Speaking (foreign language). Knowledge is supreme.

I want to generalize this theorem to arbitrary commutative ring impossible. So I want to prove, so down the base ring may not be a field. So the question is what is the dimension of polynomial ring in n variable or arbitrary commutative ring. A is commutative ring. So obviously, then now we cannot apply the normalization lemma all the techniques, because we don't have a field, we don't have a finite type K-algebra, we don't have transcendent degree and so on. So and… so I will do it for one variable first, polynomial ring in one variable over commutative ring. So we have to analyze, we need to analyze the situation like this. A. contained in the polynomial ring and this is my ring B, and what happened to the prime ideals here, whatever, we did we want to imitate it her. So if we have a chain of prime ideals here. How do we get chain of prime ideals in a polynomial ring and if you have a chain of prime ideals in a polynomial ring, and what is the difference between their lengths and so on and so on. So this is what analysis, we want to do. So first of all note that, if I have a prime ideal P in A and extend this prime ideal to B, PB, PB is the ideal generated by P in B. So the coefficients allowed are more than A. So this is precisely all the polynomials whose coefficients are in B. So that notation, I want to use this PX. So this is the notation for, this is the… precisely the set of all polynomials $a_i X^i$ i is from 0 to whatever, d, such that these coefficients $a_i$ s are in B for all i. This is clearly an ideal, because if you have such two polynomials, if I add up the coefficients are correspondingly added. So they are also in B. If I multiply by arbitrary polynomials, then the new coefficients will be in the ideal generated by the coefficients of this, so they are in B. So this is clearly an ideal. I don't have to check that if I would have said this, because this is ideal generated by P in B.

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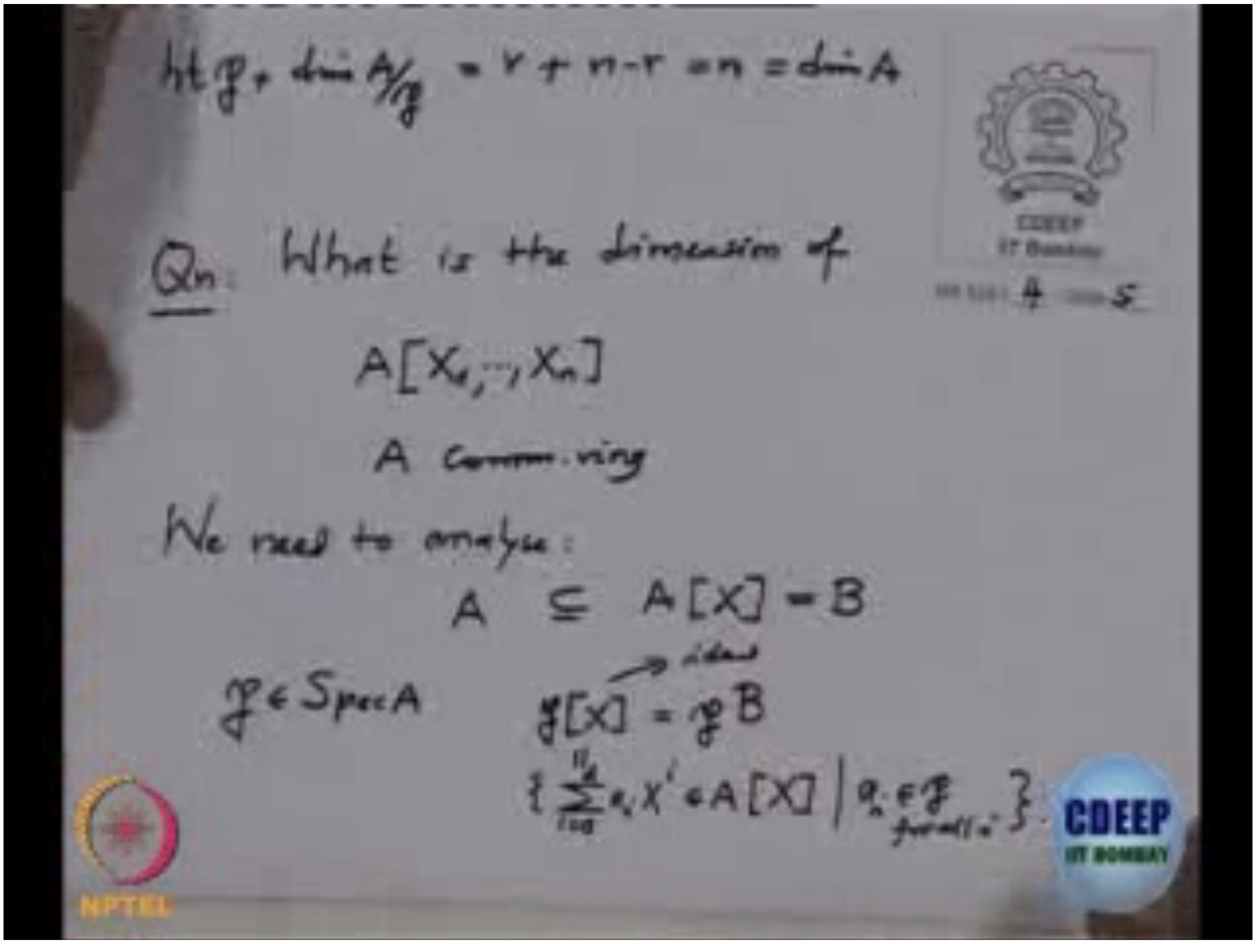


So ideal is clear and the next thing I want to check it is a prime ideal. So $P[X]$ belong to the spectrum of spec of $A[X]$. In general extension of a prime ideal may not be prime. So… and as I keep saying, the only way to check ideal is prime in the ring, you go more than that and check it, it is integral domain. So we have this $A[X]$, A may not be an integral domain, but here when you go mod it, $\frac{A[X]}{P[X]}$, we had A here and this is isomorphic to in fact $\frac{A}{P}[X]$. Now the new base ring is $\frac{A}{P}$ and this is… this ring, this residue class ring is nothing but the polynomial ring over a residue class ring of B, this is an integral domain. So therefore this is an integral domain. So therefore $P[X]$ is a prime ideal and note also that, when… so when we had P, prime ideal here, we were into prime ideal here, when I come back again, that means if I contract this $P[X]$ to A, I get back P. This is also very trivial. So I would just say check this. The checking I will live it, routine checking is do it, but when checking involves little bit new ideas, I will indicate that this needs more attention. So there is another prime ideal which we can get from the given P. So P is given. Now if I take ideal generated by P and X in $A[X]$, this also is a prime ideal that is because when you go mod, $A[X]$ mod, ideal generated by P and X, so this ring is nothing but isomorphic to $\frac{A}{P}$, that X will disappear, because X you are putting it 0. So you get an integral domain. So therefore this ideal is also prime ideal and because it contains P, therefore it will contain $P[X]$, this is contained yet. This is a prime ideal, this is a prime ideal, and 1 is contained in the other, and also they are not equal, because X is a variable, which is here and which is not here, because here only those polynomials occur, whose coefficients are in B. So also note that this… this contraction of this ideal generated by P and X is P. So P is intersection of ideal generated by P and X intersected with A. This is also clear. So therefore I have a chain of prime ideals in $A[X]$, which is one is contained in the other, but both of them are lying over the same… same prime ideal P.

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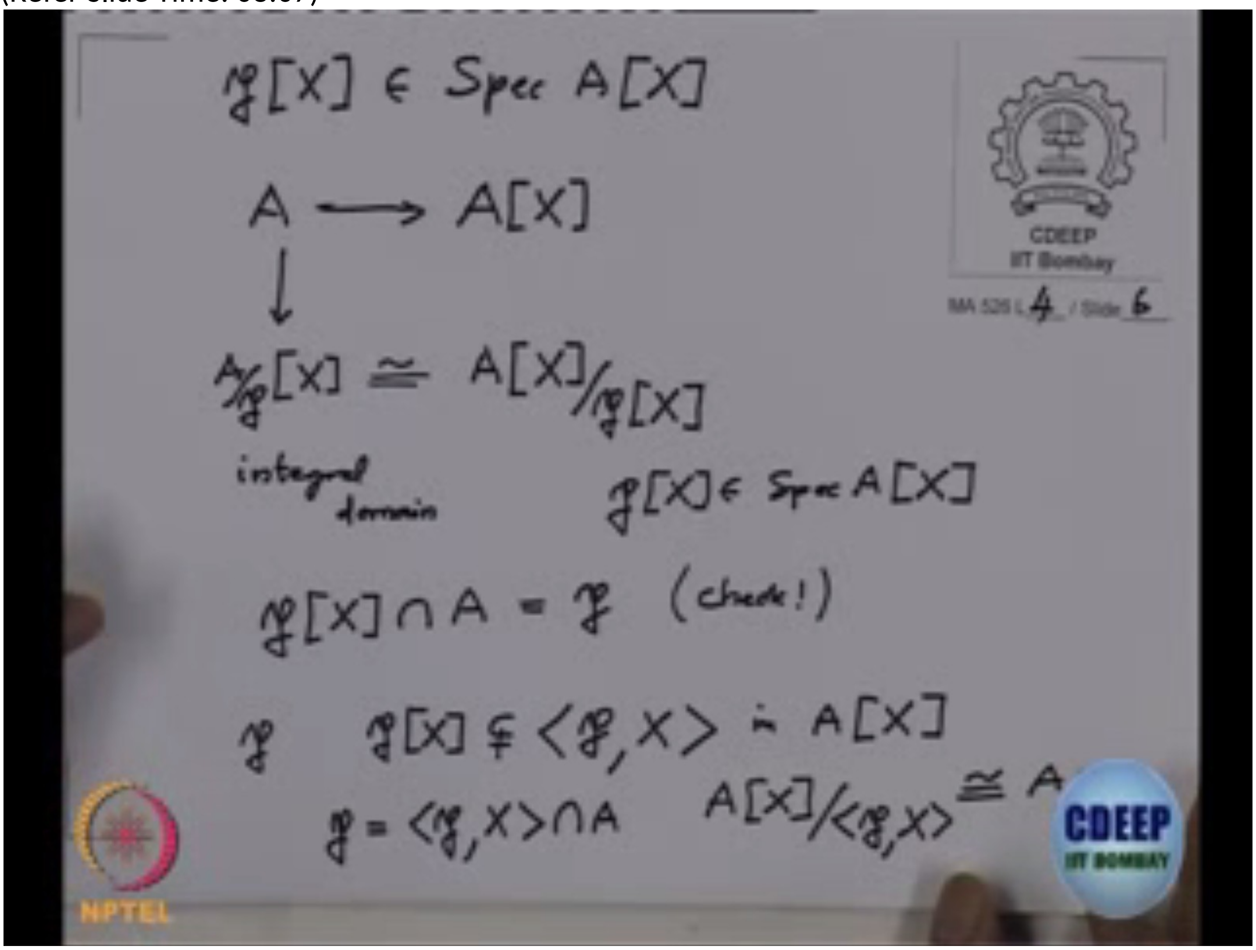


So what we, the theorem lying over theorem, which was in the integral extension case, this is not true here, no wonder, because we have transcendental element there X. So I want to write and I want to analyze, now what happens with the heights. So for example, we have just noted that this is a chain of prime ideals and it is proper. How many are there in between, or is it there any way at all or something, so I want to analyze now what happens to the relation between the heights.

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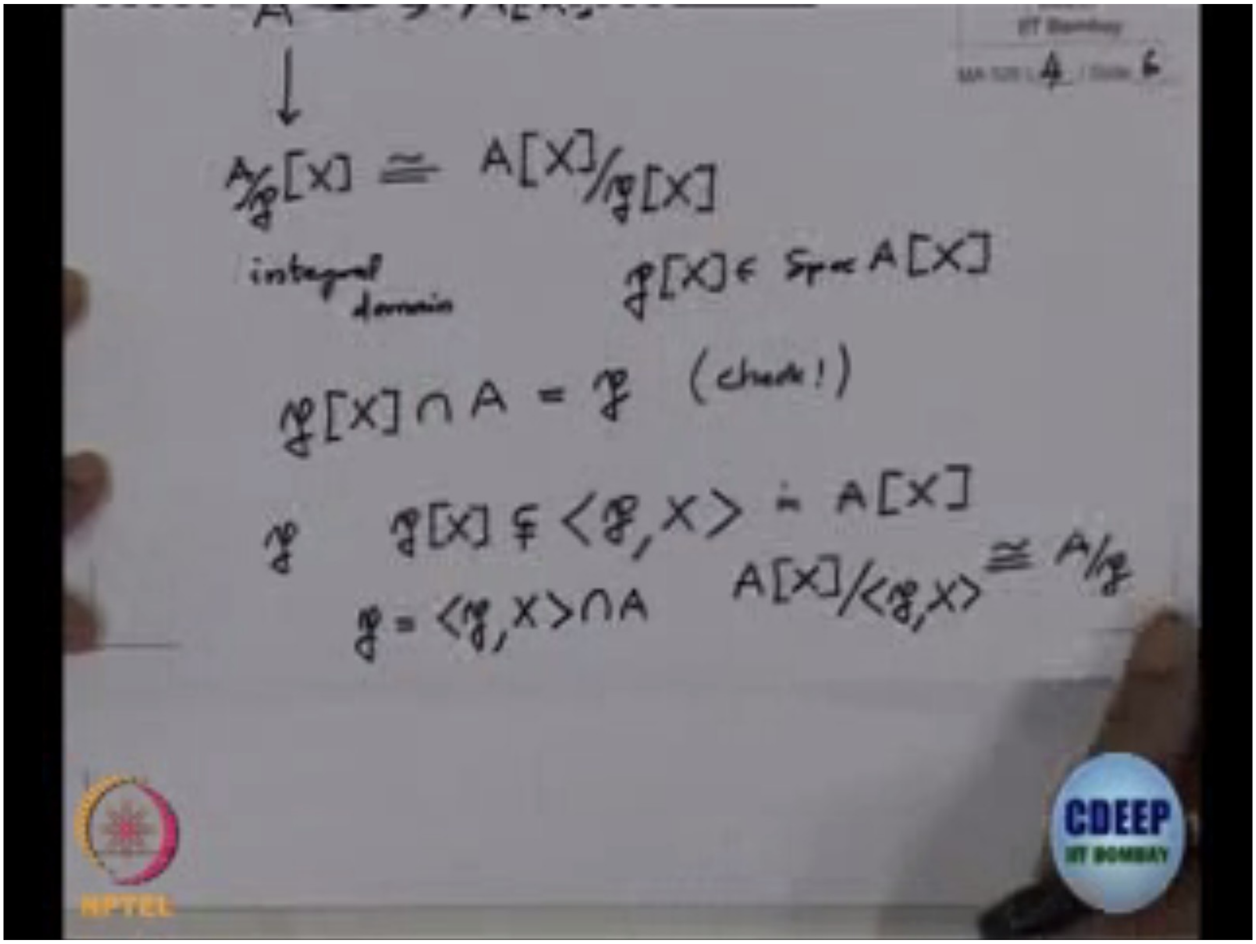


So that is what I am doing now, so… so let us write it as a preposition. Okay right now our base ring is arbitrary commutative ring. So the notation I will use now, given a prime ideal P, we have a residue ring $\frac{A}{P}$ , residue class ring $\frac{A}{P}$ , and also we have a localization A localized at P. And this is an integral domain, so A to this… this is a natural map, this is an integral domain. So it makes sense to talk about that quotient field, quotient field of $\frac{A}{P}$ . So this is a natural inclusion. And on the other hand, we have this local ring, and whenever you have a local ring, you have a residue field. So there is here $A_P$ , the residue field is $\frac{A_P}{P\,A_P}$ . This is a local ring with maximum ideal, precisely generated by P. So this is a residue field. So these two fields are same isomorphic equal. Because see these field you have got by inverting all the elements here and this field we got by going modulo this. So exactly the elements, which are outside P, we have noted. So these are… these two fields are same. So the residue field at P, so these I keep call residue field at P. So residue field at P is the quotient field of $\frac{A}{P}$ . So residue field at P is same as quotient field of $\frac{A}{P}$ . Okay, now I wanted to write down preposition, which will be relate prime ideals of $A[X]$ and prime ideals of A, and their heights, this is what I want to know. So start with the prime ideal, let Q be a prime ideal in $A[X]$ , and let us call P to be contraction, P the contraction of Q to A. This is in the spec of A. Then I want to write, the first statement I want to write, 1. Height of $\frac{Q}{P[X]}$ , note that because Q context to P, and when I extend P, that will be contained in Q, so $P[X]$ , this is an extension of P, which is contained in Q, because Q we have contacted to P, and this we have extended, so obviously the extended ideal will be contained in the original ideal, so this is it. So this makes sense. And when I write height, that is height in the quotient ring $\frac{A[X]}{P[X]}$ , which is the polynomial ring in X over $\frac{A}{P}$ , so this height + transcendent degree of the residue… this residue field, I will denote by $K(P)$ . So this is the transcendent degree of… and similarly… similar notation of $K(Q)$ . So $K(Q)$ , the field extension of $K(P)$ , and that transcendent degree, these two numbers add up to one, so that is the first assertion.

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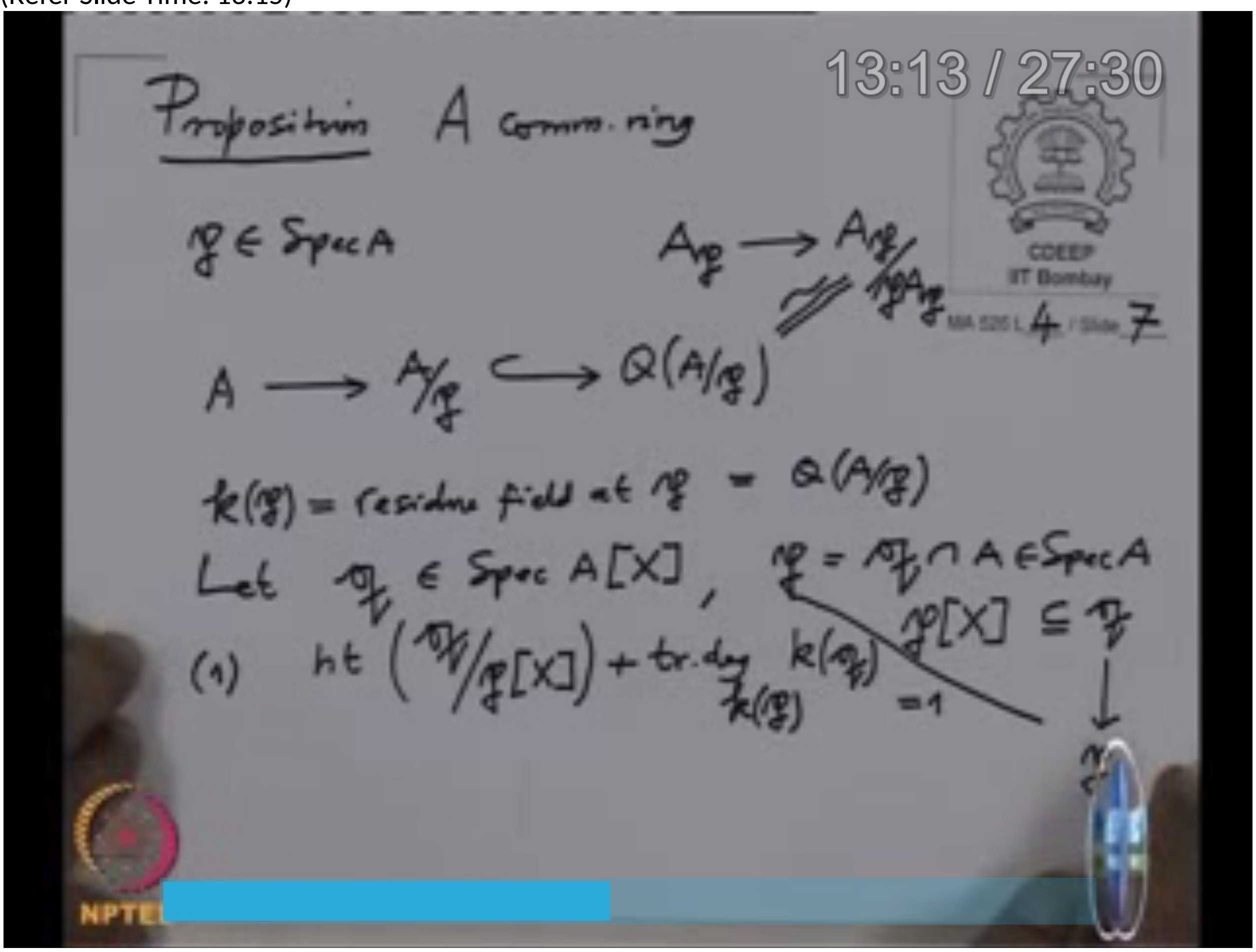


So the next one is second, second assertion, height of Q=height of P+height of $\frac{Q}{P[X]}$ , which is also equal to height of P+1-transcendent degree of the field extension $K(Q)$ over $K(P)$ . So first of all note, if I have this one, if I would have proved one that is this, this +

transcendent degree of $K(Q)$ over $K(P)$ is 1 then this height will be 1-this transcendent degree and then this equality will follow from 1, because height P is same and I have just replaced this height $\frac{Q}{P[X]}$ -transcendent degree, this… this equality just follows from 1. But this equality we… we need to prove and third one is, height of Q equal to either height P or height P+. And this case will occur if this Q is the extension of P and this will occur, if Q is not the extension of. So if I prove these three equalities, it will give you fair good knowledge of what happen to the prime ideals in the polynomial ring, when we contact to the below ring. So let us prove one first, and… so proof, well. So note that all we are concerned with… with P and not anybody beyond P. So height will not change when I localize all the chains, if all the chains of prime ideals, if I consider, they are contained in the given P, and whether I want to compute the link, whether I localize or before localization it is the same. So we may assume, the short form I will use it for we may assume, A is local with maximal ideal P. So this you achieve by replacing A by $S^{-1}A$ , where S is $A \setminus P$ . The quotient from A to A is you know, is A and then, okay. Alright, in this case now, because P is a maximal ideal, so these are ideal number. In this case $\frac{A}{P}$ the residue field, $\frac{A}{P}$ is a field, because we are assuming A… P is a maximal ideal. And the polynomial ring mod, the extended ideal than this, as was… as we earlier also this is isomorphic to $\frac{A}{P}$ and then are joining the variable. This is a field, so this is a polynomial ring in one variable over a field, which we know is a PID. So this is a PID.

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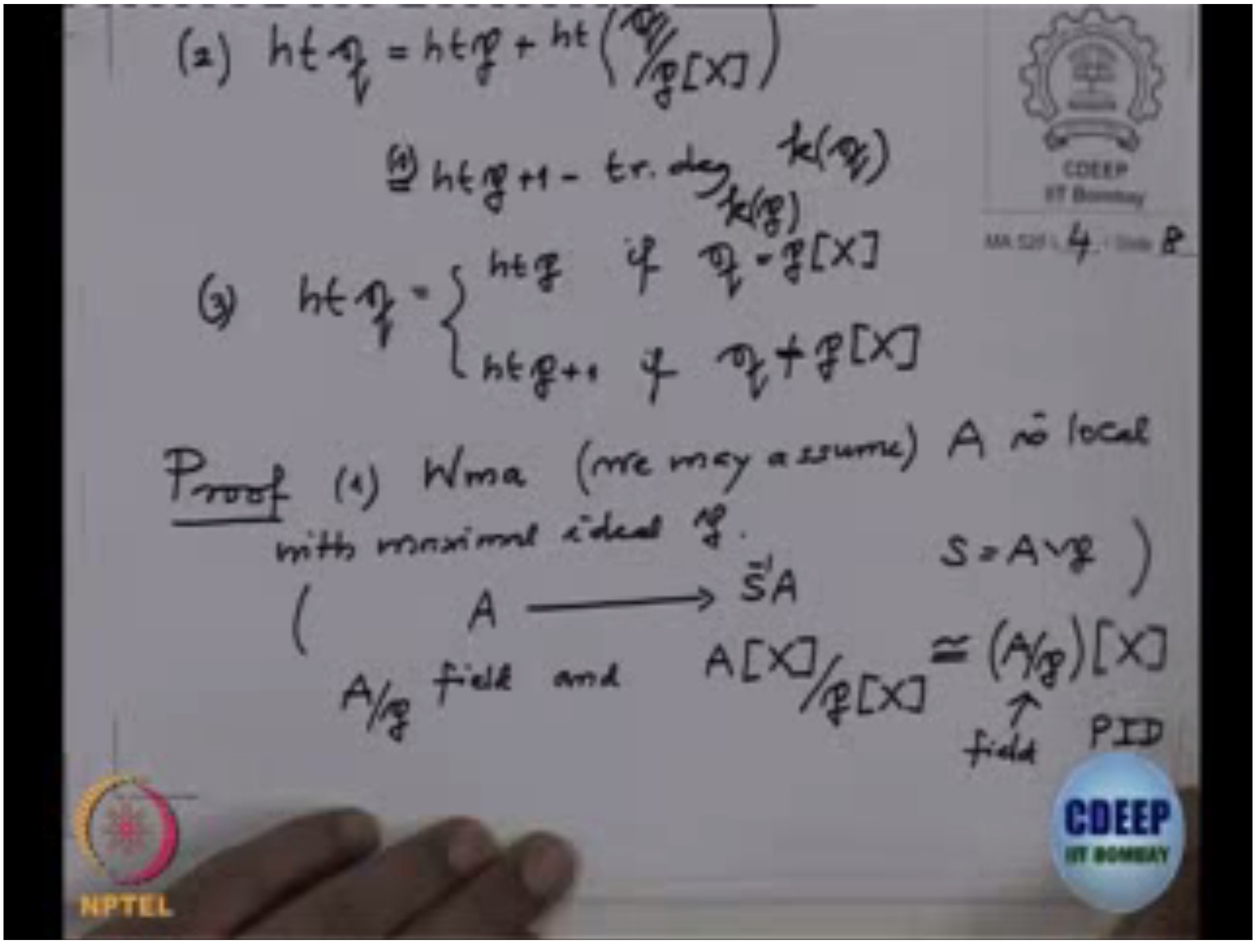


And we are considering the ideal Q mod ideal generated by extended ideal by P, this is what we are… we are concerned with. We want to compute the height of this, that is the first assertion. So therefore we are looking at this ideal, which is an ideal in this ring, which is a PID. Therefore what can be… this is a prime ideal in this PID. So there are only two possibilities, either that prime ideal is 0 or it is non-zero, and in that case it will be maximal. So there are two possibilities only.

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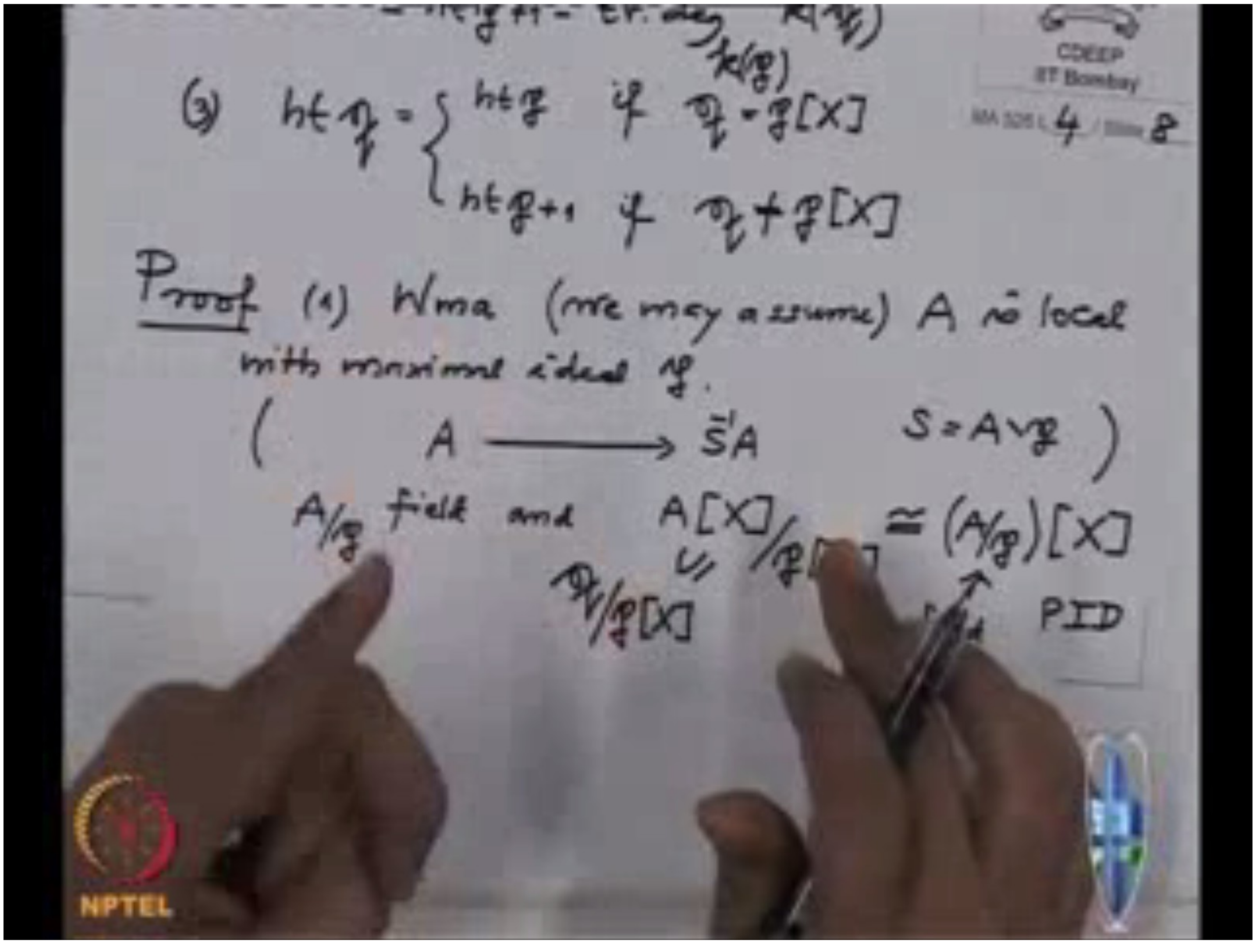


So height of $\frac{Q}{P[X]}$ can either be 0, when it is a 0 prime ideal or it is 1, there is no other possibility, because any… if you take any chain of prime ideals in a PID, it can have length at most 1, 0, and then it stops, the next stage. So it can either be 0 or 1. So now we will analyze when it is 0 and when it is 1. So for example, if it is 0, height is 0, means what? 0 ideal, 0 is a prime ideal. So this case, this will imply, Q is $P[X]$ , 0 ideal means they are equal. So Q is $P[X]$ and therefore height of… height… and what is the residue field $K(Q)$ , see we… the other number we want to relate, height is a transcendent degree of the residue field at Q and residue field at P. So let us write down what is the residue field and residue field at P. So what is KQ in this case? $K(Q)$ is nothing, but how do you… what I the recipe for finding the KP or $K(Q)$ … $K(P)$ . So you first go mod P and take the quotient filed. So first go mod Q and take the quotient field that means.. this I have to do this, $\frac{A[X]}{Q}$ , and then I have to take the residue… I have to take the quotient field, the quotient field of this, Q of this. But what is this residue… residue ring. This is nothing, but K… so this is nothing, but… this ring is nothing, but the polynomial ring over a field and when I take the quotient field, I get the rational function field over $K(Q)$ … $K(P)$ , this is this ring, is that clear, and this…. and we want to know, what is in this case, what is the transcendent degree of $K(Q)$ over $K(P)$ , but this is over this residue field $K(P)$ , this is a transcendent degree 1, because it is generated by the variable 1, so this is 1, so height is 0, then the transcendent degree is 1. So that means they add up to 1, so that was the assertion. Now in this case, we should prove that if height of this $\frac{Q}{P[X]}$ is 1, then the transcendent degree should be 0, that is what we have to prove, right. So let us prove that.

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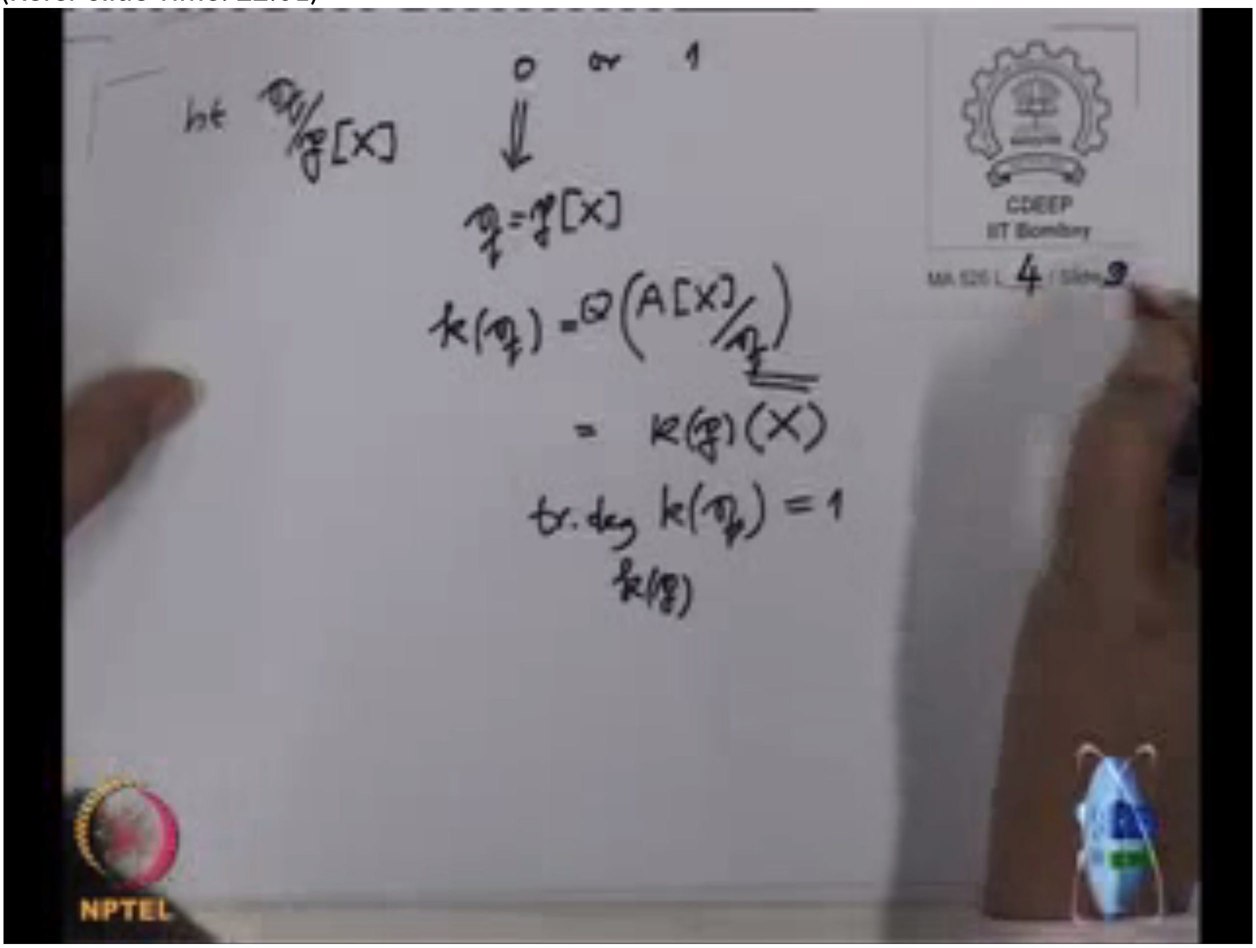


So now we are in the situation, where height of $\frac{Q}{P[X]}$ is 1. Note that we are working in the ring, working in a PID, $\frac{A}{P[X]}$, this is a field by our assumption that P is maximum. So we are working in this, so this is ideal there, prime ideal there and we are assuming its ID is 1, so it's a maximal ideal. So this is a maximal ideal in this ring, in this PID. And you know very well that the ideals are principle, so it is generated by 1 polynomial, the minimal polynomial, right. So there exist a monic polynomial F in $A[X]$, F is in Q, such that $\frac{Q}{P[X]}$ is generated by $\overline{F}$, where $\overline{F}$ means, you read this polynomial F mod P. We know it is generated by one element. So take that element and we can assume without loss monic and you lift I, lift in such a way that keep the polynomial monic, that means when you do lifting, one bar you should lift to 1, because you have lot of choice. So keep the monic as intact. So there exist such a monic polynomial F, so that mod $P[X]$, overline{F} generates this ideal, but now when I go mod Q, so $A[X]$ ... when I go mod this ideal, then I get this ring, because this I have to go mod... in this ring, I have to go mod this ideal, and then you get a field, because this is a maximal ideal, so you get a field and in that field... this is a field, and in this field, that this $\overline{F}$, F... when I take F and plug it in X bar, small x... this small x, small x is a image of capital X in this residue class ring. When I... in this F, when I plug capital X = small X, this is 0 in this ring, because it is in Q, therefore this small X actually satisfies integral equation. So this means, so that is, X is integral over A, and this was the residue field, this is precisely $K(Q)$. So we have checked that this residue field is algebraic over $K(P)$. Because when you read this coefficients, mod this Patient, reading mod P, so you get coefficients in $K(P)$. So therefore this is algebraic over $K(P)$. So once... once it is algebraic or $K(P)$, that means the transcendent degree is 0, because by definition is a cardinal degree of the transcendent basis and for algebraic extension empty shade is a transcendent basis. So therefore transcendent degree of $K(Q)$ over $K(P)$, this is 0, and that is what we wanted to prove, this was in the case, when the height is 1. So our first assertion was, when you put... when you add up the true numbers that is height of $\frac{Q}{P[X]}$ and transcendent degree of $K(Q)$ over $K(P)$, the sum should be 1. So that proves it is 1.

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