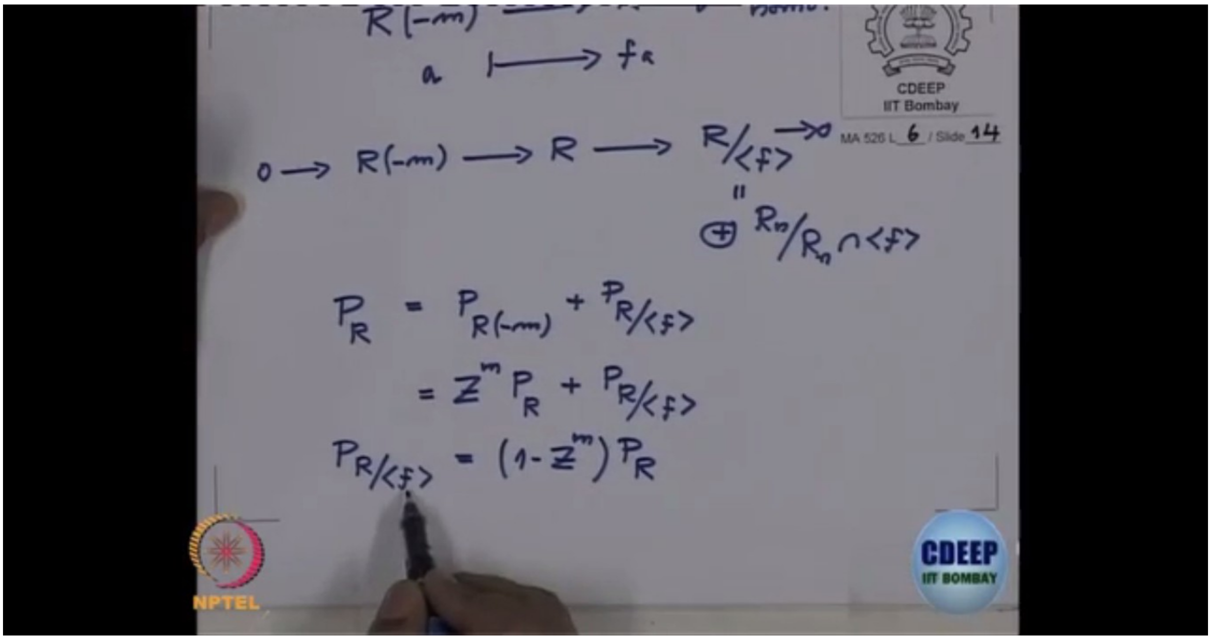

So let us write in the form of Lemma now. So this is Lemma which will be used again and again for computational purpose. So I will write more precisely. So let M be a finite graded R-module, R is $\mathbb{N}$ -graded of finite type over a field K with the R_0 is positive dimensionally. And of course finite. And if and f is R_m homogeneous element of degree m, which is a non-zero divisor for the R-module M. Then Poincare Series of $\frac{M}{f\,M}$. Note that f time same as a graded sum module of M.

And therefore $\frac{M}{f M}$ is also graded. The graded homogeneous components are preciously, the homogeneous components of M divided by the homogeneous components of, f times M. So this equal to $1-Z^m$ time the Poincare Series of M. the proof is same. Proof: We just have to look at the exact sequence 0 to $M(-m)$ to M to $\frac{M}{f M}$ to 0. This map is a multiplication by f. any element x goes to f times x, and this is a natural map. So this sequence is exact sequence of, is a short exact sequence of graded R-modules. And therefore you write down the Poincare Series and whatever we did is the same calculation shows it is this.

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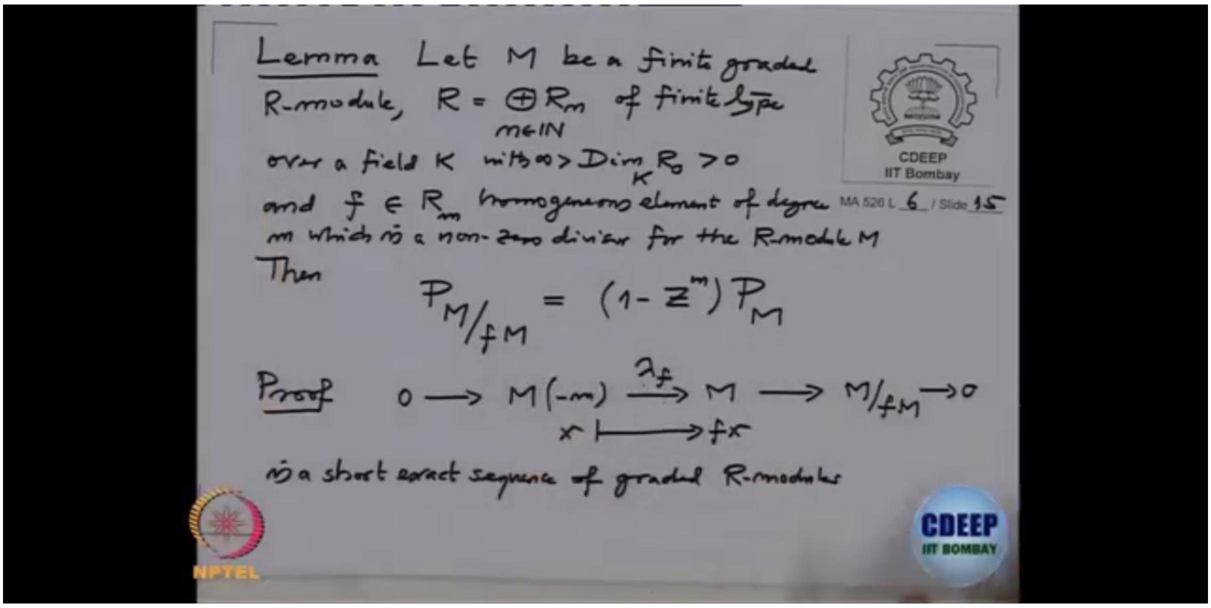

Okay, now I will state the theorem and we will continue to prove next time. So this theorem is very fundamental theorem. And these were the already actually proved by Hilbert. But Hilbert had a very complicated proof of this. The theorem I going to said. So theorem, and again if one states very precisely that will be clear what to do in the proof. So R and M as above. That means R is $\mathbb{N}$ graded ring with R_0 algebra over a field, which is finite algebra that means finite dimensional spaceand R as R_0 algebra generated by finite leaving elements. They may not be of degree one, they may be different degrees. Then, so let me just for your remembrance, to remember write this R as R_0 generated by x_1 to x_n , and this x_i belong to R_{γ_i} , i is on 1 to n. That means R as a R_0 algebra generated by x_1 to x_n , which are homogeneous element of degree $\gamma_{1,} \gamma_2, \dots, \gamma_n$. Okay, then the Poincare Series of M, P_M , is same as some Laurent polynomial Q divided by, it's a rational function of the type. The numerator is not just a polynomial it's a Laurent polynomial Q and then the denominator is $(1-Z^{\gamma_1})\dots(1-Z^{\gamma_n})$, where Q is a Laurent polynomial, Q also depends on M actually, so I should really write here M. Q_M is, Q_M belongs to integer coefficients and Laurent polynomial, so $\mathbb{Z}[Z^{\pm 1}]$, so Laurent polynomial with integer coefficients. And that denominator is nice. So in particular, when, for example, when you want it to do for R, then it will be and suppose you are polynomial ring then all gamma's are 1. So you can, so it will be one minus Z power somebody, if all gamma's are 1.

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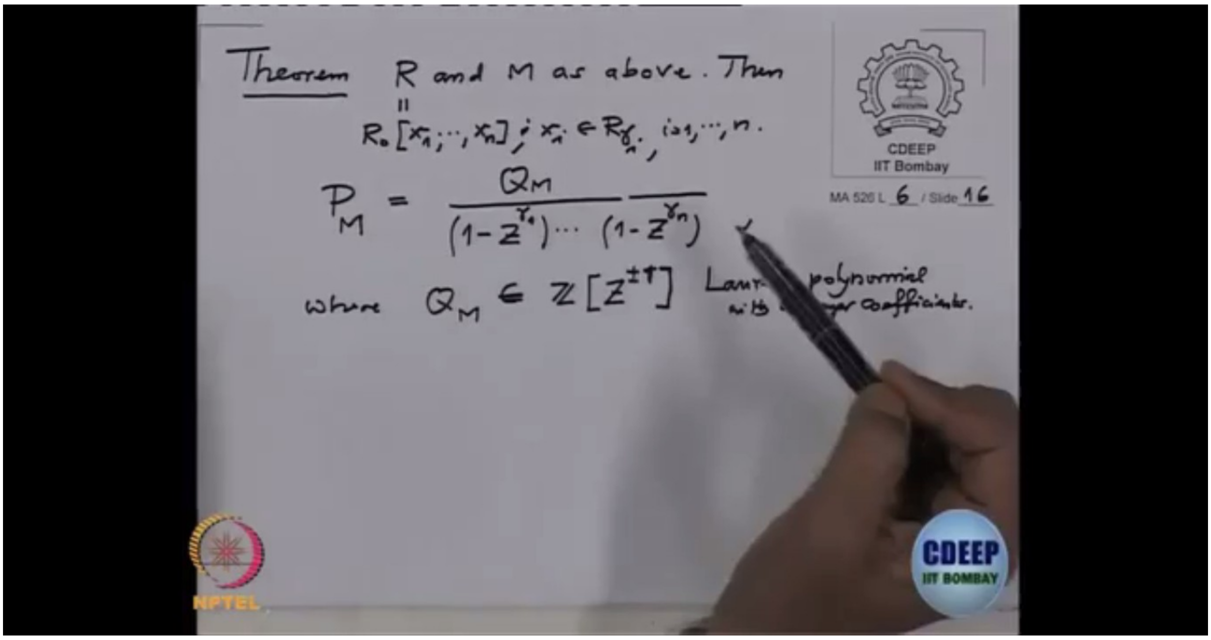

So we will prove this next time and reduce consequences from here. So prove next.