Week 4

Lecture – 20

Commutative Algebra

On

Language of Algebraic Geometry

Gyanam Paramam Dhyeyam. Knowledge is supreme.

In the general case, in the general case, let us denote, let h and t be natural numbers such that height of q is at least h, and transcendent degree of the residue fields $k(q)$ or $k(p)$ be at least t. Choose, these are two non-negative numbers, so I will choose some natural numbers where this is at least h, and this is at least t. h and t could be zero. So, this height q at least, h means, there is a chain of length h in B, in the spectrum of B. So, there is a chain q_0 contain in,contain in, contain in q_h in Spec B. And now, this is contained in q. Right, q is at least h, so at least there is length of chain, this could be equal length then.

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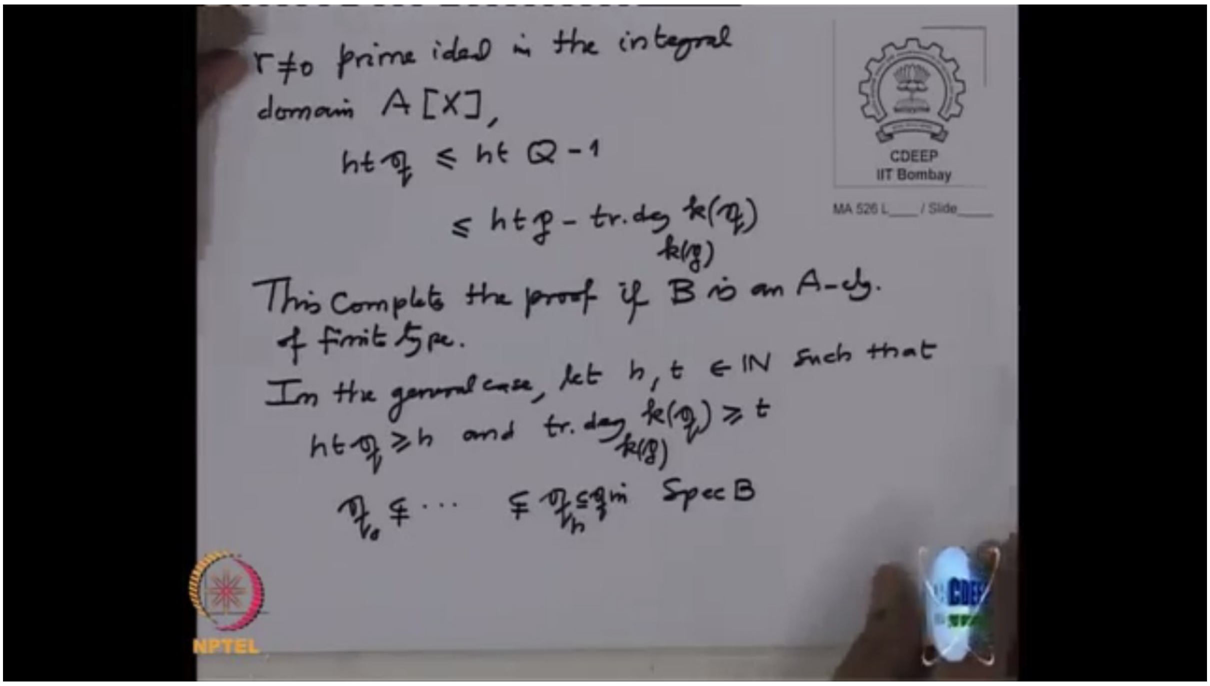

And similarly, about the transcendent degree,the transcendent degree is at least, t means, I have elements. So there are elements, b_1 to $b_t \in B$. And there exist, b_1 to $b_t \in B$, such that, they are images in the residue field, $\overline{b_1}$,... $\overline{b_t}$, in $k(q)$ are algebraically independent overkp. See, this is what I'm using, thetranscendent degree of this residue field at q over $k(p)$ at least t, means there are elements which are algebraically independent over this residue field. So the elements here are coming from B. So, those are, that is what I call it b_1 to b_t . All right, now choose, because the chain is proper chain, I will choose at each stage an element which is not coming from the earlier one. So choose $c_i \in q_i$ which is not in q_{i-1} . This we choose it for i equal to 1 to h. Okay. We can choose an element here, which is not earlier, and keep doing it. So there the c's. And now I consider they are in B' . This B' is the algebra generate by this c's and these b's. So, b_1 to b_t , and c_1 to c_h . Look at this algebra. See, what the idea is, from a given, situation, you create a situation, so that, you prove the, you know, that for the sub ring, which is finite,take algebra and then, that's how the data is captured, know, so that is the idea. So, now this b isA algebra of finite type, and the chain remains there and, so you contract the chain to be B' , so that means you have here $q_0 \cap B'$ contain in, and these containment remains proper because I have made surethat the elements are also in $B^{'}$. So this is in $q_1 \cap B^{'}$ and so on. This is $q_h \cap B^{'}$. That is why we put all the elementshere in $B^{'}$. So, this is correct. And also, the residue field, if I take this prime ideal, of course, this contains, this is containing q intersection b, and if I takethe residue field at this $B^{'}$. Residue field at this-- there are actually contain all these b's also, because b were in q. So, this $\overline{b_1}$, the image is here, $\overline{b_t}$, they are all algebraically independent over $k(p)$ .I don't even need this. They are there, so I don't even want to use this. So, what do I know from the early situation, finite take k algebra case, that the height plus the transcendent degree is less equal to height of this contracted ideal, $q \cap B^{'}$ plus the transcendent degree of over $k(p)$, $k(q \cap B^{'})$. These I know, this is the case from finite type case, because I am applying it to $B^{'}$. But is less equal to height p plus transcendent degree of $B^{'}$ over A, which is less equal to height p plus transcendent degree of B over A, because this transcendent degree may go up.

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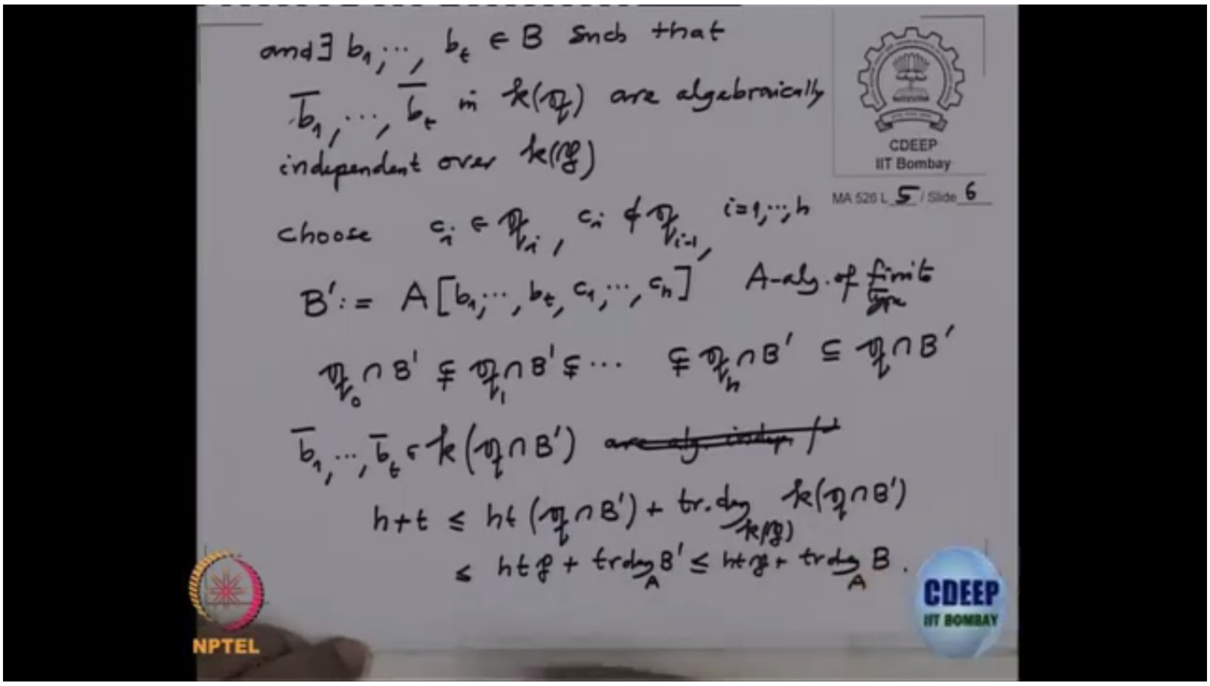

So, that trues the inequality. Okay. So, I will remember that we have two state true that, the one inequality that-- for that I will need, little bit more local algebra and thing like, system of parameters and thing like that. So, when I come develop that, I will come back to that proof. Now, I want to spend some little more time for the geometric part which we will keep-- which I have used in these exercises also. For example, the Hilbert's Rule telling that,so, the numbering here in the exercises, so what I have done is, at little miracle. So, the idea of the-- so this recall this from-- this is the beginning of the algebraic geometry. This-- so for example, the problem is-- so algebraic geometry is study of solutions of finitely many polynomials, infinitely many variables. So, it's a study of Solution Sets of equation like this, f_1 and f_1 is a polynomial in X_1 to X_n , f_m in X_1 to X_n . So, and we want to equate, we want to find the common solutions. So the standard notation for that is and where? This is very important, where you are taking, where are the co-efficients and where you want to look for the solutions. So, I don't want to assume in the beginning itself that the base field is algebraically close. I don't to assume that, because then you cannot certify, for example, equations or rational numbers which are very important.

So we want to allow base will to be non-algebraically closed. But the solution we will look for in algebraically closed, because if you don't do that, then there may not be any solution. So, the typical situation, what I will consider is, K is the given field, and I will take a bigger field than K. K is if L, or K is a field extension. And in this, I will assume that bigger field is algebraically closed. L is algebraically closed. And, therefore, when I look for the solutions, I look for the solutions in L, it coordinates in L. So therefore, the notation one uses will be V_L. Just to remember, that we are taking the solutions in L. V_L of f_1 to f_m. This is by definition. All those points $a=(a_1,\dots,a_n)$ in L^n such that all f_i is vanished there for all i. This is some subset of L^n. And this is called algebraic, K-algebraic set defined by f_1 to f_m.

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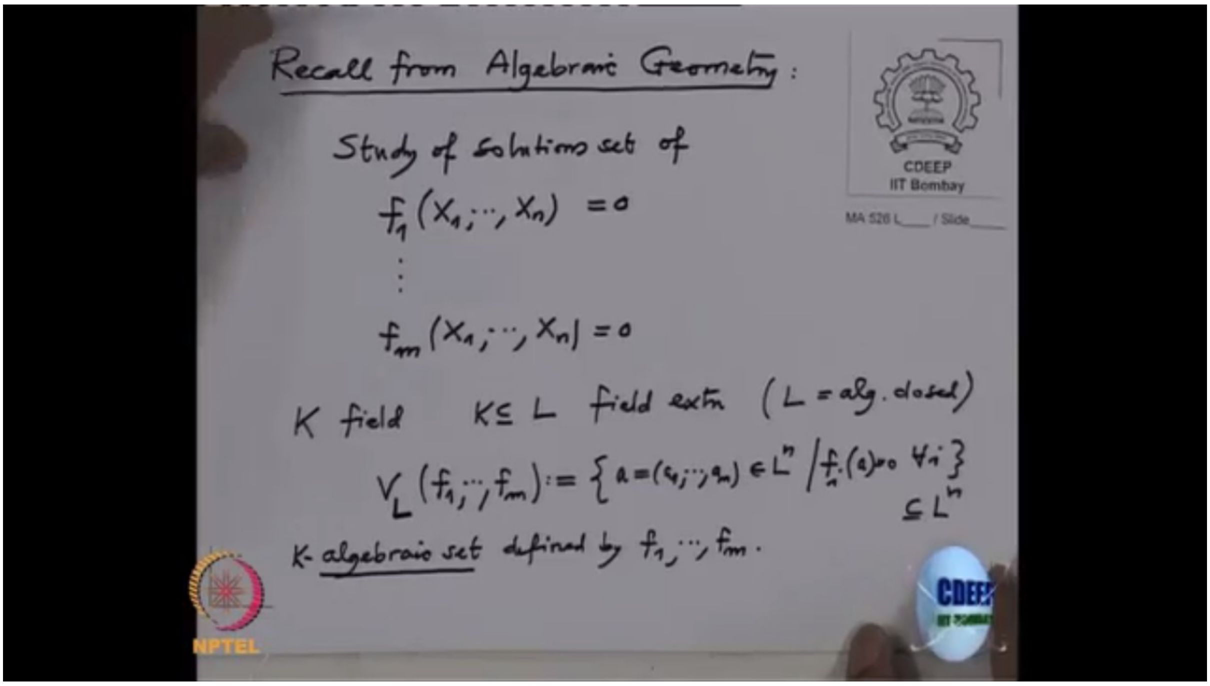

We are also keeping track of the coefficients where they are lying. So, for example, for example, the Fermat's Last Theorem… So that was asking, what we know is or see they are in Infinitely Many Solutions. But over $\mathbb{Q}$, the solution where we wanted to show that the over $\mathbb{Q}$, they are only finitely many and which are they. So that was a problem. So, later on, I would even want to drop this condition where K is a field, and replaced by integers or arbitrary ring. But those will require more, more sophisticated tools, we will see it at the end if it is possible. So, V_L also, now obvious things, you might have checked earlier that this V_L depends only on the you ideal generated fn to fm. This will not change. The solutions will not change, because any other polynomial in the ideal is a linear combination of that. So, in other words, I can always consented only the ideals in a polynomial ring in variable. So, therefore, we are considering only ideals in $K[X1,\dots,X_n]$. And we have given a map V_L from these ideal set to K-algebraic sets in L^n which you think it is a subset of the power set of L^n. Alright. And also, you knew it that this L doesn't only depend on the ideal, it depends on the radical of that ideal. So you might change these ideals to radical ideal, because once you know for the radical ideal, you know. So, therefore, I will consider this L map from radical ideals $K[X1,\ldots,X_n]$ to K-algebraic sets in L^n. And, also, it is, you might have checked earlier that these K-algebraic sets form, it defines a topology on L^n where you declare the close sets in that topology are precisely the algebraic cells. So those things I'm not checking. So, I'm achieving those. So, this is our V_L map. And we have a map in the other direction, namely, so given a... Actually, the map is, again, like this map. It is from arbitrary set. If I have arbitrary set W of L^n, then we have the ideal I_K of, now I have to use this I_K also in the notation because we want to keep track where the equation are. This is by definition, all those polynomials f in the ball ring such that, F is 0 at every point of W. F of a is 0 for a in W. Now when you check that this is an ideal, that is obvious, because fm vanishes, g vanishes, then, f + g also vanish, and arbitrary multiplication by arbitrary polynomial also vanish.

So this is actually an ideal. Not only an ideal, it a root ideal. That is also clear, because if power of a polynomial vanishes at a, then that polynomial also vanishes because, see if f^n is $f(a)^n$ and in the field we are, so therefore, fa is also 0. So therefore, this is a root ideal. Sometimes, they will radical or root ideal, that is same as this. So therefore, we have a map in the other direction namely this I_K.

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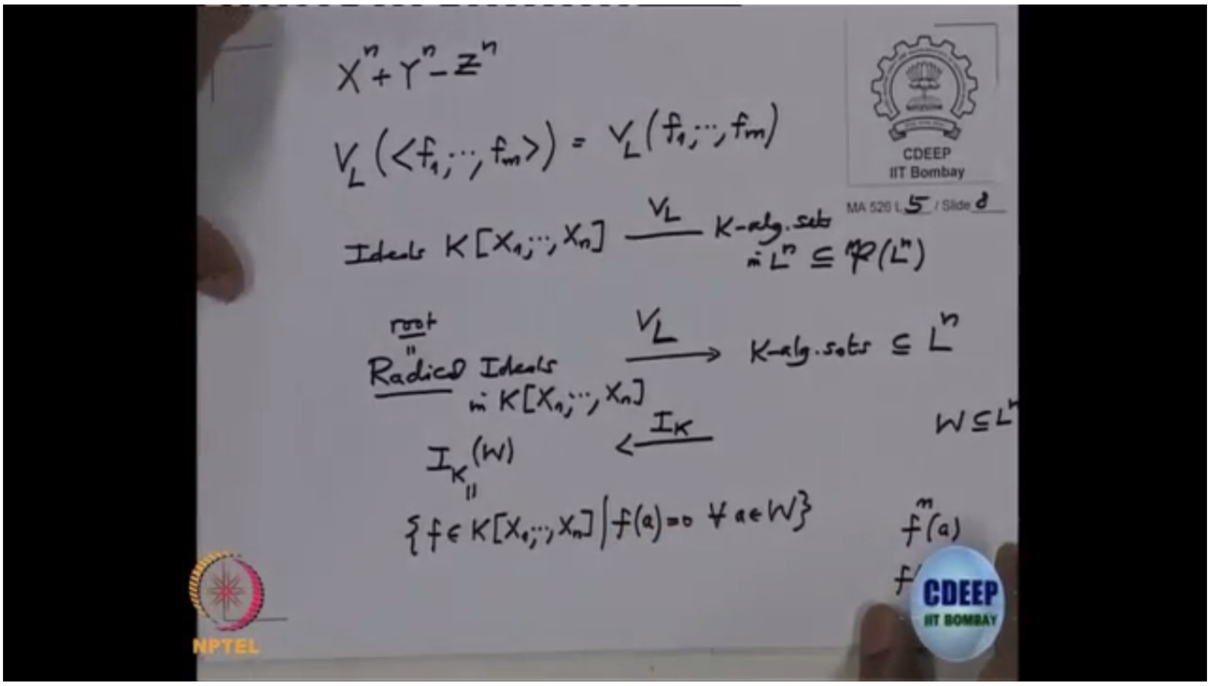

And then, we want to know when these maps are inverses of each other. And that is precisely when? If L is algebraically closed, then these maps are inverses of each other. So when L is algebraically closed, then V_L and I_K are inclusion reversing inverses of each other. Obviously, this L, if an radical ideal is containing radical ideal b, then V_L reverse the inclusion, and similarly, this ideal. Okay, then, to this, we write the consequences namely, so what I call, this HNS 1. So, for example, what is the criteria in that given any ideal, V_L of that is non-empty. See, it's not clear whether

V_L of an ideal in non-empty. So HNS 1 says, if L is algebraically closed, then, and V_L of a is non-empty if a is a proper ideal. That means there is at least a common, one common solution. So this is like linear algebra. You see, if all were linear polynomials, if all these fis were linears, then we know from linear algebra that the system of linear equation is consistent if some condition, right? So, similar to that. So this is a proper ideal. Also, I forgot to mention that, any ideal is finitely generated, that is why Hilbert proved the Hilbert Basis Theorem, that only finitely equations are needed to define an algebraic set. Okay, so HNS 2, that says, if I take V_L of an ideal a and I apply I_K to that, then this is some ideal and this ideal is, if a is a root ideal, then this ideal is a. Otherwise, we'll have to write root a here. So this, so let me write root a here. So this is usually called, if you see the books, it is called a geometric form. The first is, I don't know, it's called a weak form. And then HNS 3, so this is, again, under the assumption L algebraically closed. And both these are not true if you drop the assumption L algebraically closed. Okay. And third one, third one is simple. Third one is more easier to state, easier to prove. So E over K as the field extension, L is, K is over this field, E over K the field extension, and E is K algebra of finite type. Then K is-- E is algebraic over K in particular finite. E is the finite over K. That is the dimension as the vector space is finite. So actually, this formulation HNS 1 equivalent to HNS 2, HNS 3 and equivalent to HNS 2. They are equivalent. This also, I am not-- I don't know whether you ever know, you know the proofs individually probably but you don't know the proof of equivalence. So I will write in the notes, you can read there.

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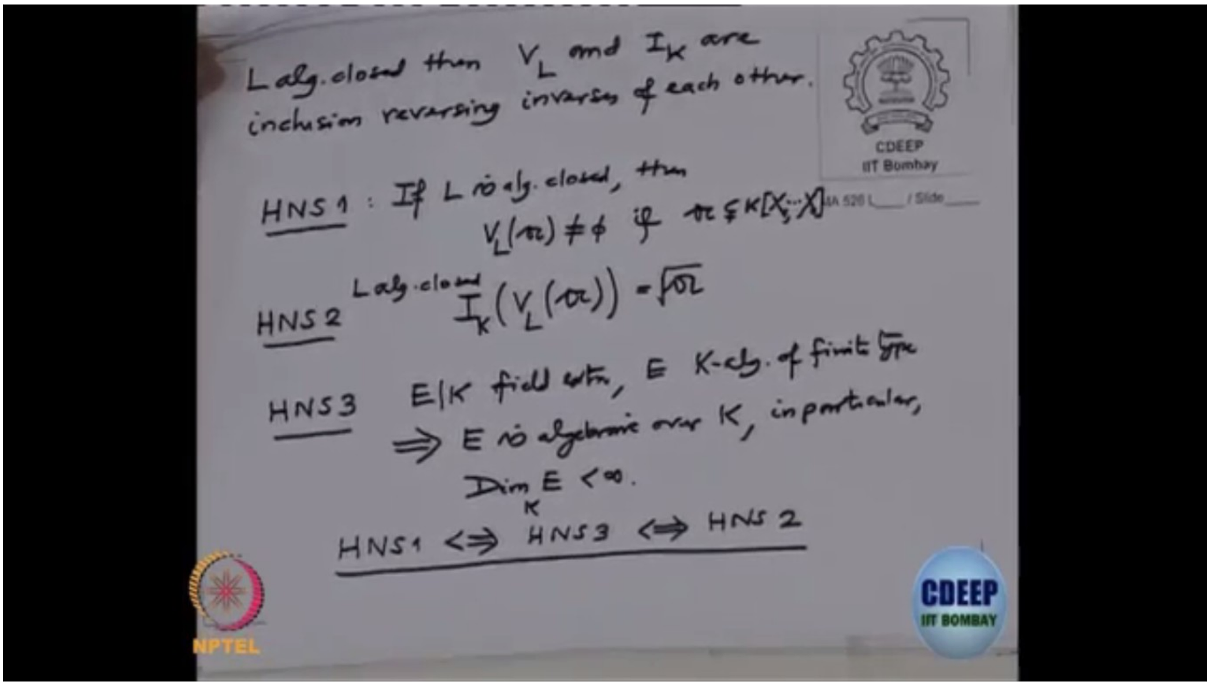

And also, if you remember we have stated the fourth one there in the exercise sheet. HNS 4. You remember, if K algebraically closed, then Spm, $K[X1,\ldots,X_n]$, and K^n . So this is a map, obvious map is in this way. Given a point a equal to a1 to an, you take the ideal generated by X_1-a_1 et cetera, X_n-a_n . This is the maximal ideal at point a. And this map, it's obviously injective. This is injective, that is clear. But, if K is algebraically closed, then this map is actually bijective. And if K is not algebraically closed, this is false. For example, if I take K equal to real numbers, n equal to 1, and if you take the ideal generated by X^2+1 , this is a maximal ideal. But it is not coming from a point, but not in the image of above map. You see, no under, this polynomial doesn’t have a 0 in R. That is why HNS 1 also fails, HNS 2 also fails and HNS 3 also fails. And HNS 4 is also equivalent actually, HNS 3. Once they are proved to 3, you see, the easiest is to prove is 3, because it's a field theory.

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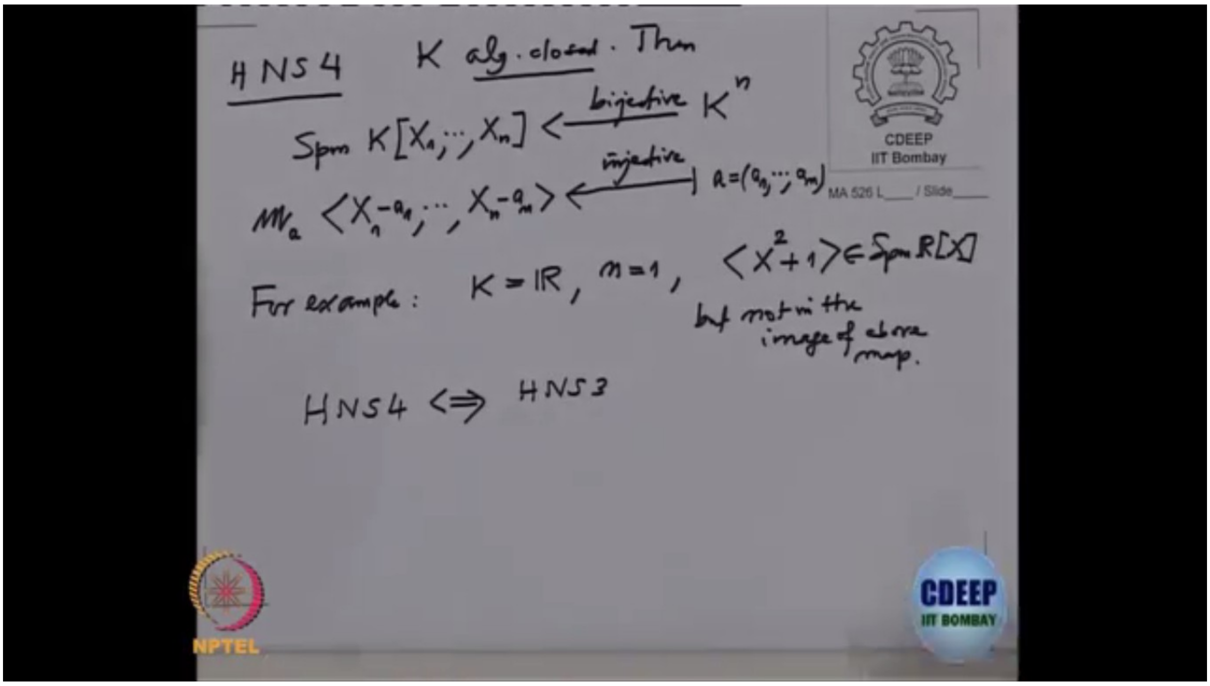

So this also you can—it’s written as an exercise here. So you can try to check.