# Week5

# Lecture – 20

# Commutative Algebra

# On

# Computational rules for Poincare Series

GyanamParamamDhyeyam: Knowledge is supreme.

Today, I'll develop a theory of Hilbert Samuel polynomials and study the dimension of graded rings in general. So just to motivate few sentences in the beginning remember that when K is a field and polynomial algebra or K in n variables. We have seen the dimension of this ring is n.
K- vector space. $K[X_1,\ldots,X_n]$ and when I write it's affix m. These are the homogeneous polynomials, monomials, vector space of. Monomials of degree less equal to m. So these vector space are dimension. Dimension of these vector space. Let me just abbreviate this $K[X]_m$ . These vector space consists of all homogeneous-- all polynomials of degree less equal to m. And it is obvious that monomials of degree less equal to m form a K basis of this vector space. And number of monomials of degree less equal to m are precisely, this is precisely $\binom{n+m}{n}$ . And this is a polynomial, is a polynomial in m of degree n.

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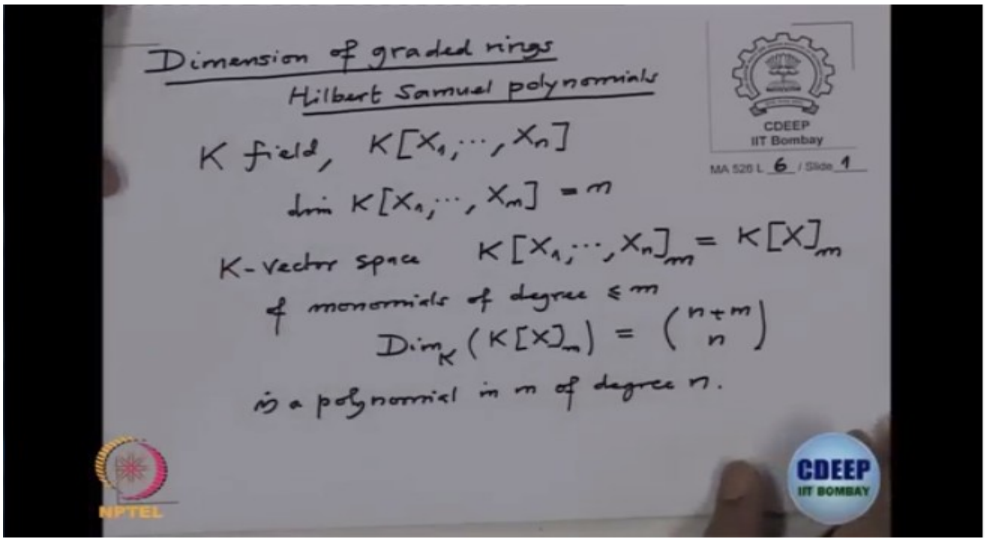


So in general also I want to do that given a graded ring, so more generally what we want to study is, when you have a graded ring. Graded ring means, the model is the polynomial ring and grading is given by the degrees. So ring is called Graded if it as decomposition like this. $\oplus_{n\in\mathbb{N}} R_n$ , . So this is the decomposition as additive groups and also it satisfies this obvious if I take $R_n$ multiply by $R_m$ , this is containing $R_{n+m}$ . So in particular these $R_0$ is a subring and all these $R_n$ s, all these abelian groups $R_n$ s. R, $R_0$ -modules. So such a ring is called graded ring and also to be more precise it is $\mathbb{N}$ graded. So $\mathbb{N}$ is index for the gradation. More generally one can also talk about grading numbered by a monoid. So this is the simplest monoid and $\mathbb{N}$ is the simplest monoid with addition.

See, it's a ring. So if the decomposition is abelian groups, 'cause it is a direct sum of these abelian groups. There is the multiplication there, so to multiply the elements in R, it is enough if you multiply by elements in $R_n$ and $R_m$ . So that multiplication is given to us and with the property that it is-- if you would take a multiplication of an element from $R_n$ and $R_m$ , it goes inside $R_{n+m}$ , . So this is the typical general relation of the polynomial rings. So for example, if you take a homogeneous polynomial of degree n and homogeneous polynomial of degree m then the product is homogeneous of degree n + m.So imitating that generally one define the graded ring as a direct sum of $R_n$ where n is varying over $\mathbb{N}$ and with the property that $R_n . R_m$ is containing $R_{n+m}$. So in particular this gives $R_0$ is a subring of R and $R_n$ are R modules. So in this case, in the above polynomial case $R_0$ is field actually. Therefore we could talk about the vector space dimension and so on.

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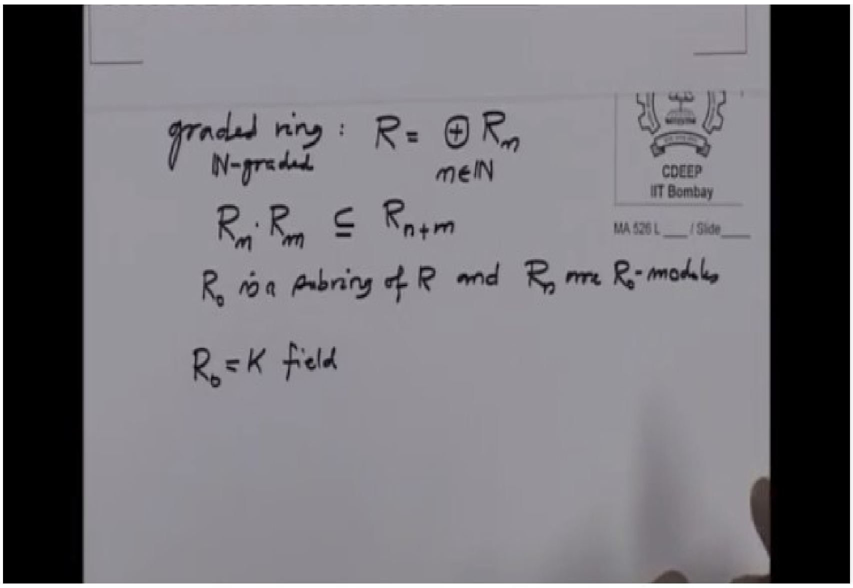


So let us do little bit quickly about graded rings and modules.So let us diagnose about graded rings and modules. Module example is polynomial ring. So let us define definition. A graded ring A is a direct sum decomposition of the abeli group, $(A, +)$. So A is direct sum $A_n$. So graded ring A is a ring, together with a direct sum decomposition of the additive group A, plus this. And which satisfies such that $A_n . A_m$, this is the multiplication given in the ring that is containing $A_{n+m}$ for all $n, m \in \mathbb{N}$. So in particular, $A_0$ is a subring and all this $A_n$ s are $A_0$ -modules. Elements of $A_n$ are called homogeneous elements of degree n.

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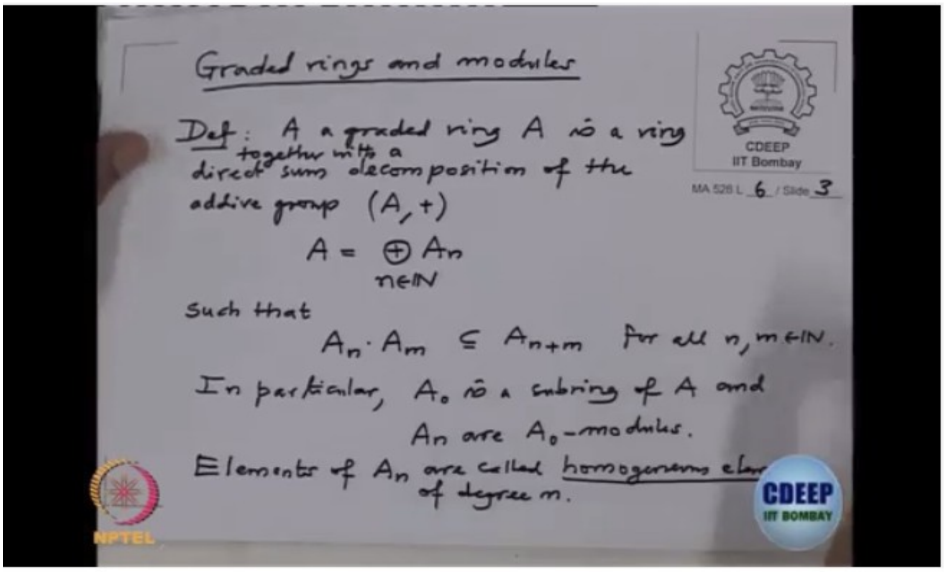


So typical example is, A polynomial ring or any ring. $A_0$ is any ring and if you take the polynomial ring $A_0[X]$ , the grading is given by now. This is $\oplus_{n\in \mathbb{N}} A_0 X^n$ . Only one variable you take. So this is-- these are the graded, and obviously multiplication is the same as $X^n$ , and so if you know this then you can generate the whole multiplication. Okay. Now, as I said, it is convenient to consider. This is called $\mathbb{N}$ -graded ring. $\mathbb{N}$ - graded. Sometimes it is convenient to consider $\mathbb{Z}$ -graded, or more generally any $\Gamma$ -graded, where $\Gamma$ is arbitrary monoid. So the numbering will be denoted by some monoid in that. Okay. So, I don't know whether I will have ocassion to take generally, general monoid. But certainly, $\mathbb{N}$ -graded will be useful. Okay. So now another definition is graded-modules. So, to define graded module, you need to underline ring to be graded. So, let A is $\oplus_{n\in \mathbb{N}} A_n$ , be graded ring. An A-module M together with direct sum decomposition of the additive group of M, is called a graded A-module, if $A_n . M_m$ and it should be containing $M_{n+m}$ for all n, m natural numbers. All right. What more do I need. Now, when do you call sub-module of the graded module to be graded. Before I forget, these elements are similar to the graded ring. Elements of $M_m$ are called homogeneous elements of degree m. Okay. So, if sub-module, an A sub-module, so now graded ring is fixed in the notation. So, an A- submodule N of M and M is graded is called a graded submodule, if I take the intersections of N with the homogeneous components $M_m$ of M then these direct sum is precisely everybody. So in other words, if you look at this direct sum, this additives subgroups of N, that direct sum is precisely N. That means, every element of N, you can write it as sum of homogeneous elements from N in a unique way. So simple Lemma will illustrate.

So Lemma, so M graded A-module over a graded ring A and N is submodule of M, then the fallowing conditions are equivalent. One, N is a graded submodule of M. Two, N is generated by homogeneous elements. And three, whenever if Y belong to N and certainly it belongs to M. So, y will have a decomposition like this. $y_0 + y_1 + ... + y_r$ with $y_i \in M_i$ . So if you take y, then these small $y_i$ 's are called homogeneous components of this y.

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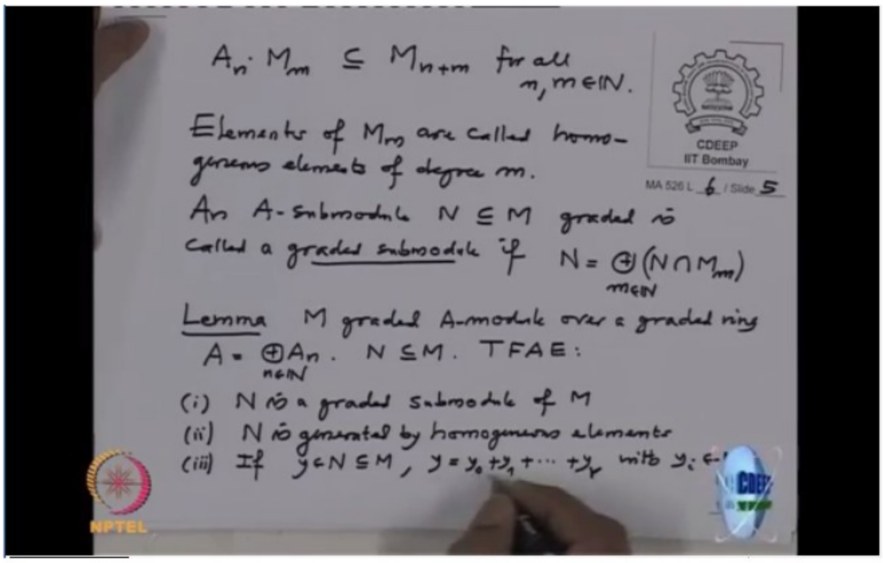


Then all the homogeneous components $y_i \in N$ for all i= 0 to r. Then it is graded submodule. So I will not prove this. So proof is very simple, just check from definition. Easy verification from definitions. All right, so graded submodules of the ring itself, they are submodules of ring are ideals. So the graded submodule are precisely graded ideals. Or also I will use as a homogeneous ideals. So let's see couple of examples. If I take A to be the polynomial ring, let's say $A_0[X]$ or if you want you can simply take $A_0$ to be K. So polynomial in one variable over a field. Then the ideal generated by X, this is homogeneous ideal. Because it is generated by homogeneous element. Whereas, if I take ideal generated by something like this, $X^2 + X$ , this is not homogeneous ideal. Because these element has two homogeneous components, one is $X^2$ and the other is X, one has degree two, the other has degree one. And neither of them belongs to this ideal. Therefore it is not a homogeneous ideal.

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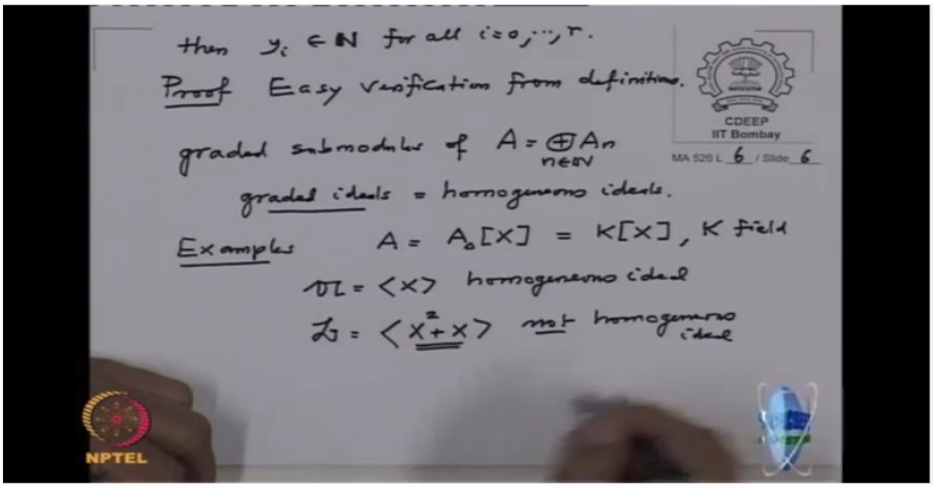


And you could do it more variables also. So this, you can do this for a polynomial ring in many variables. And just to say some, the grading we have considered here are so far is standard grading.Standard means, that means to the polynomial we have attach, their usual degree as a grading index, but whereas one can do a different grading. For example, for example, another example let us do it. Suppose I take polynomial ring in two variables X and Y. And suppose, I now give a grading, so degree of X, I say it is 2. And degree of Y is 3. Suppose, I declare that then you can, for any polynomial you can decide what the degree is. So that is the grading index. For example, now $X^2$ will have grading index what? 4. And if I take $X^2$ or $X^3-Y^2$ this monomial will have degree 6 and this also will have degree 6. So this will become a homogeneous. Homogeneous of degree 6. So in these, when I tried to write the decomposition for this ring as $A_n, n\in\mathbb{N}$ , this $A_n$ should be generated by $A_n$ is a vector space, K vector space. with basis whose monomials have degree n but weighted degree.So weight 2 to X and weight 3 to Y. So this some, for example, there is no degree 1 element in this. $A_1$ is missing. $A_1$ is 0 actually. And so and so, sometime this is also useful. This giveorbitary natural numbers to the variables and consider the grading. Or not even natural numbers, even the integers. So sometimes even useful to have minus weighting. So these, for example these polynomial is not homogeneous in a usual way but in these new degree it became homogeneous. So that is useful for computation of dimension or other invariants of the ring. So I think I will recall whenever I need more about graded rings and modules. I will not do exhaustive thing now and then what I want to do is, I want to do, I want to attach a polynomial to a graded module. So what we would like to do is? To each graded module. So our graded ring is fixed to each m which is graded. And grading of m, I would allow to be negative integers also with the same definition. That definition doesn't-- so it's a only thing we need is, this condition. $A_n . M_m$ is containing $M_{n+m}$ for all n, m. n natural number and m to be integer if makes sense. So each such module, I want to attach a polynomial with coefficient in rational numbers with rational coefficients. And the degree of that polynomial, I want to relate to the dimension. That is a whole idea of this full section. And as a result, as a immediate corollary we'll see it is finite because it is degree of some polynomial. And therefore Krull dimension will be finite. Whereas it was not clear from a definition of a Krull dimension that it is-- If at all it is finite. So we approved it for a polynomial ring over a field or finite k -algebras, it's finite. But in general definition of a Krull dimension doesn't immediately say, that it is always finite. And in fact, it may not be finite but for these class of rings it will be finite. And that is what the theory I want to develop now. So only assumption I want to make on this module is for large negative integers, it is zero. It doesn't go to on the left side to the, all the way non-zero.We will assume here that $M_m$ is 0 for m less-equal to from fixed stage $m_0$ . So that means and the-- it looks like that. On the negative side only finitely many terms can be non-zero. So it like, Laurent polynomial ring, okay. I will first consider our ring. Let me write, instead of A notation. I will write R as a ring. R is $R_{0,} R_1$ et cetera. This is the gradation. And $R_0$ , this $R_0$ , I will assume it is finite dimensional, or $R_0$ is finite K-algebra. Where, K is a field. That is the first I want to do it. Finite as a module. It's a K-vector space. So K vector space. Finite dimensional K vector space. So that is dimension of $R_0$ , this is finite. So if one gets struck, one should always go back to the polynomial ring example. In that case $R_0$ is actually K and $R_1$ is generated by the $X_1$ to $X_n$ as a vector space. $R_2$ will be generated by homogenous monomials of degree two and so on. All right, so and I will assume that as a $R_0$ , R is finite type $R_0$ -algebra. $R_0$ is a sub-ring of R. So $R_0$ , R is a algebra or $R_0$ and as a algebra, it is generated by finitely many elements. So in our standard notation, R looks like, $R_0$ and generated by-- now I will write the generation says, $x_1$ to $x_n$ . And this $x_1$ to $x_n$ are homogenous elements. So first of all, R is finite type, $R_0$ algebra. So R is or $R_0$ generated by finitely mini elements and each element is a finite sum of homogenous elements. So I will add all the homogenous components and I assume that $x_1$ to $x_n$ are all homogenous. And therefore, they will have some degrees. So those degrees I will call homogenous elements of positive degrees, $\gamma_1$ to $\gamma_n$ . So for example, if all $\gamma$ s are one.So one example, if you'd see here, typical example will be, or polynomial algebra. R is the polynomial algebra. So this is our $R_0$ and these guys are homogenous of degree one. So in this case, whole $\gamma_1$ , all these $\gamma_n$ are 1. So such situation in, if all $\gamma$ s are equal to one then one call, it is a standard graded algebra. Standard means all the generators are all, we need some degree one. All right, now you take and remember we are considering, I want to act as a polynomial, that is our aim. Two are graded module, we want to attach some polynomial with rational coefficients. So take a module M, M would have some negative homogenous components but not too many. So and $M_m$ is 0 for all m less than $m_0$ for a fixed $m_0$ . So then we look at these $M_m$ s, so dimension, all these $M_m$ s are $R_0$ modules and I'm assuming that, we assume that all these $M_m$ s are finitely generated $R_0$ modules. So therefore and $R_0$ is a finite dimensional vector space. Therefore in turn all these $M_m$ s are also K-vector spaces of finite dimension. So dimension of $M_m$ as a K vector space, these integers. This is some integer, this I want to denote by $H_M(m)$ .

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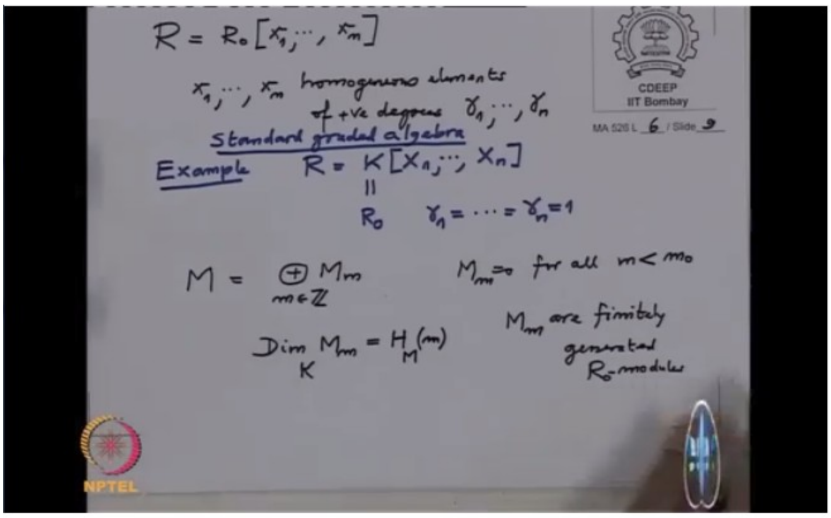


So that means, I have defined, this means, I have defined a map $H_M$ from these integers, to natural numbers. Any m it goes to dimension of $M_m$ . For large negative, this function is zero actually. And I consider now the power series. So it is here to, you will see, it is here to compute with the power series. Because that, if you take that as a generating function, combinatorial formulas will be easier to change if you write in terms of power series. So look at these power series. This is m in $\mathbb{Z}$ , $R_0$ at m and I need a variable for power series. Right, so $Z^m$ this is a power series with integer coefficient and some may come negative powers. Right? So this is actually, this is belongs to, let me write and then we will explain, integer coefficients, power series in Z, and maybe polynomial in $Z^{-1}$ . So it will start with some few negative terms and then the power series. So it's a power series with-- Power series coefficient with and polynomial in $Z^{-1}$ . So this is called, this I want to introduce this is $P_M(Z)$ . So this is called a Poincaréseries. Series of M.

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