Week 4

Lecture – 19

Commutative Algebra

On

Dimension Inequalities

Gyanam Paramam Dhyeyam: Knowledge is supreme.

I have left some proofs incomplete last time. So this continues more consequences of NNL. So the proposition we're trying to prove last time about the height of prime ideals in the extension A to polynomial $A[X]$. So Q is a prime ideal, so A is commutative ring and Q is prime ideal in $A[X]$ and P is the contraction of Q to A. Then we want three statements about the heights. Then 1, height of $\frac{Q}{P[X]}$, $P[X]$ is the extension of P to the polynomial ring. This, this plus transcendent degree of the residue field at Q or residue field at P, this sum is 1. Two, two had two equalities. At the height of Q, equal to height P, plus height of $\frac{Q}{P[X]}$ and then if you use this equality 1, this becomes height P plus 1, minus transcendent degree of the residue field. And third one, height of Q equal to height P, if Q is the extended ideal of P and otherwise, it's height plus 1, if Q is not extended and we approved one and also these equality follows from one. So, now we have to put these two and this three. We'll first, or first of all note that in these three follows from one and two. So, these three follows from one and two. Three follows from one and two. So, we only, I put only one equality, that is this one.

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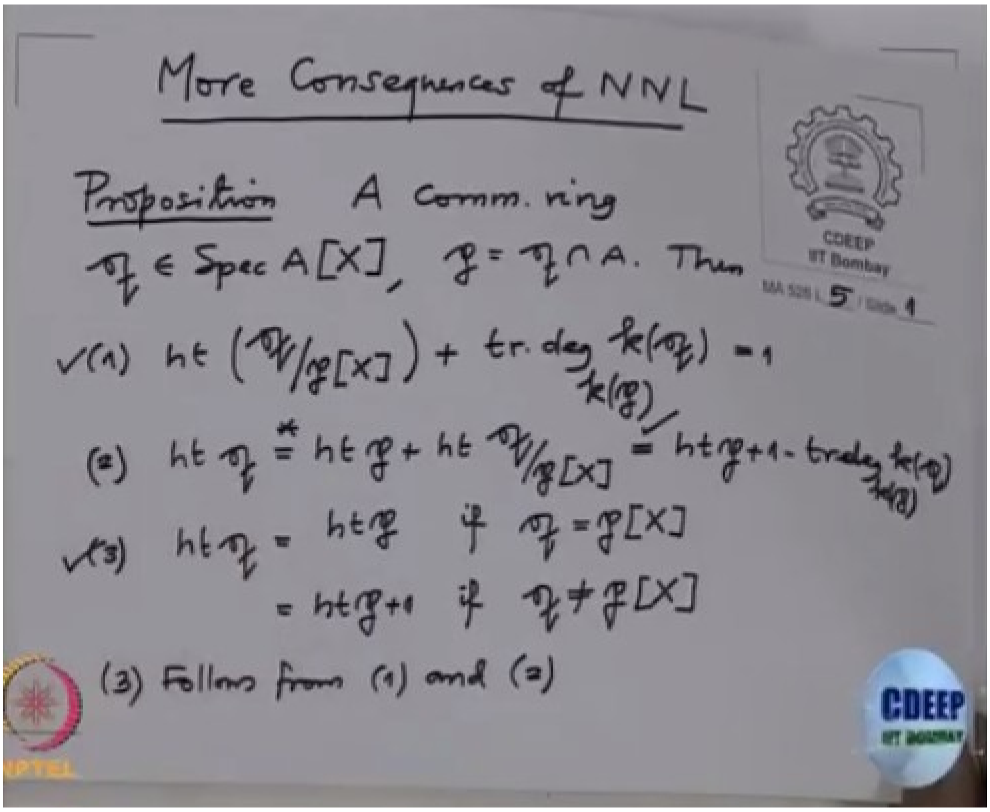

But this equality, I will postpone. Because I need some more machinery from my local algebra which I will postpone and I will assume for the time being and we will continue. So, one immediate corollary is, if A in noetherian then the dimension of the polynomial ring in n variables is n more than the dimension of A. But noetherian is very important. So proof, we'll use induction on n. So it is enough to prove that for one variable. Enough to prove that dimension of $A[X]$ is one more than the dimension. So, 1, now, if you have a chain of prime ideals in the A, so P_0 contained in P_r , this is a chain of length are in the spectrum of A. No. Take, taking in $A[X]$ and then therefore not P, but I'll call it $P^{'}$. And when I contract to A, I will get P_r . This is P_r containing P_0 . And we've checked that, so this means that height of $P_r^{'}$ is at least r which is less equal to height P_r which is height P_r plus one that is because of these formulas from here. We hide P at least equal to P or this, so it can at most increase by one. So therefore, the end result is less equal to dimension of E plus one. So, therefore, one inequality is cleared, dimension of $A[X]$ is less equal to dimension A, plus one.

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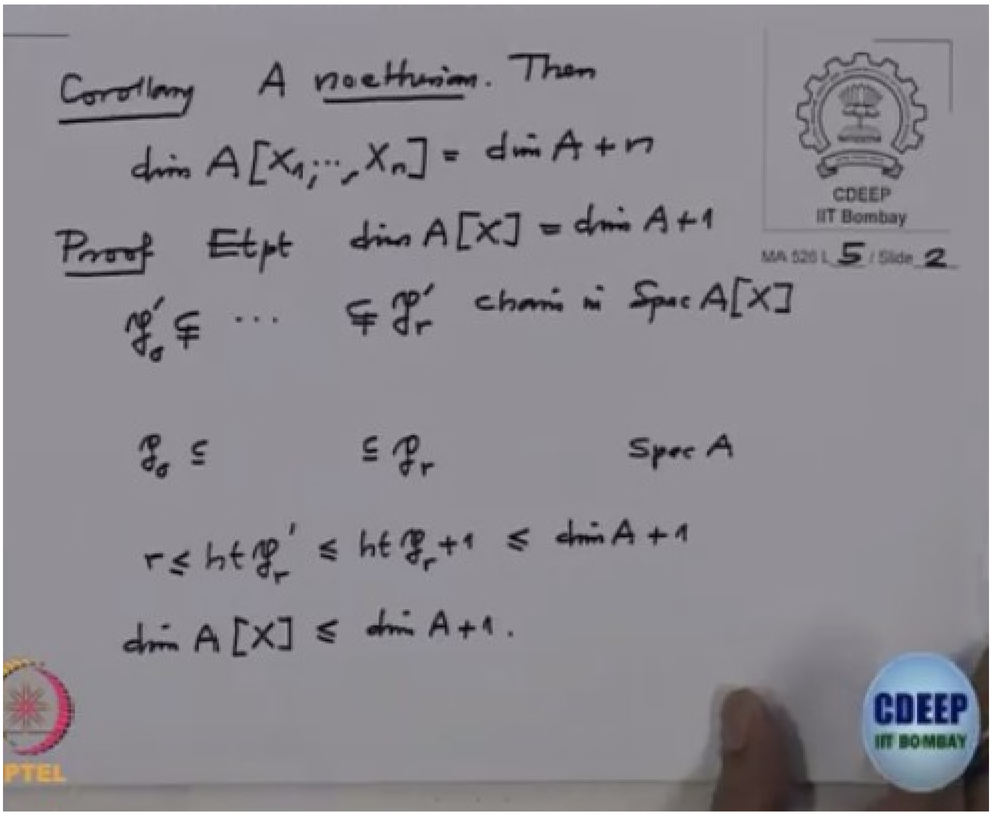

For the converge, if I have a chain of prime ideals in the A, P_0 containing, containing, containing P_r in the spectrum, then they extended once $P_0[X]$, containing not equal to P, $P_1[X]$, these, these inclusions are also proper because we have checked that if I take this and intersect back to get back, do you get back P. So, these are, this is a chain in Spec of A[X]. Therefore, and these are, I can exchange it further by one element in P_r ideals generated by P_r and X. And this is proper. Therefore, from a length of the chain of length are in spectrum and it produce a chain of link $r+1$, in the spectrum $A[X]$. So that means the dimension of $A[X]$, is at least $r+1$ now. But, is R is arbitrary R? Chain with arbitrary. So this is bigger equal do dimension A plus one. So that would be equality. Okay. So, the next theorem is also very important for, for the geometric reason. So this is called dimension inequalities. So, A, we will assume A is containing B and they are integral domains and assume A noetherian. And we will take a prime ideal Q in B and contracted to A, P is a contraction. Then, height of P, height of Q, height of P, plus the transcendent degree of the residual fields. Residue field at Q, over residue field at P, this is less equal to height of P, plus transcendent degree of B over A. When you say B over A that means, which means this is the transcendent degree of the quotient field of B, or quotient field of A. That is the definition of this transcendent A. Yes? All right. Let me check. One side it is Q, so this is Q. This is Q.

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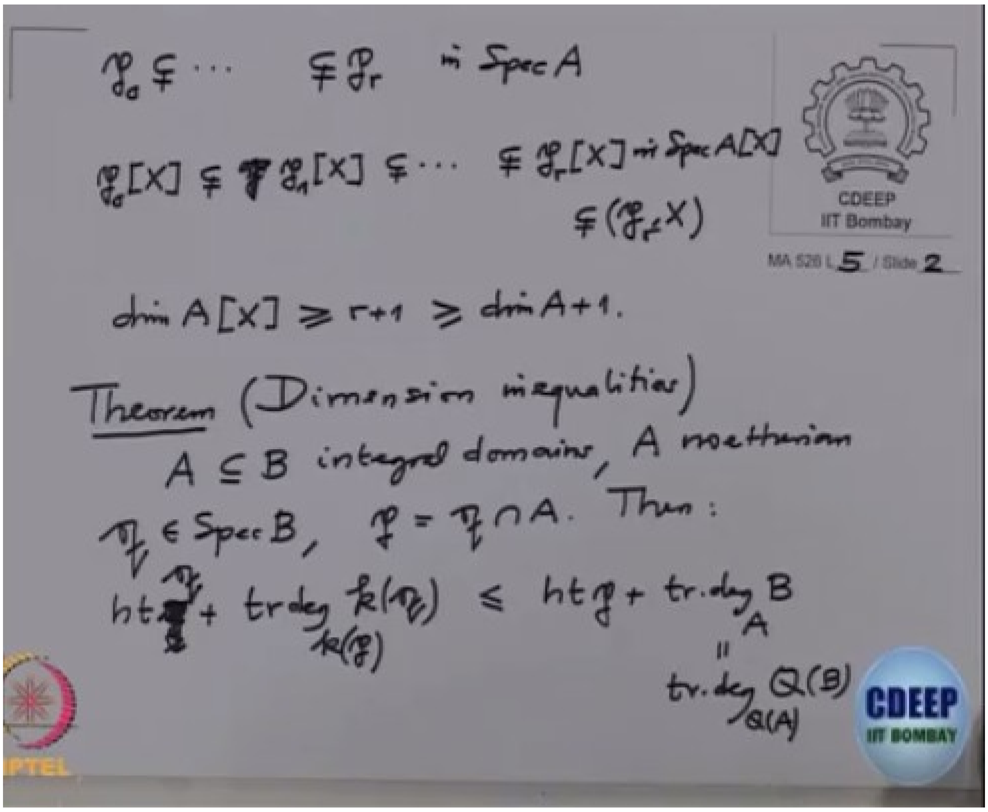

So, I wanted to write to some more statement there. Moreover, if B is a polynomial algebra, then the equality hold. Then, the void inequality is equality. Is an inequality, is an equality. Okay, let's prove it. First, I will assume that B is actually a finite type, a algebra. A algebra. So let us assume first. So proof, first assume that B is aalgebra of finite type. And then, in this case, I will pull the assertion by the number of algebra generators. So, that is, B is like this. Like $A[x_1,\dots,x_n]$. This x_1 to x_n are the algebra generators of B over A. And we will prove the assertion by induction on n, or n is equal to zero. There's nothing to prove. But anyway, let us prove it n equal to one, so n equal to one. So that is B generated as an algebra or A by one element, in such case, I will also keep saying that B this is cyclic, is a cyclic A algebra. In this case, we will first prove the assertion mainly the inequality first and then we will see we're do we near to the equality. So to prove inequality, I will assume by replacing A by A localized at P. We may assume that A is a local ring. A is a local ring with maximal ideal P. Because this inequality involves only P and Q. And therefore the bigger ideals are not playing any rule, so I may localize at P. And in this case, the residual field $K(P)$ is now in the new set-up if this is the residual field of A that is K which is $\frac{A}{P}$. And note that B there will be a domain and B cyclic algebra. So B is $A[X]$, modulus on prime ideal. Let us call it somebody. Let us call it r, where r is the prime ideal in the X. Now, I will divide the further proof into two cases. Namely this r is zero and or not zero. If r if zero is the polynomial case and r is not zero is un-polynomial case, so the two cases we will consider. Case 1,r is zero. And case 2, r is not zero.

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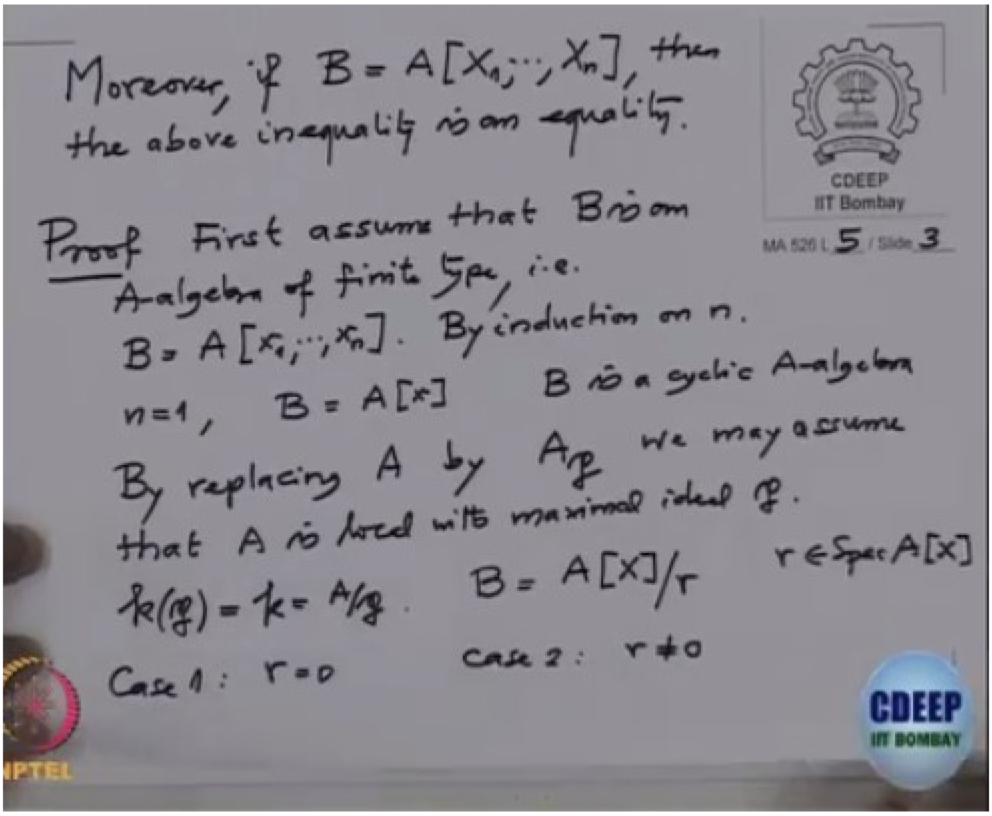

Alright. So in case of r is zero then B is, P is polynomial ring and at transcendent degree of B over A in the end is one, because the, when you say these, that is the quotient with the, at the quotient we will also actually the X is the indeterminate. So, transcendent increase with early one and height of Q because just analyse what happens in a earlier preposition, height of P, the first, second inequality. This one I want to use. Height of Q is equal to, we are in the, precisely in the same set-up. Height of Q is height, P plus one minus transcendent degree. So this is equal to height of P, plus one, minus transcendent degree of $K(q)$ over $K(p)$. This is by earlier preposition. So these I feel proof of what we want. So what we want to prove is theorem of, we wanted to prove this height of Q, plus transcendent degree that's equal to height B plus transcendent degree. But height Q plus this transcendent degree is height P, plus one. And actually into the equality that is what our assertion also say that moreover, if the polynomial ring then it's actually equality. So, this proves both the things, inequality, which is equality. maximum over that side. Now case two, r is non-zero, that means it's non-zero prime ideal. So therefore the transcendent degree of B over A has to be zero. Because in this case, the X is algebraic over the quotient build of it because that is early that r we contain pron non-Zero polynomial and that we will give a algebraic relation. So this, so write, this and this r is the prime ideal in B which will be of the form. No, Q. In the given Q, we have given Q. So Q is the prime ideal in B, but B is polynomial ring modular this ideal r. So therefore, r will look like some prime ideal in Q, capital $\frac{Q}{r}$, where Q is the prime ideal actually in the polynomial ring. See, B is, $A[X]$ is here. And B is residue class ring of X by non-zero ideal. This is B. This is objective up here. And Q was given prime ideal here. So this Q will come from some prime ideal here that I'm calling it capital Q. That simply means, this Q, not r, Q looks like Q/r . For some Q in the spectrum of $A[X]$ and also note in this case, $K(Q)$, residual field of Q, and residual field at capital Q, they're same. Because residual field means you won't want that and then take the quotient field. But when you go Mod, q or go mod capital Q, they are the residual class ring are the same. So therefore, it is, if the residual are the same. Now, I want to apply the case 1 to the ideal Q, which is ideal in the polynomial ring. So therefore, by case one, applied to Q. What do we get is height of, in that K , there's equality. Height of Q plus transcendent degree of $K(Q)$, then I write K capital Q, or K small q it is the same. $K(q)$ over $K(p)$, that is same as height P, plus one. That is the case one.

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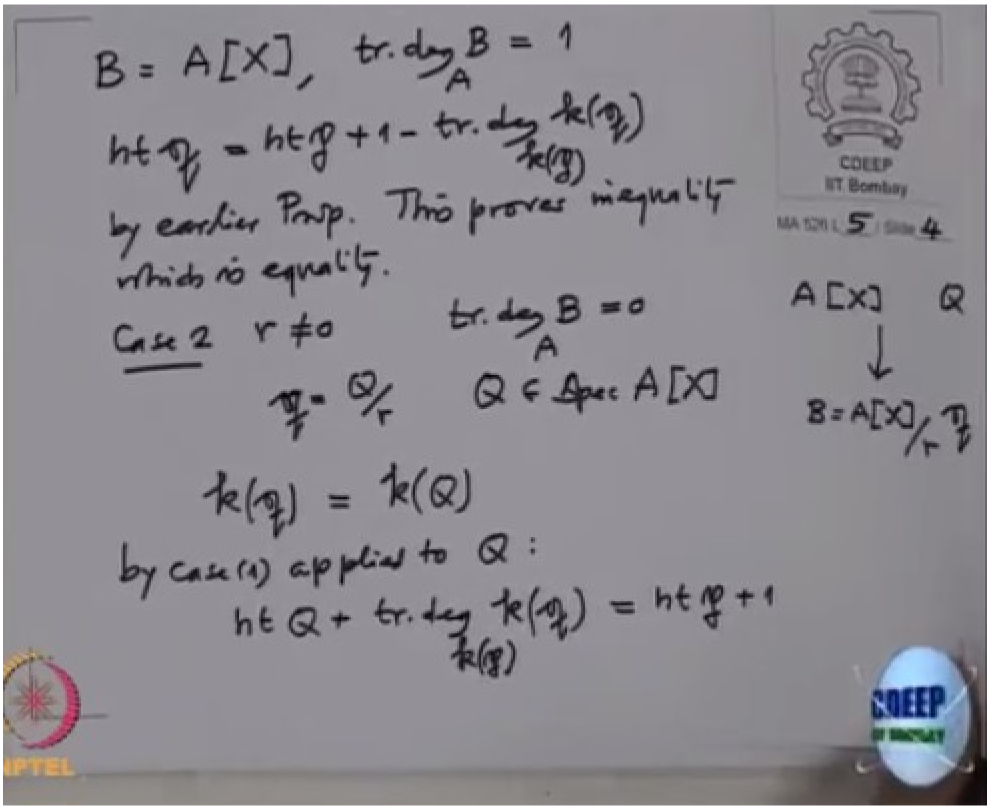

Because the other transcendent degrees. This transcendent degree is zero. Okay. So once we you have done this. Now, r is the non-zero prime ideal in r is a non-zero prime ideal, r is non-zero prime ideal. In the integral domain, $A[X]$. Therefore height of Q, height of small q is smaller equal to height Q minus one, because you can allow the extended H in for Q by putting r in the beginning. So combining this, with this, we can simply get, I'll height Q by this smaller than. So therefore, this is smaller equal to. So, this we get the, what we want. Which of this gives height P minus transcendent degree of $K(Q)$ or $K(P)$, because other transcendent degree zero? Okay. So this proves the theorem for when B is finitely generated A-algebra, so this completes the proof if B is an A-algebra of finite type.