INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

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Lecture No. – 02
Zariski Topology and K-Spectrum

Okay so let us continue our study of K spectrum, so what we did I'll just recall briefly that we have defined a map V_K from ideals in the polynomial ring $K[X_1,\ldots,X_n]$ to the power set of K^n, any ideal A goes to $V_K(A)$, $V_K(A)$ is by definition all those points in K^n so that all polynomials in the ideal A vanish at that point.

Now actually this I could have taken they're radical ideals, not only all ideals but I can restrict this two only radical ideals, so I will write here $r-I(K[X_1,\ldots,X_n])$, r is for radical ideals, because ideal and its radical ideal will give you the same set of points, so to illustrate it, see suppose you take let's take a polynomial N = 2, so polynomial ring in 2 variables and suppose I take ideal generated by $\langle X_1^2\rangle$, this is ideal, so what will be the picture there?

Now the picture I have to draw in K2, K2 is the plane, affine plane, and what will the picture for, that means X_1^2 is 0, but X_1^2 is 0 means that is this, but line twice, but the picture is same, so this is $X_1=0$, more generally all this is same as, picture it will look same but what, we are losing something, we are losing multiplicities,
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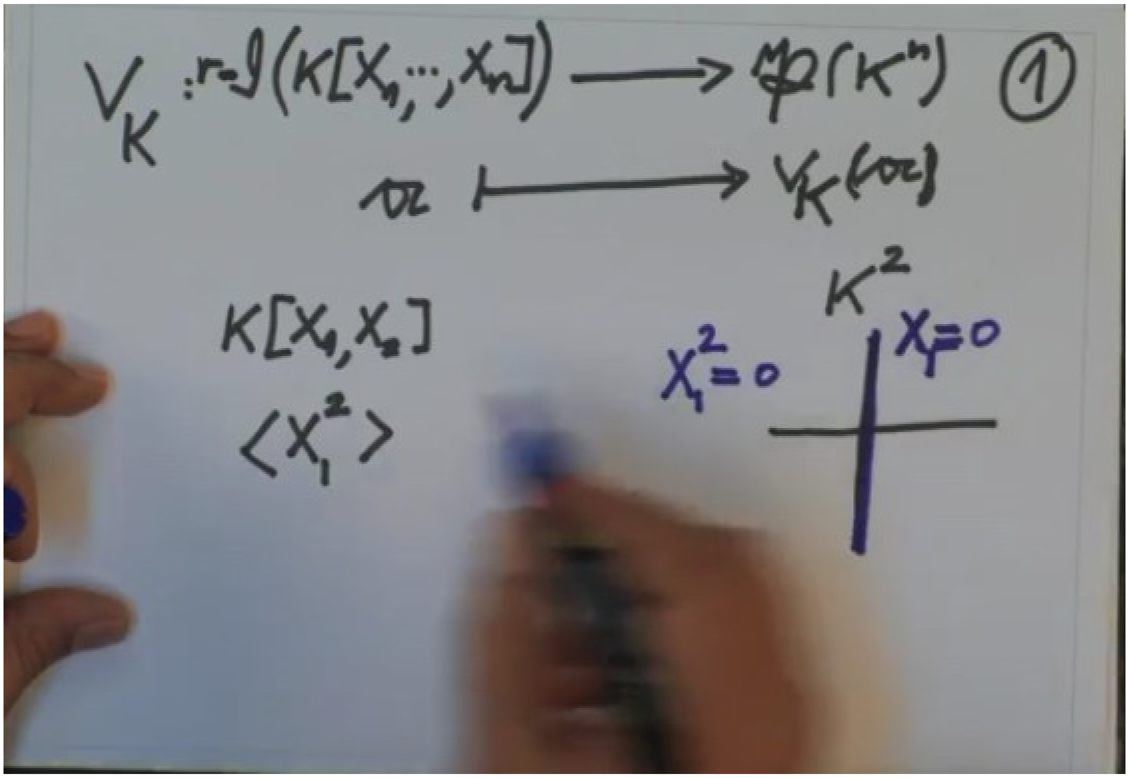

so when you go from algebra to geometry something is getting lost, this information is getting lost that this is double line, so this is very important and this will cause also lot of concern later, first of all also note that not every subset is in the image of V_K , so for example if K is infinite and n = 1, and suppose so now what you have to do, we have K here and this is radical ideals in the polynomial in one variable.

Now if I take infinitely many points where K is infinite field, so if I take infinite subset, if I take W to be infinite, it cannot be V_K of an ideal, V of any ideal A cannot be this W because ideals here, ideals in these are principal so they are generated by one polynomials, so therefore ideally generated by one polynomial only f and this V_K will be $V_K(f)$ that means this 0 is a set of one polynomial in one variable and that we know it is a finite set, and therefore it cannot be the W, therefore all subsets are not in the image, so when we were hoping that this is a bijective map that is not correct, so we have to adjust it, so we have to see which subsets come into the image, for that I want to define a map in the other way,
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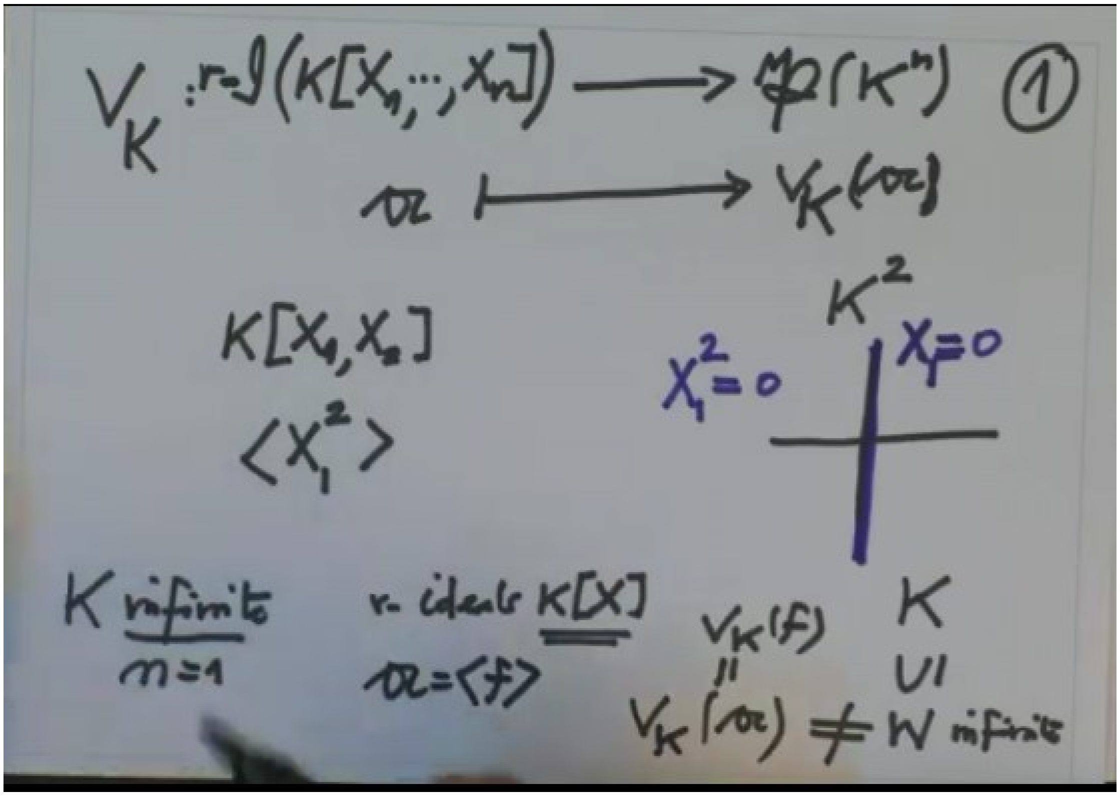

so for this we define a map now in the other way, so I want to define a map which will be called an I_K , this map will be from the power set of K^n to radical ideals of the ring $K[X_1,\ldots,X_n]$, and what is the map? The map is take any subset W and map it to, now I want an ideal so that is $I_K(W)$ is by definition, this is by definition all those polynomials $f\in K[X1,\ldots,X_n]$ such that f on any a in 0 for all $a\in W$, all those polynomials which vanish on every element of W, you put it together, now you check that this $I_K(W)$ is an ideal in $K[X_1,\ldots,X_n]$, not only ideal it's a radical ideal, why that because what do we have to check, if I have two polynomials here, then the sum and difference is also there but that is clear because f(A) = 0, g(A) = 0 then f + g and

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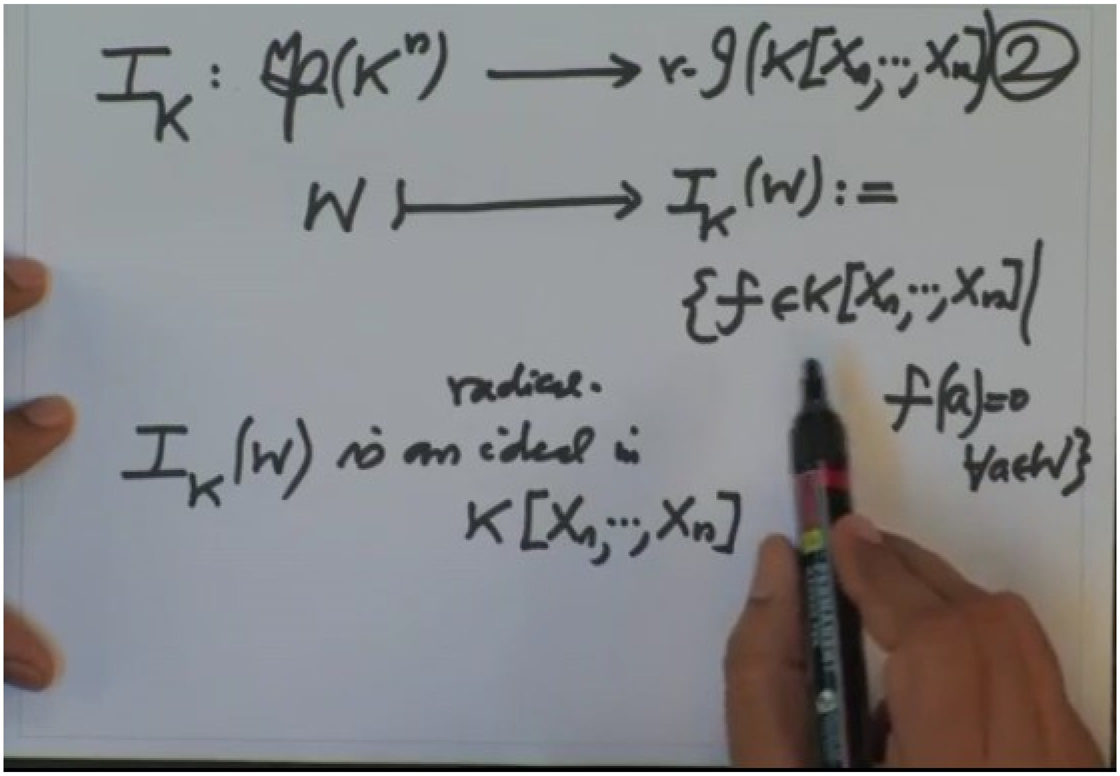

evaluated a is same thing as f(a) + g(a) or f(a) – g(a), and both are 0 therefore it is a subgroup, then you have to check that if I have another polynomial, arbitrary polynomial H in n variable and if H time set also there because H time set evaluated any A and W, A is HA times FA, but f(a) is 0 therefore, so it's clearly an ideal.

Also similarly it is radical ideal if f is there that means f(a) is 0 for all in W, so if somebody is on radical then the power will be here, but then the power $f^n(a)$ is 0 then f(a) is 0, so therefore it is a radical ideal, so we have defined therefore a map from a power set of K^n to power set of other, to the radical ideals of this.

Now the big question is, I want to say that this I_K and V_K , what is the relation between the maps I_K and V_K , and that was given by the strong Nullstellensatz, so this is Hilbert's Strong Nullstellensatz that says that if I take,

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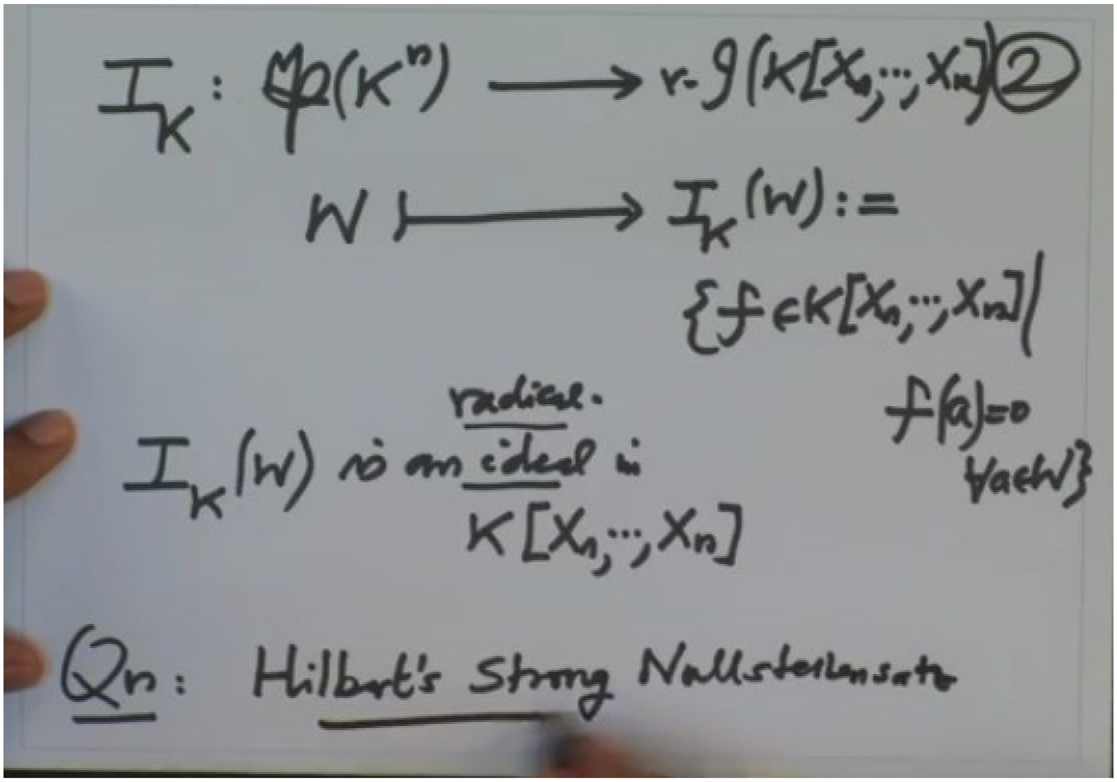

if I take any ideal A and then take V_K, and apply I_K then I get back A, therefore that mean what that mean this I_K and V_K they are inverses of each other, on what? On so, so V_K, on the image of V_K, so V_K is a map from the radical ideals of $K[X1,\dots,X_n]$ to the image, I want to take only the image of this, so these are called affine K algebraic set.

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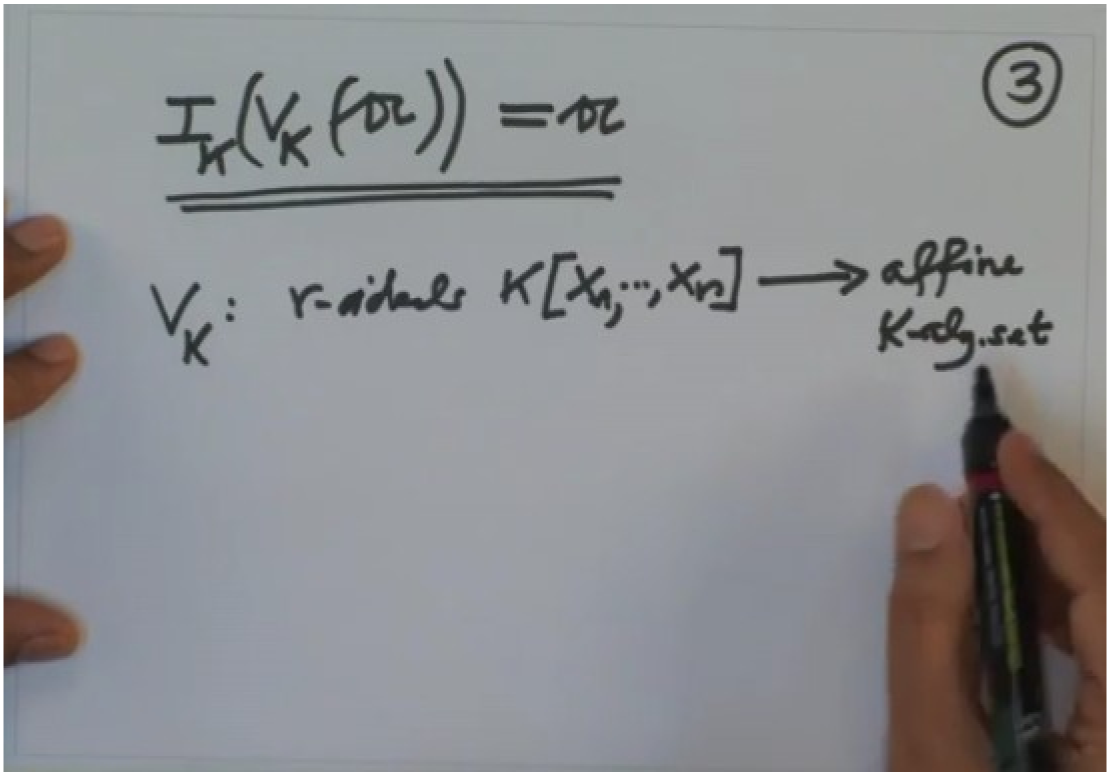

What is affine K algebraic set? It is also form V_K of some ideal, so this is also denoted by $AFF_K K^n$, these are affine algebraic sets in K^n, this is a subset of, power set of K^n, so what are these by definition? These are precisely, so affine algebraic set is precisely V_K of some ideal A, where A I can assume to be radical ideal, so this map has inverse namely I_K, but that is under the assumption, this is under the assumption that K is algebraically closed, this was proved by Hilbert in 1863,

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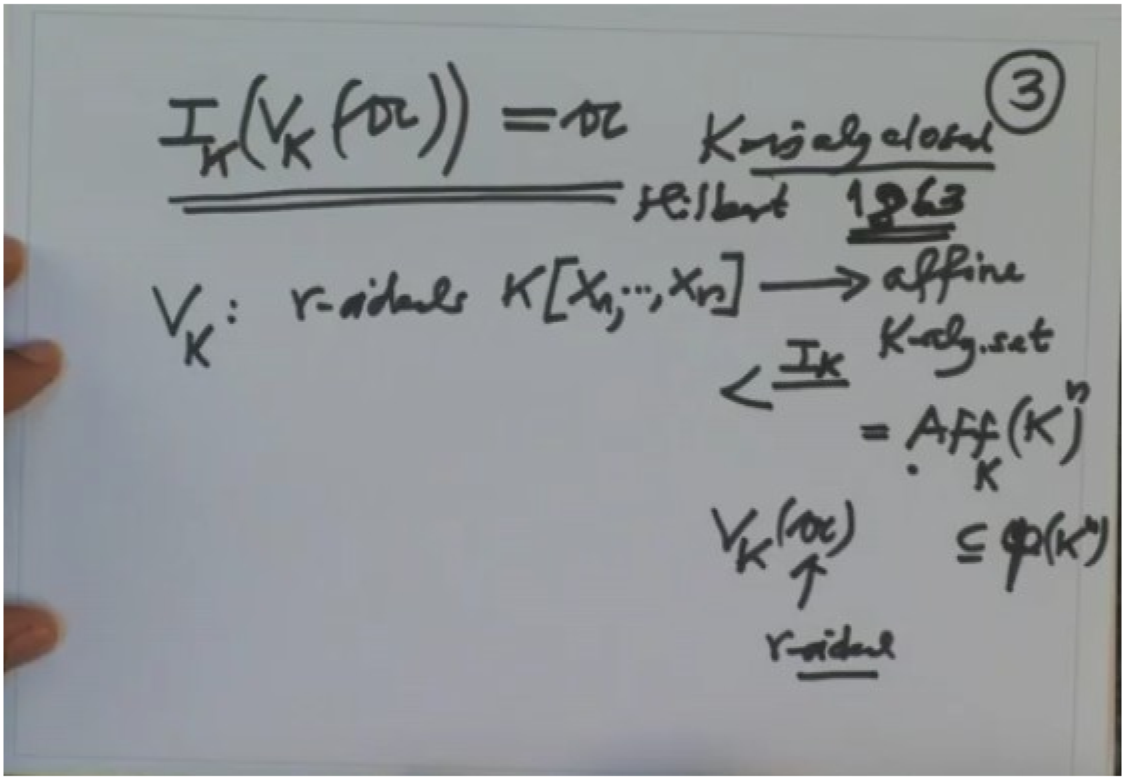

so that gave a way to study affine algebraic sets in K^n with the help of radical ideals, so therefore from that onwards study of ideals in a polynomial ring, maximal ideals etcetera that became very, very important because that corresponding to study of the geometry, this geometry is called affine algebraic geometry, that study now equivalent to studying ideals in a polynomial ring, in the polynomial ring $K[X1,\ldots,X_n]$, so this gave a two way traffic, because we want to study ideal then you study corresponding and then so on and so that became, this became, this is known as classical algebraic geometry.

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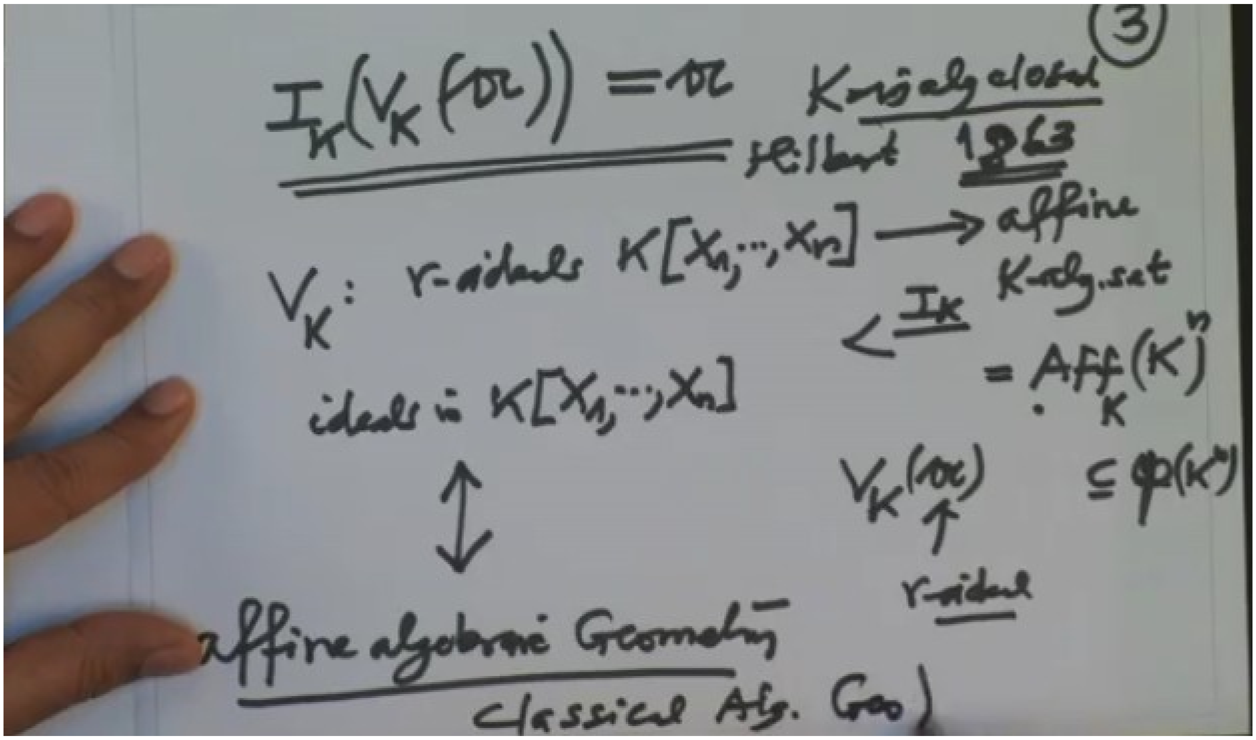

But then remember this as a big assumption K is algebraically closed, so for example one cannot study real algebraic geometry with this, we need a different tool for that, alright, so this was what the motivation for Hilbert Nullstellensatz which will deduce Hilbert Nullstellensatz from the normalization lemma, normalization lemma is more general.

And now in the remaining time I want to now generalize this also, so remember to each, suppose I have a field K, what do we understand from this study is for a field and for R, where R is K algebra of finite type, to study this we have attached a set to this, that is K spectrum, K-spec R, K-spec R is by definition $Hom_{K\,algebra}(R, K)$. And this also we could identify with the maximal ideals M in SPM R such that the residue field $\frac{R}{M}$ is K, this is the K spectrum, this is obviously a subset of the maximal ideals Spm R, and we have seen example there are maximal ideals which are not here, for example when K is real numbers then X^2+1 which is a maximal ideal, ideal generated by X^2+1 is a maximal ideal, but it is not K spectrum, it is not in the K spectrum, and this is further enlarge to spec of R, this is a set of all prime ideals, the set of all ideals in R.
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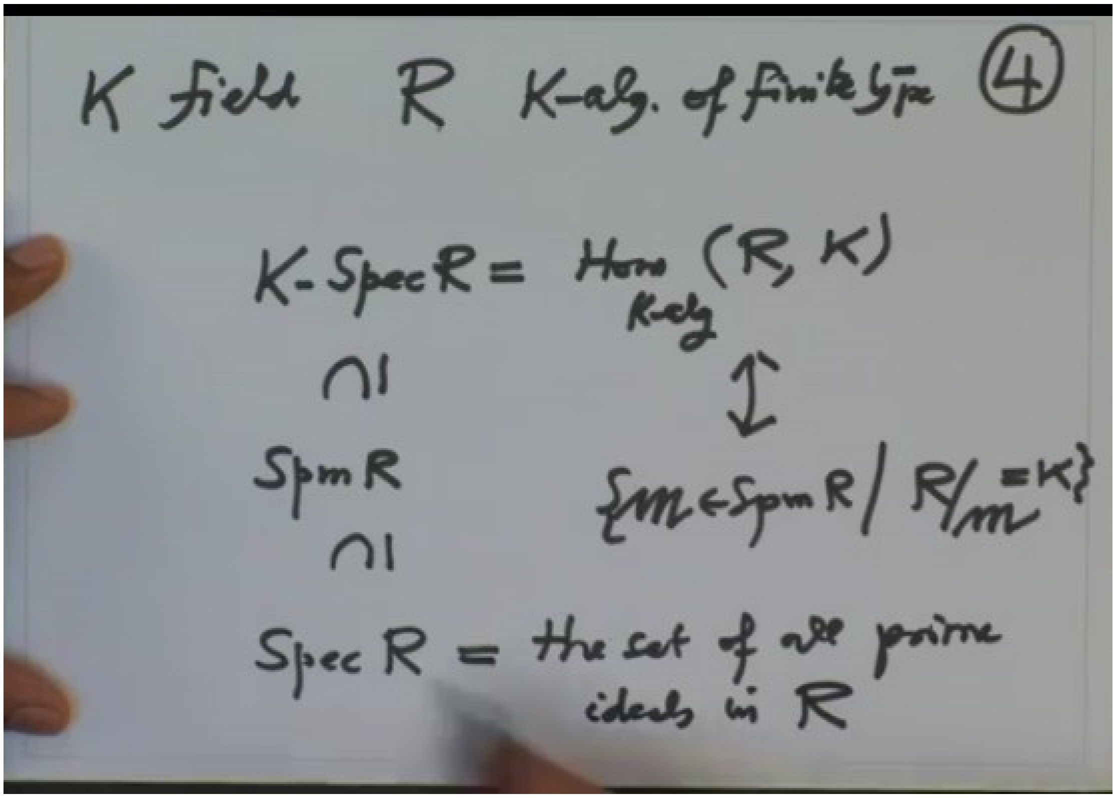

Now we have only defined a topology here and that too we have an assumption that these R is K algebra of finite type, so in general we don't have K field, ring may not contain a field and they may not be finite type over field, so that we want to define more generally topology on this bigger set, on the spectrum set, and therefore I can take induced topologies here, this is not there so more generally what do we have for any ring mark, if R is any commutative ring, any commutative ring we have these two sets Spm R which is contained in the spec R, and we know this set Spm R is non-empty, then R is nonzero, this is under the assumption that R is nonzero, this is what known as Krull's theorem.

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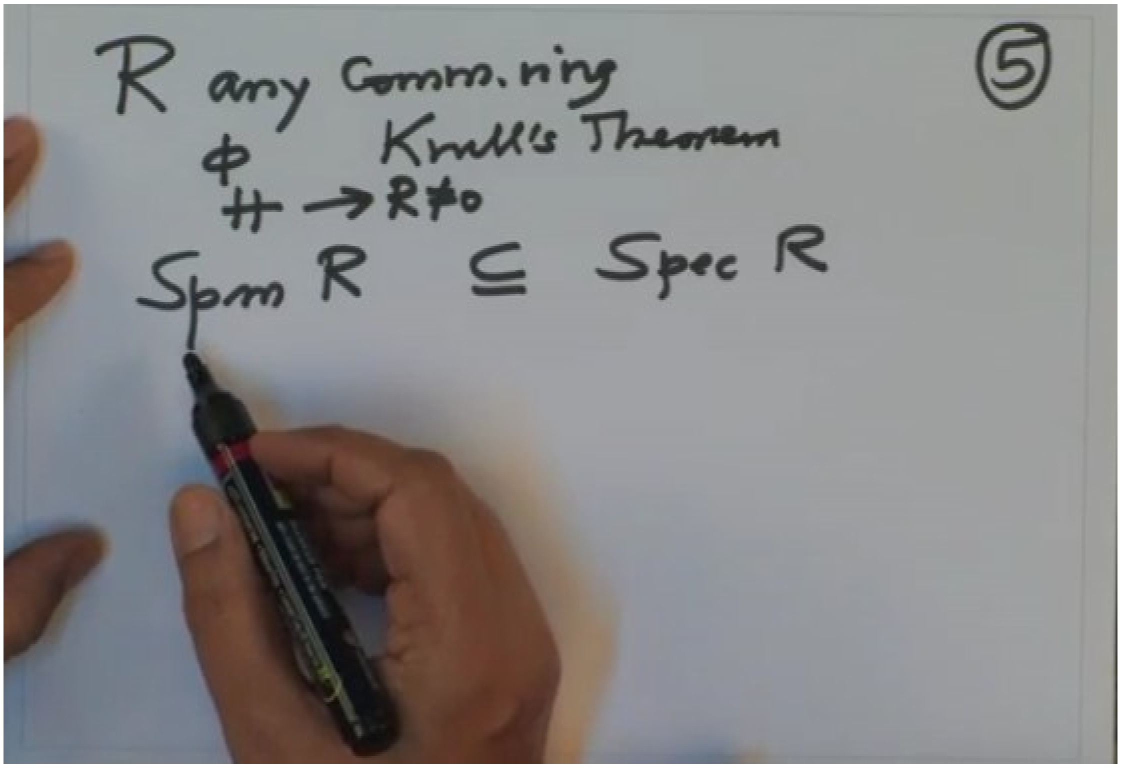

Krull’s theorem says that if you have a nonzero commutative ring then there is at least one maximal ideal in R and this was proved using Zorn’s Lemma, and therefore there is a prime ideal, there may be many prime ideals.

So we want to define topology on this bigger set, spec R, so that means what we want to define, we want to give you a collection of subsets of this which satisfy properties of the closed set in a topological space, that means we want to give a collection which has empty set, which has a whole space and which is closed under arbitrary intersection and which is closed under finite union, if I have such a collection then that will define a topology on that set, and this sets, this collection, elements of this collection will be precisely the closed sets in the topology.
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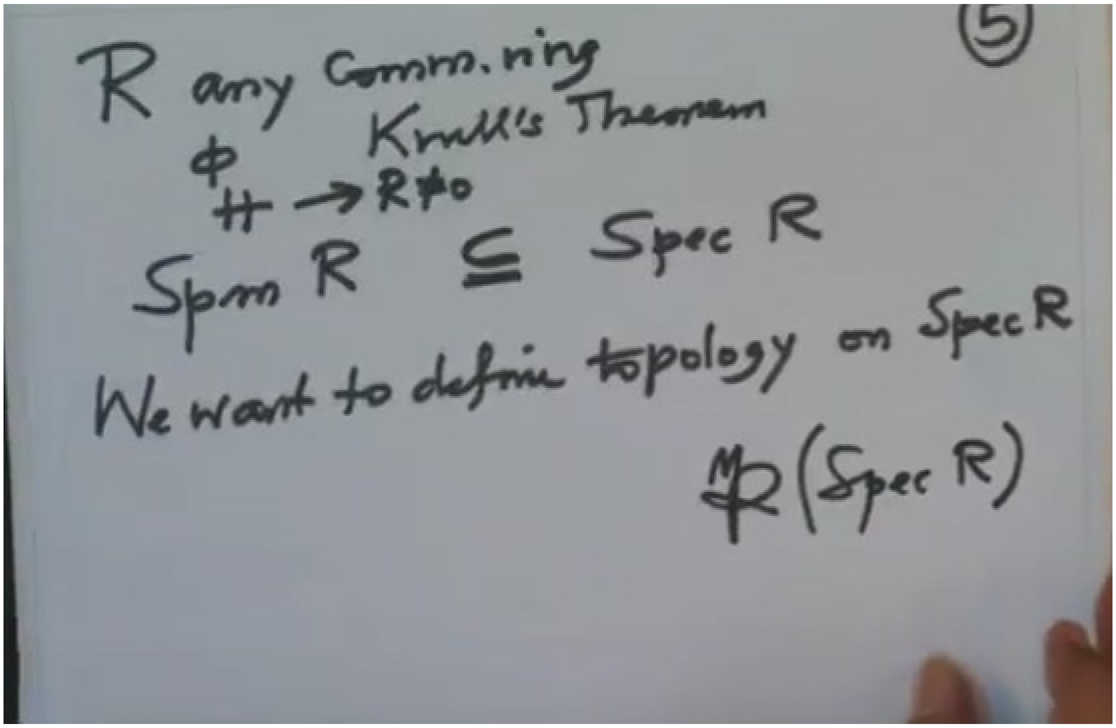

So we are looking for collection of, so we are looking for subset of the power set of the spectrum, okay, and what is that now? I'll now define, so obviously the ideals will come to the help, so take any ideals, ideals in the ring R and I want to define a map from this to the power set of spec R, and this map I'll call it V, so take any ideal A and now I want to define V(A), and what is V(A) should be? A subset of a spec of R, so what is this by definition? This is by definition all those prime ideals P in spec R such that P should contain the given ideal A, so I would define V(A), now there is no suffix K here, because there is no field.

Now first of all $V(A)=V(\sqrt{A})$ they are same, because if P contains A, then P will also contain radical of A because P is a prime ideal, radical of, recall that radical of an ideal is by definition, all those elements F in the ring R such that f^n belongs to A for some n, so this is the radical. And in general radical of the ideal contains A, if it equality holds here than one call that ideal A to be radical ideal, so because this map ideal A and radical of ideal A they go to the same set, so I will,

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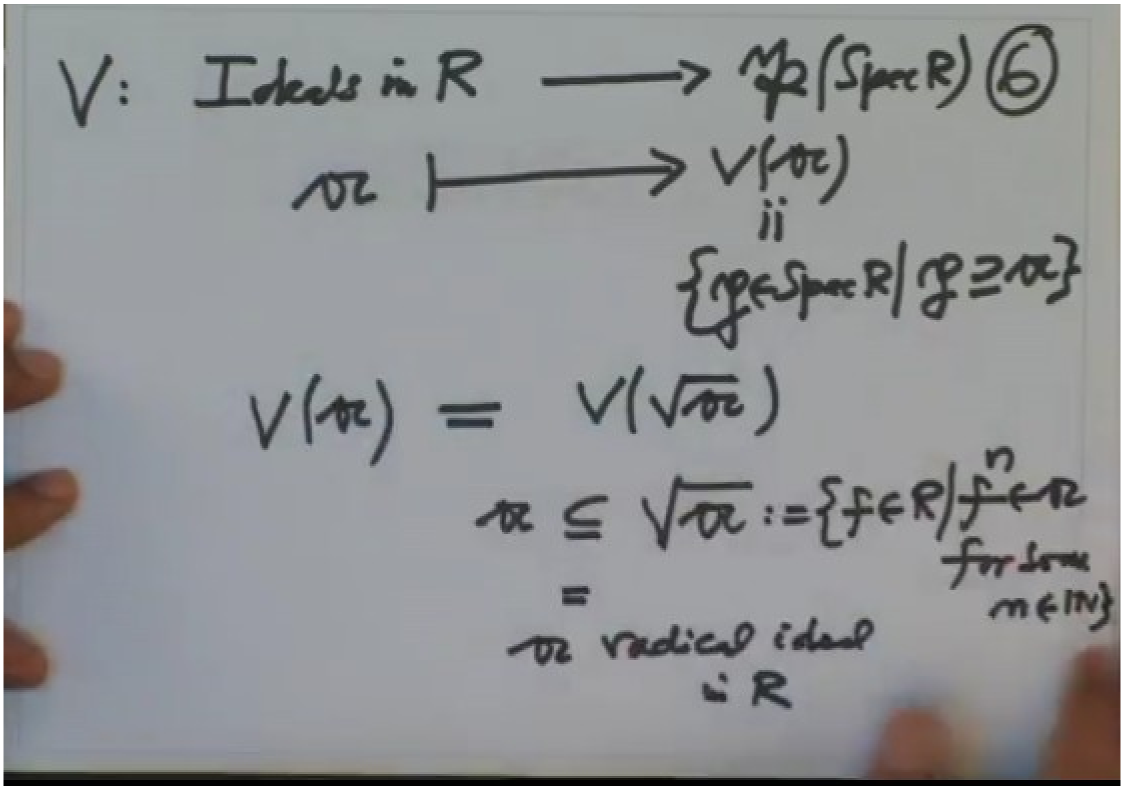

we will concentrate on the radical ideals only, so therefore think V the map from radical ideals in R to the power set of the spectrum, A going to V(A).

Now again we only consider the image of it, and then try to define a map, we would try to define a map in the other direction that will be map I, so what is it? We have given any subset Y of the spectrum and we want to define I(Y), I(Y) is by definition, we look at all those prime ideals in Y so intersection P, P where is in Y, so therefore this makes sense, this is an ideal, not only it is an ideal it's intersection of ideals and intersection of radical ideals, therefore it is again a radical ideal, so this indeed belongs to the radical ideals in R.

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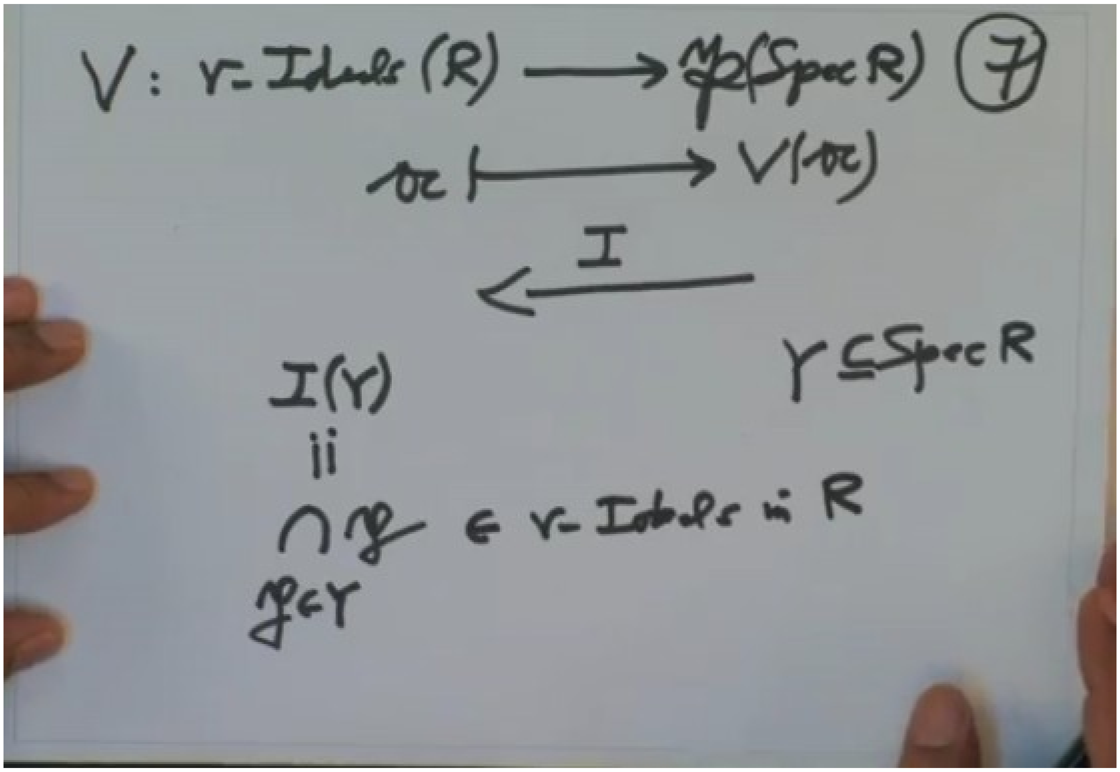

And now like we have checked the properties of these V, what is the property we should check? We should check so this is the proposition, this will check that the collection V(A) where A varies in the radical ideals of R, this collection satisfy the properties of closed sets in a topology, what is that mean? That is one V(1) is empty set, V(0) is whole spectra, so that means the whole space is there, empty set is there, secondly, we have to check that it is union of V of some ideal arbitrary family, this is intersection of V(I) so that means given V(A) is, family V(A) is index by I then there intersection is also of the same form.

And third one that if I take the product ideal A times B, which is same thing as intersection ideal, which is union of A and $V(A) \cup V(B)$,
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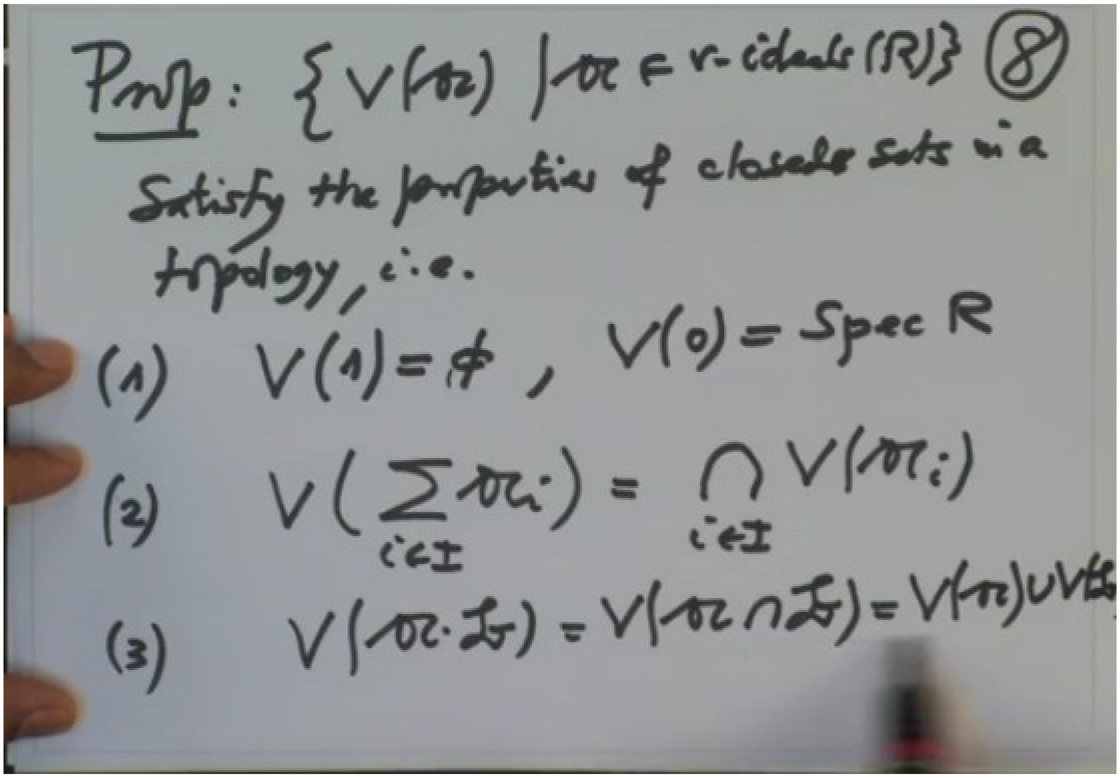

so this is also easy to check that see if you translate it what do you want to prove? It is very easy this means if a prime ideal contains the product of two ideals then it contains either of them, so similarly if prime ideal contains all this guys, then it contains it is belong to the intersections, so these are precisely the, these are the properties that closed sets satisfy in a topology, so therefore this topology, so this defines a topology on spec R and it's called the Zariski topology on spec R, alright, so one has to check these properties, but one possibility to check this, I want to give that notation,

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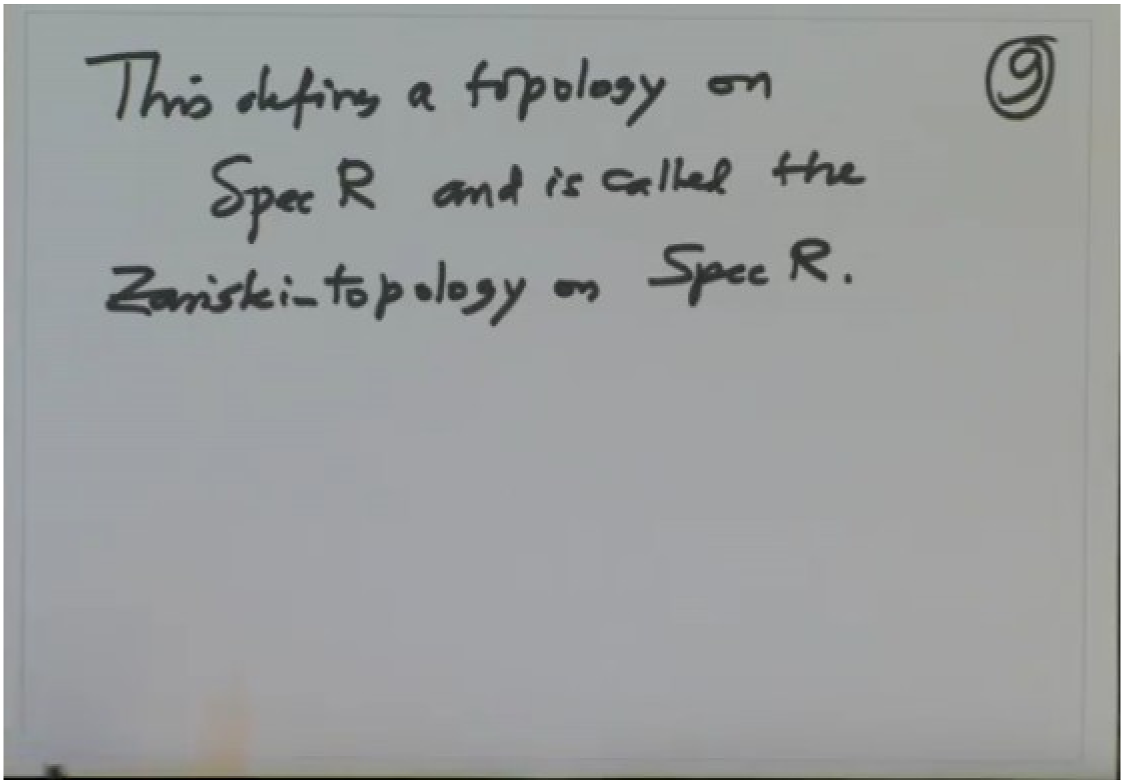

so what are the open sets? So open sets are precisely the compliments, open sets in spec of R are precisely the $spec\ R \backslash V(A)$, also these are called D(A). So what is this by definition? So all those prime ideals P in R such that P is not here, that means P does not contain A, then it is there,

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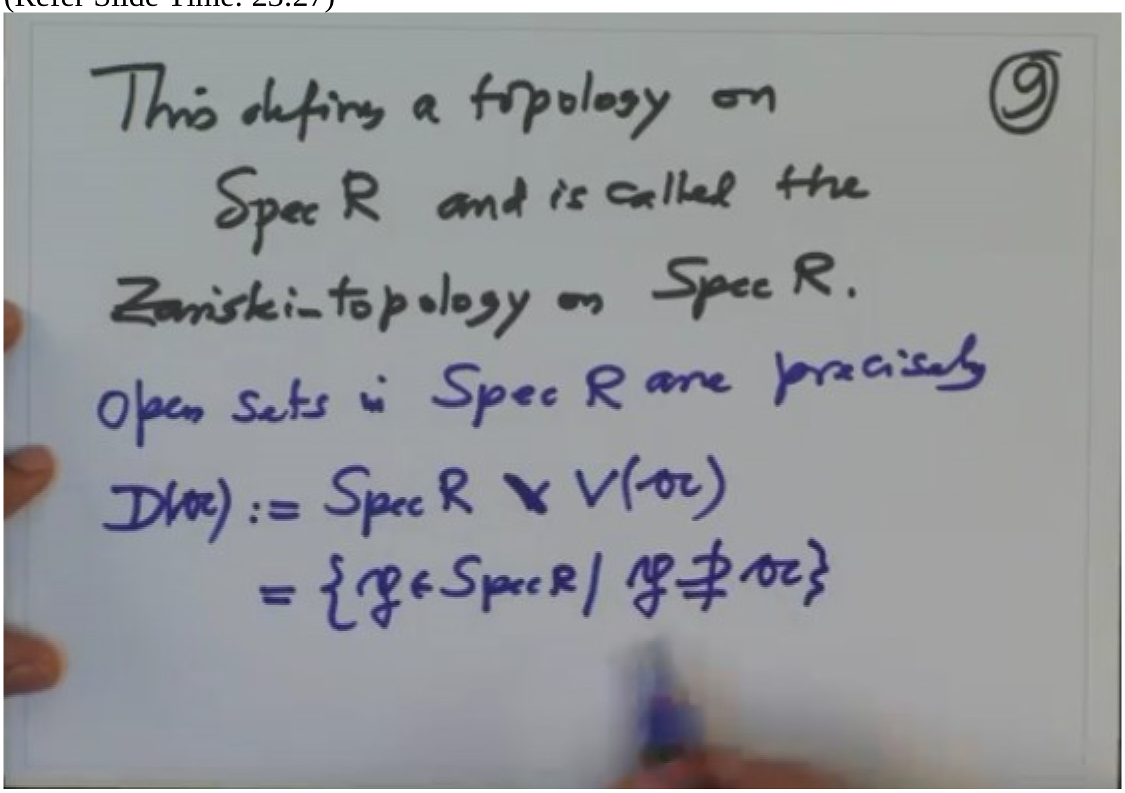

right, so these are precisely the open sets.

Now what is V(A)? Another important thing one should note here is that, so when the ring is noetherian, when R is noetherian then every ideal A in R is finitely generated, but then this V(A) and V of a generating set are same, finitely generated so that is A is generated by $f_1,\ldots,f_r$, then V(A) is same thing as $V(f_1,\ldots,f_r)$ and therefore we only have to consider finitely many elements at a time, so therefore what would be the complement of this? So D(A) which is $spec\ R - V(f_1,\ldots,f_r)$, so these are precisely all those prime ideals P in R such that P is not here, that means at least one of the f_i is not in P

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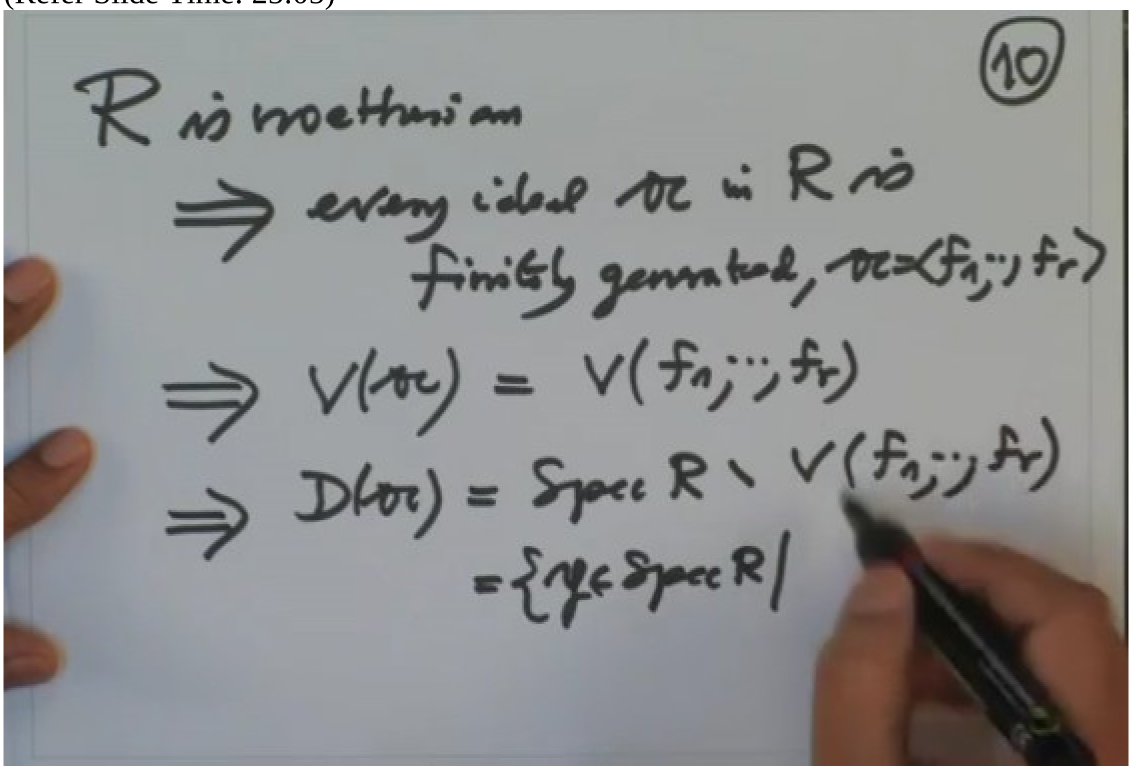

so f_i is not in P for some i from 1 to r, so for example what will be D of single element f? This is $spec\ R \setminus V(f)$ which is all those prime ideals P such that F is not in P, so these are the basic open sets,

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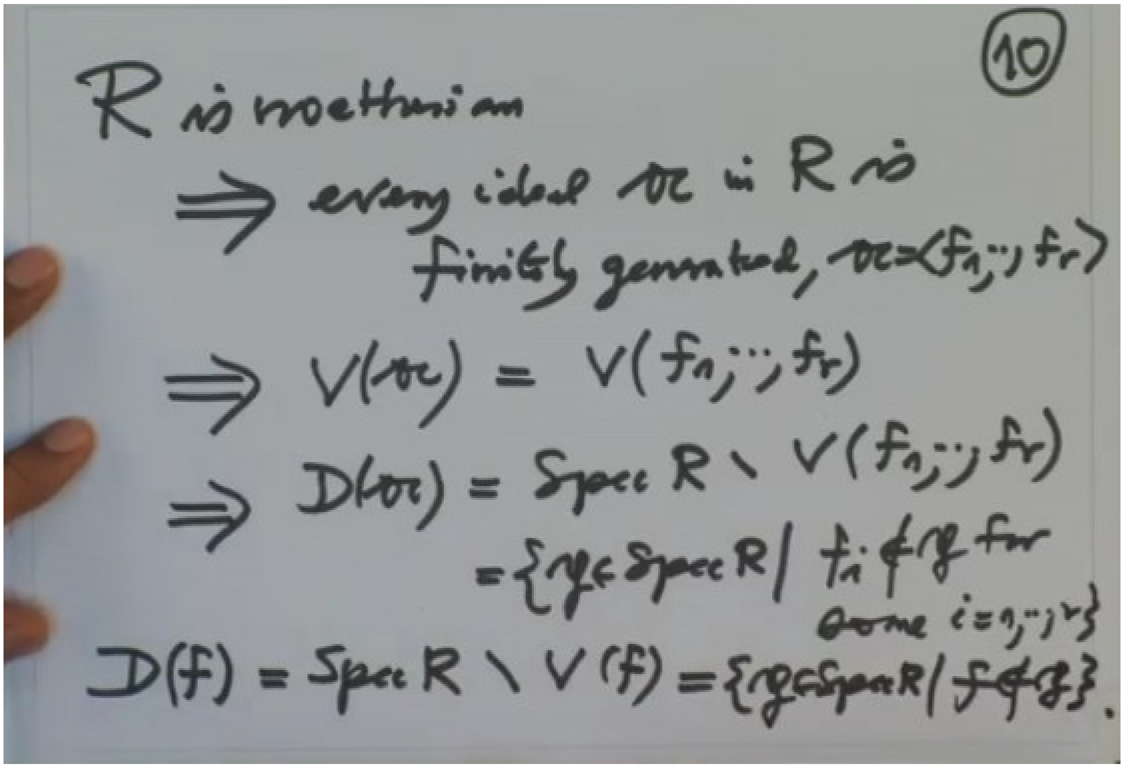

so therefore what we noted is this collection D(f), where f varies in R form basic open sets for the Zariski topology on spectrum, spec R.

So to check things this the following notation is very very useful so let me write that, let me explain how does one write neatly, so first of all for psychological reasons one like to have one notation for a topological spec to be capital X so given a ring here R I have this spec R, this is a set of all prime ideals, these I would like to denote this by X, because we are used to topological space by capital letter X, okay, and how we are going to change our notation? When we want to stress upon the property the prime ideals then we will write the notation P, and when you want set the stress about the points in a topological spec then I will write it x, but just to keep track these P corresponds to this X, I'll write P_x just to remember that this P we have identified with that x, alright.

Now therefore in these notation, okay, so I have the ring R here, I have $\frac{R}{P_x}$ here, now this I have performed the algebraic operation therefore I want to write it as PX, so this is a residue map, this is modulo P_x map, this is integral domain and therefore it has a quotient field, so quotient field of $\frac{R}{P_x}$, this quotient field I want to denote by κ_x, because it depends only on x, x means P_x, so this is a field, this is a field and we have a natural map here, see this is a ring homomorphism, this is also ring homomorphism because this is embedding of the integral domain in the quotient field.

Now if I have an element f here I denote that in general elements of the ring by letters f, g etcetera like we denote the polynomials, because I always want to think rings as polynomial rings,

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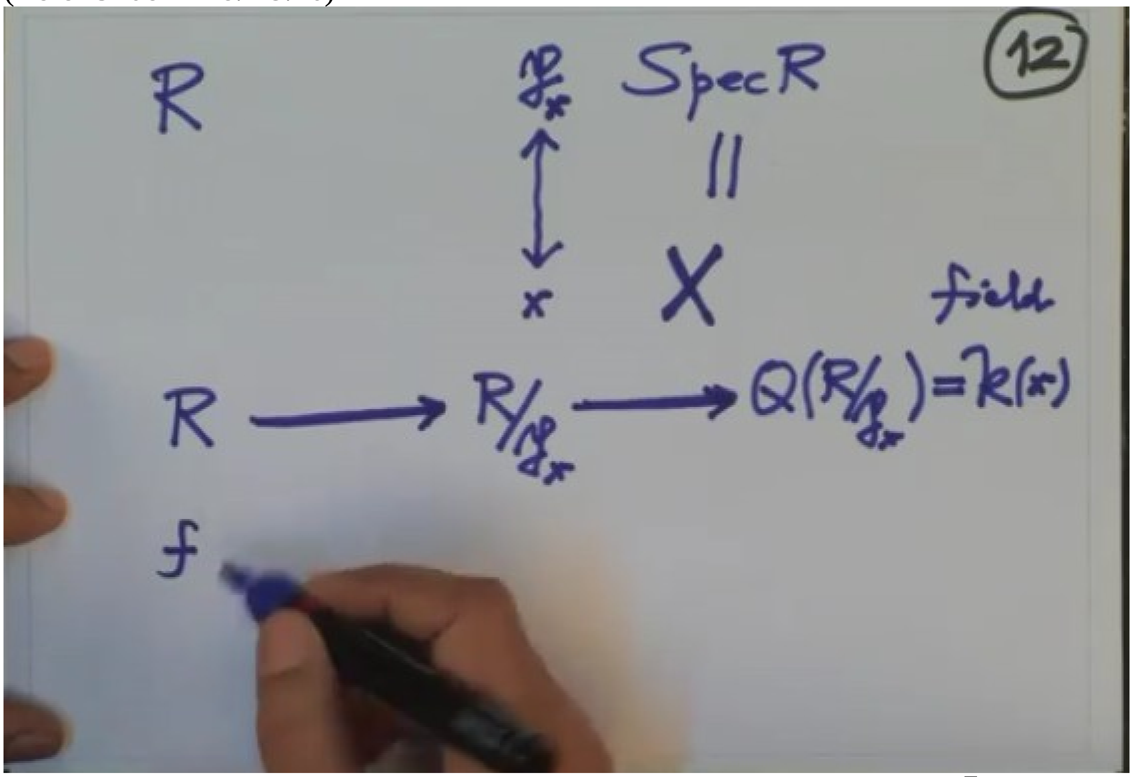

and therefore the choice of notation is from that implication, so f goes to $\bar{f}$ here by reading what? P_x , and this $\bar{f}$ will go to somebody here that image I am going to denote by f(x), this is the image of f in this quotient field, it comes $\bar{f}$ and then this, now this is a ring homomorphism, this is a ring homomorphism means what?

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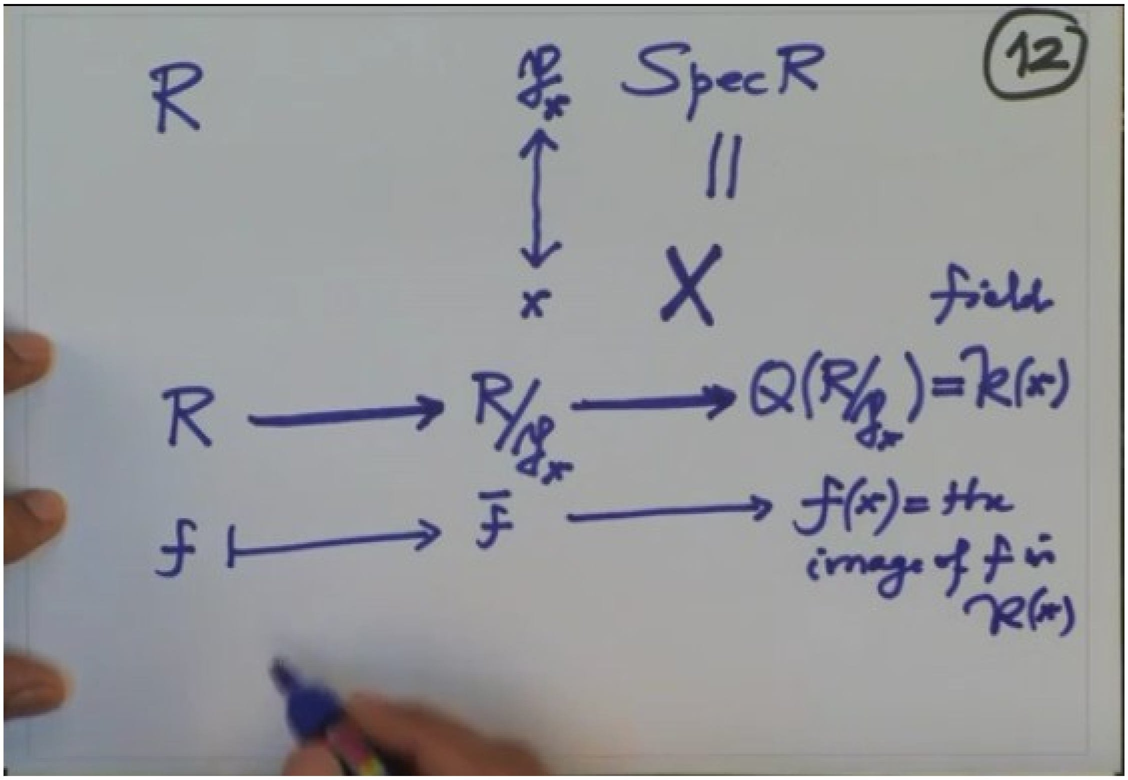

If I take f and another element g, f+g and take its image directly or I take the image of f and image of g and then this is equal to this plus this, because this is a ring homomorphism.

Similarly f time g evaluated at x, I'll write it to the evaluated x, this is f(x) times g(x), so if this product is 0 then one have them has to be 0 because we are in a images are in the field, so therefore what will be my definition of V(a) now let us,

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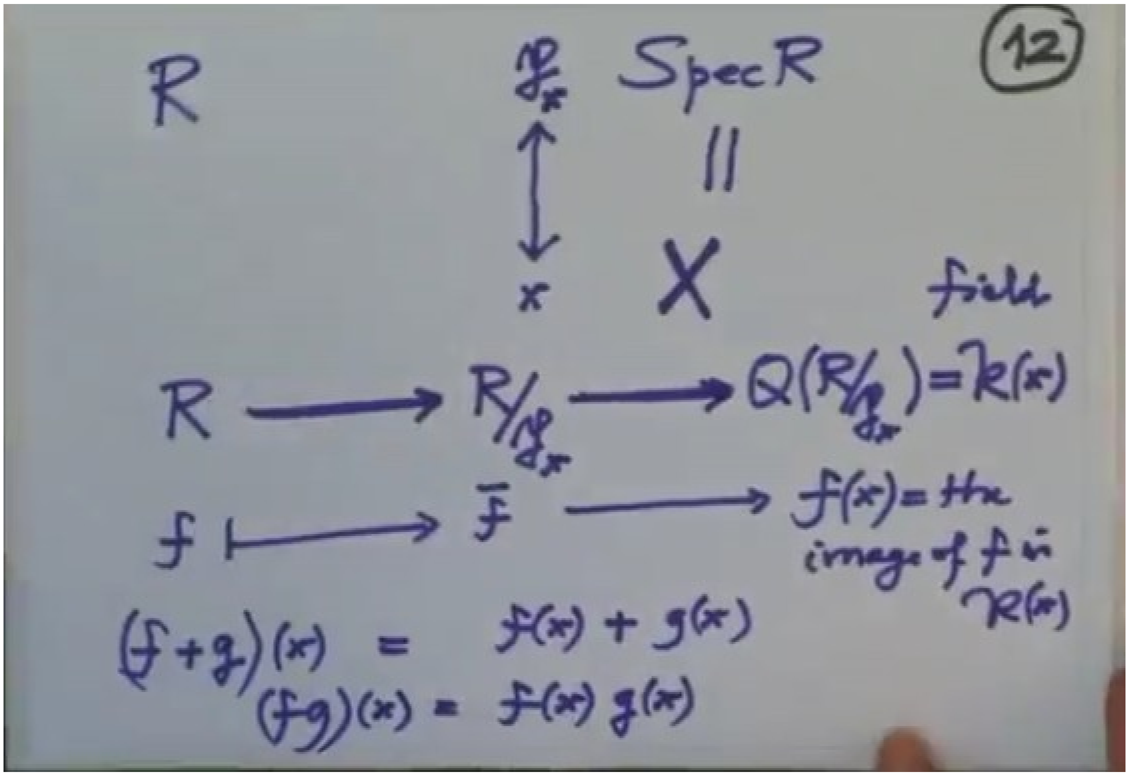

for example V(f) will be what? These are all those prime ideals P such that P contains f, that means f belongs to P_x, so this is a same thing if I want to write in the new notation now these are all those elements x in X, remember this P_x we have identified with x such that, now this will be what? f belong to P_x so that means f(x) is 0, this is the image of f in the residue field κ_x, so therefore this becomes, this gives the same feeling as what we did it for K spectrum, and therefore if you adopt the same checking bit will be very easy,

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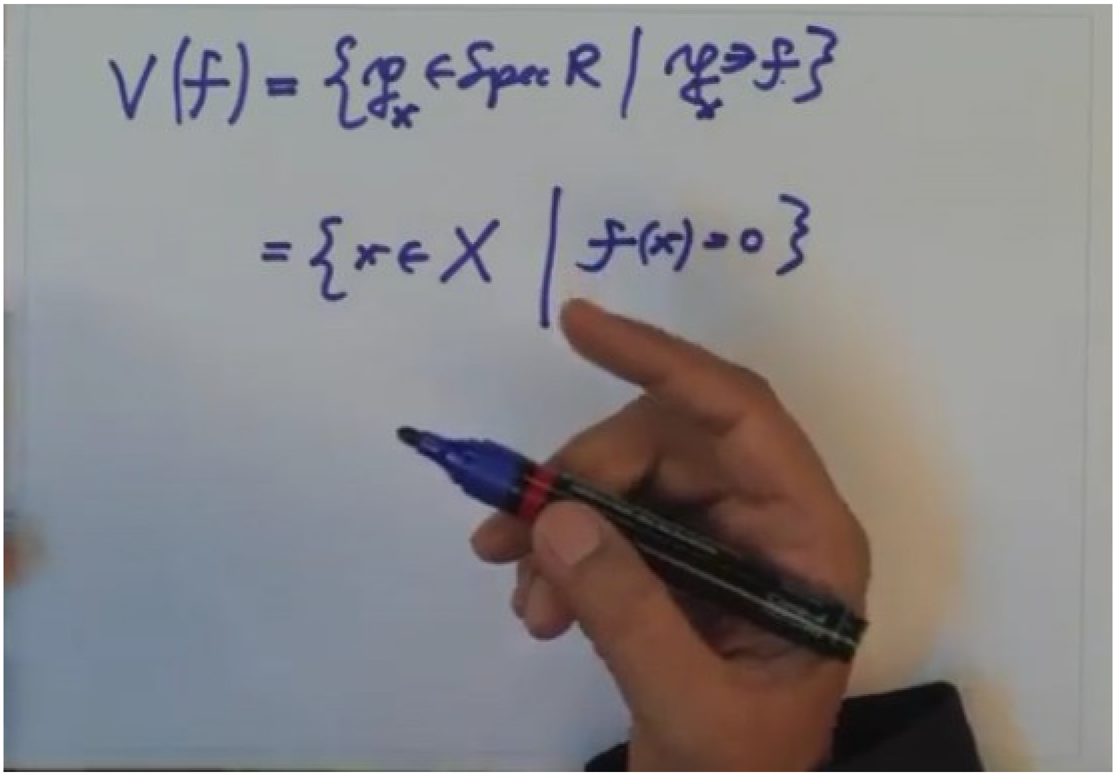

so I suggest you adopt that for checking that they satisfy the properties of the closed set, this notation it will be very, very helpful, this notation which is due to Grothendieck,

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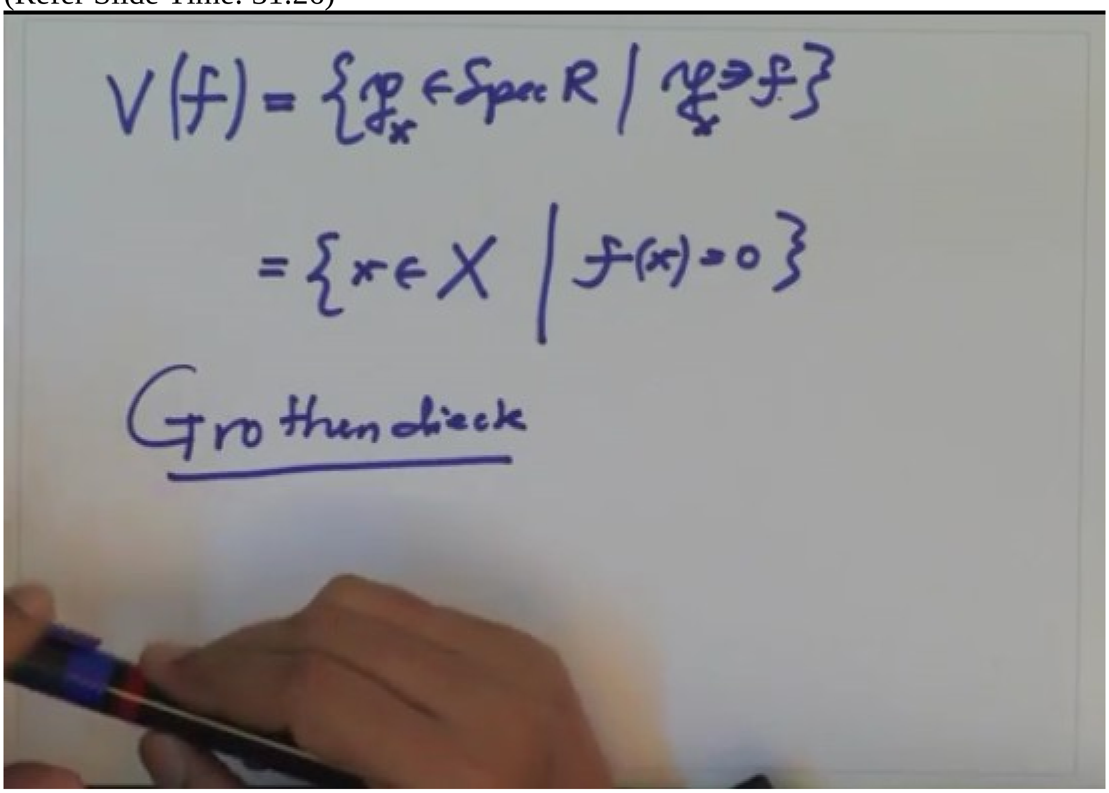

Grothendieck is the one who change algebraic geometry to very abstract algebraic geometry and it gave lot of important results. With this he proved lot of important results, okay so with this I will stop.