Speaking (foreign language). Knowledge is supreme.

Today I will have more consequences on the refined version of NNL. The first theorem we will prove is, if K is a field, then the polynomial ring in the n variables over K as dimension n, that means you need to prove that there is a chain of length n in the spectrum of this polynomial algebra and every chain of prime ideals in this should have length at most n. So first I am trying to produce a chain of length n in the spectrum of the polynomial algebra. So for that observe that 0 is a prime ideal, ideal generated by X_1 is a prime ideal, ideal generated by X_1, X_2 is a prime ideal, and so on. Ideal generated by $X_1, \ldots, X_n$ these are all prime ideals. The easiest way to check that these are prime ideals is just go mark that and check the ring is an integral domain. And in each one of these case actually the residue class ring is actually the polynomial ring. Therefore it's an integral domain. So... and the chain is at no place there is equality. So it’s a chain of length N. So therefore its immediate by definition of the dimension that dimension of $K[X_1, \ldots, X_n]$ is n less bigger equal to n. Because this is a chain of length n. So the dimension is supreme, therefore this dimension is at least n. Conversely I want to prove that every chain has length at most n. So start with arbitrary chain P_0 contained in P_1 and so on, contained in P_r , this is in spec of $K[X_1, \ldots, X_n]$, this is... this has length R and we want to prove $r \leqslant n$. so to prove $r \leqslant n$. Now I am going to apply Nagata's version of NNL to this chain. And the ring A is the polynomial ring. This... and this is a chain in that spectrum. So Nagata generally say that I will find an elements Y_1 to Y_m in these, which are algebraically independent and when I contract this chain of prime ideals to that, generated by the variables. So there exist... so B is $K[Y_1, \ldots, Y_n]$, so that this is integral and Y_1 to Y_n are algebraically independent, such that P_0 intersected with B, this is generated by Y_1 to $Y_{h(0)}$, I will use the same notation $h(0)$ then contained in $P_1 \cap B$, this is also generated by the variables Y_1 to $Y_{h(1)}$ and so on. And because this is an integral extension, if this chain is proper than the contracted chain is also proper, because the integralness.

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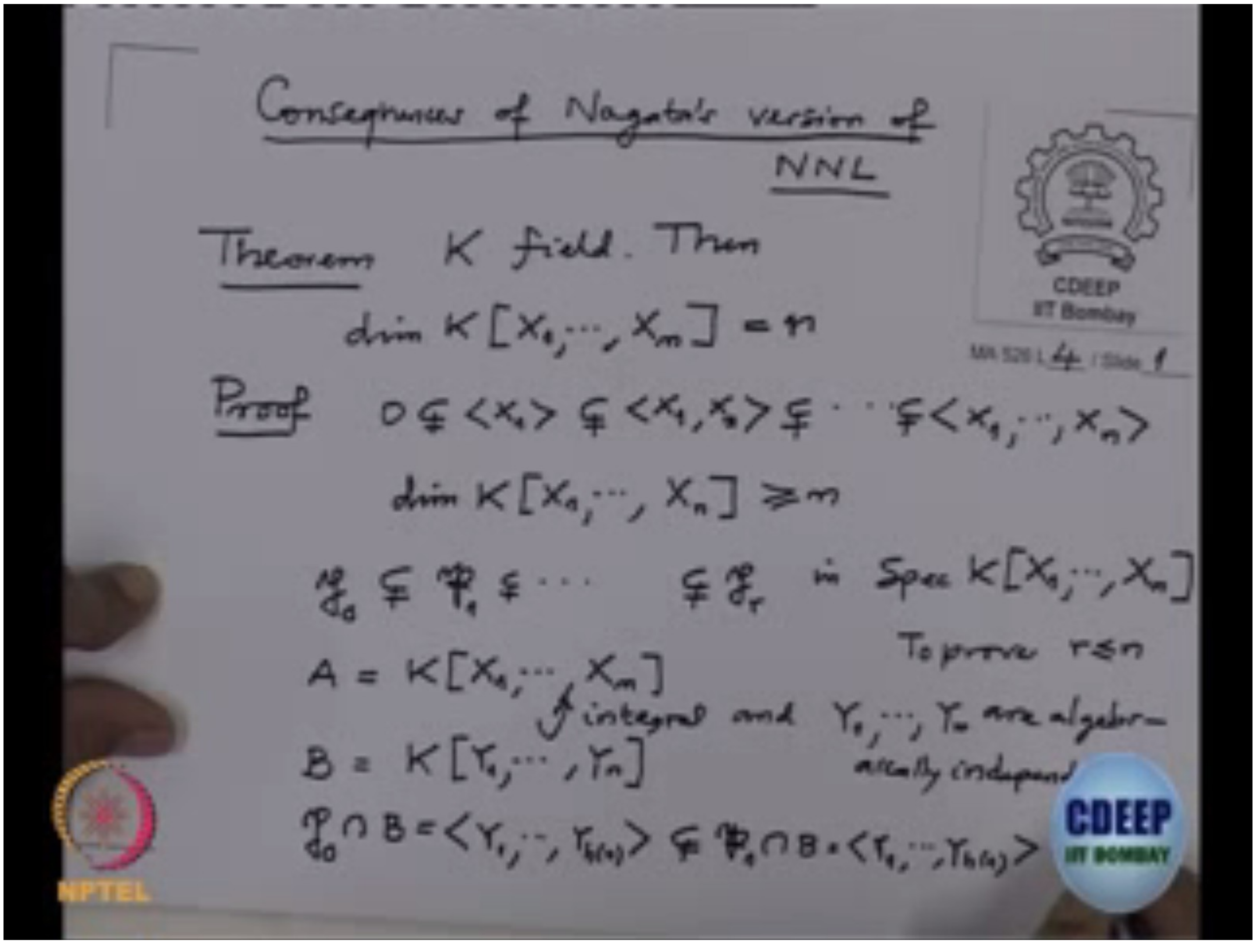

So that is, so the last one is $P_r \cap B$, this is Y_1 to $Y_{h(r)}$. So here just observe that I have used the same n, that I explained last time, because the, because of this integralness, the transcendent degrees are same, therefore it is the same n.

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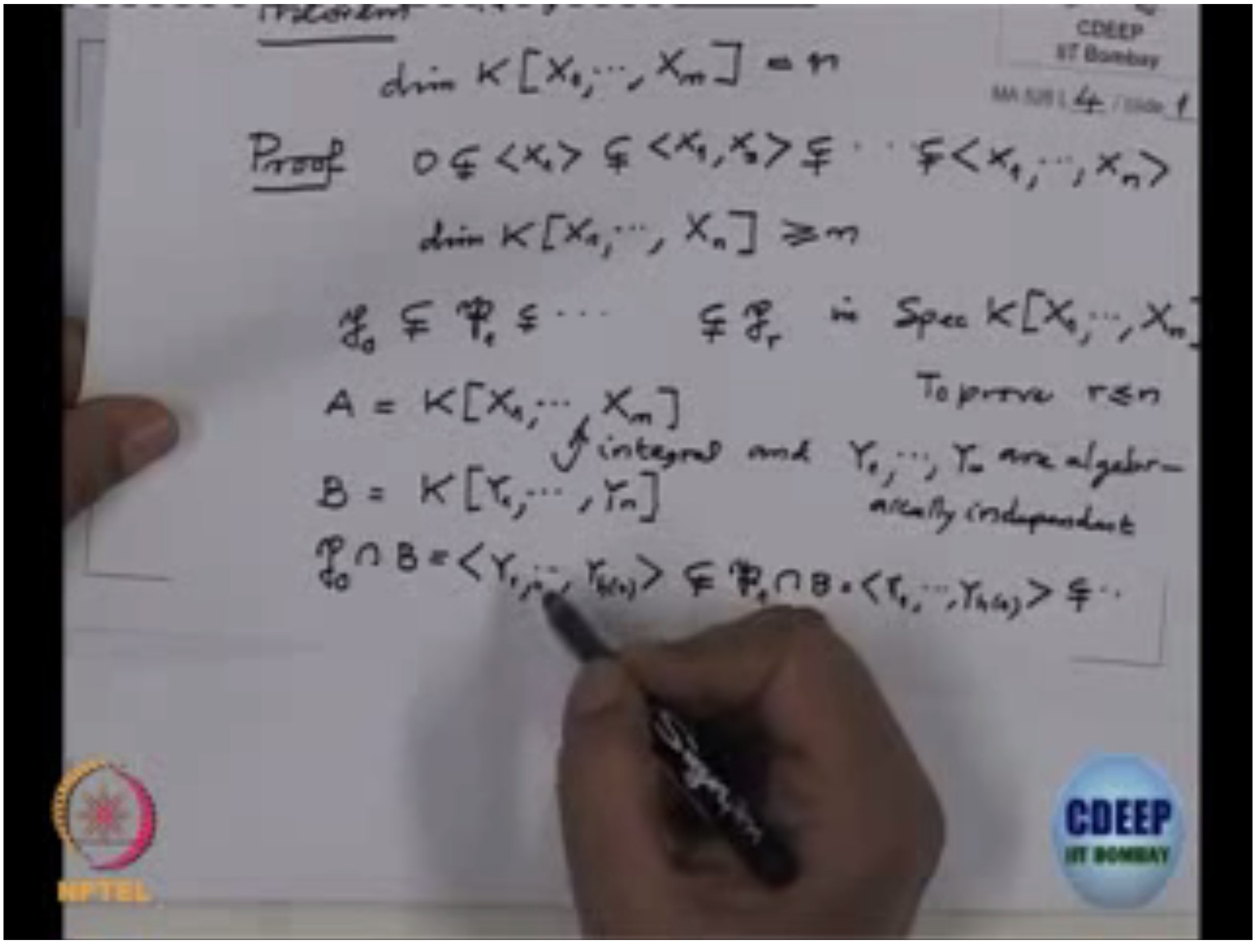

So therefore, because the chain is proper, the h are increasing $h(i+1) > h(i)$ for every i = 0 to whatever r. So we get a chain like this 0 less equal to to $h(0)$, strictly less than $h(1)$, strictly less than… strictly less than $h(r)$, which is less equal to n, because the number of variables is at most n and h are increasing so the one… therefore r is less, $r \leqslant n$ and therefore dimension of A is less equal to n. So that finishes the proof. Right so thing to note that is from the… the classical version of normalization Lemma, you could not have reduced this, because we needed refined, where for the chains, that was the reason I proved that first, okay. Now the next observation is, this is also very important, this… this in fact shows that the dimension of a finite type K algebra is finite. From the definition of a dimension, it is not clear, a dimension is finite or not. But, so… A is K algebra of finite type. Assume that A is an integral domain, then the dimension of A is transcendent degree of A over K, when I write this, this is by definition transcendent degree of the quotient field of A over K, quotient field of A is an extension field… field extension of A… K and last time I recalled what is the definition of a transcendent degree of a field extension. It is the number of algebraically independent elements, so that when you joint them, you get an algebraic extension of, quotient release and algebraic extension of this purely transcendent extension. So proof. So this… this in particular says that the dimension of finite type K algebra is finite. So because this transcendent degree is finite. Because the quotient field is finitely generated field extension. So therefore the transcendent degree can at most be the number of K algebra generators for… for the algebra A, okay. So by normalization

Lemma, we have, we can find algebraically independent elements $K[X_1,...,X_n]$, such that this extension is integral and these are algebraically independent elements. So this is a polynomial algebra. Let's call this as B. We have noted last time that whenever we have integral extension, then the dimensions are same, so therefore dimension of A equal to dimension of B, but B is a polynomial algebra, because X_1 to X_n are algebraically independent. Therefore this is n by earlier theorem. And this n is nothing, but the transcendent degree, because in this case, the quotient field of A, when I go to quotient field level.

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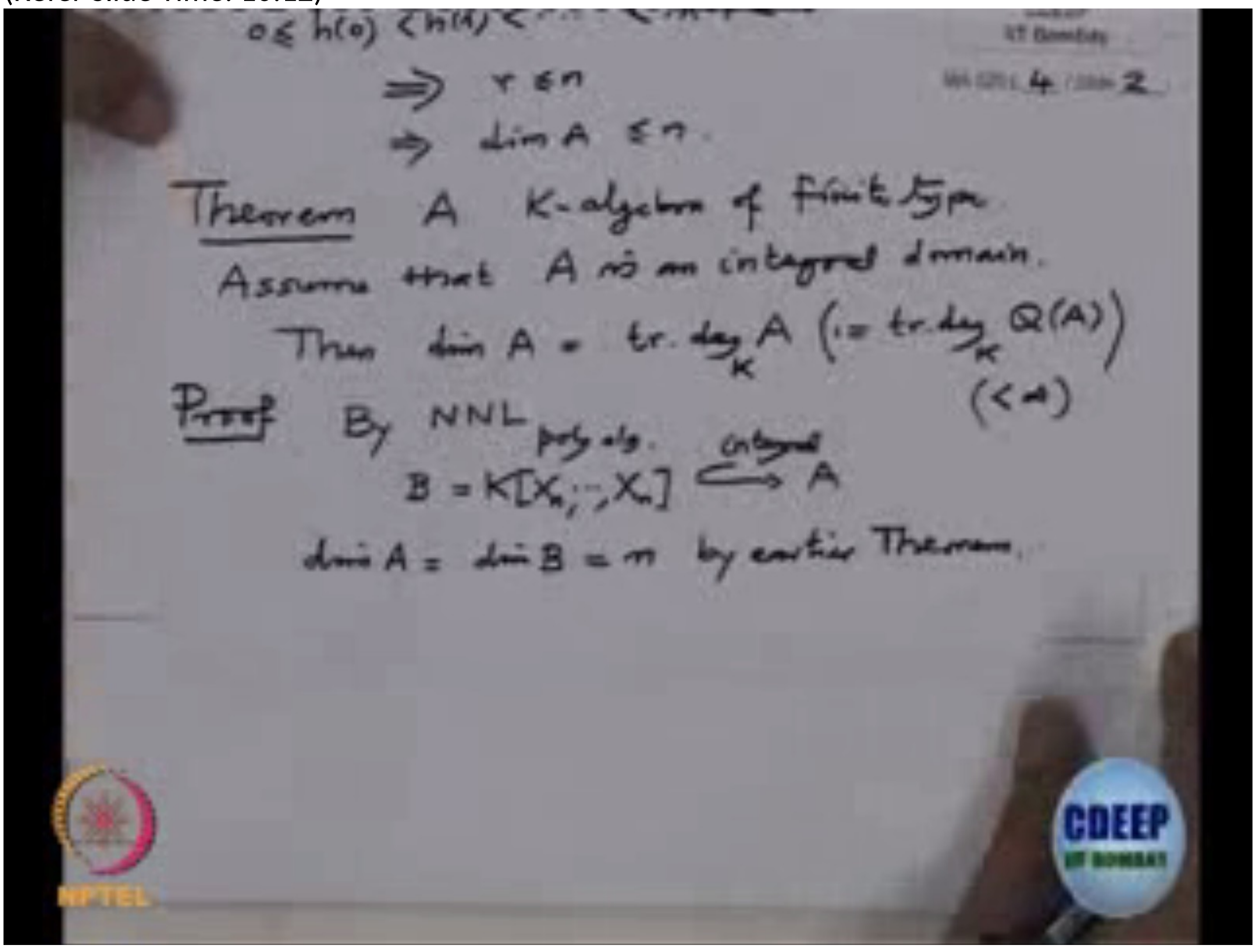

quotient field of B, this is rational function field in n variables and quotient field of A is here and this... because this A is integral over B, this is algebraic extension. Therefore indeed this X_1 to X_n is a transcendent basis. So X_1 to X_n is a transcendence basis of $Q(A)$ over A and we have by... we have stated that any two transcendent basis of the same number of elements, so therefore transcendent degree of $Q(A)$ over K is n and just now we have proved this that n is the dimension of A. This is very-very important for those who want to study algebraic geometry. When... when we have enough language, next time I want to do little bit of language from algebraic geometry, because one cannot really avoid it. So it is better that we... we do it from the basic search and then some more consequences, which are geometric in nature, that we will... we will write down, okay. Now the next one is... next one is also very important theorem for many things. So that... so again K is a field and A is finite... K algebra of finite type. This is also sometimes I call it affine algebra over K. And supposedly I have a prime ideal P, let's assume also A is an integral domain, assume that A is an integral domain. So let P be a prime ideal in A, then we have these two things, two numbers attached to this P, one is height P and the other is dimension of the residue class ring. So height P is by definition, you know, you take the chains of prime ideals, which will end at P. So P_0 contained in P_1 etc. etc. P_r , which is P... this is a length of r... length is r, ending at P and you take the supreme HR, that is called the height of P. On the other hand, we have also this dimension of the residue class string, that means you start with a chain at P and take, this is a chain of prime ideals in A... in spec A. And if you take their supreme here, then you get a dimension of the residue class. So we want to know what is the relation between the three numbers, dimension A, height P, and dimension of $\frac{A}{P}$ and obviously one, if I take the chain, which ends at P and continue with the chain, obviously I am going to get a chain of length, their sum, so therefore dimension is at least the sum.

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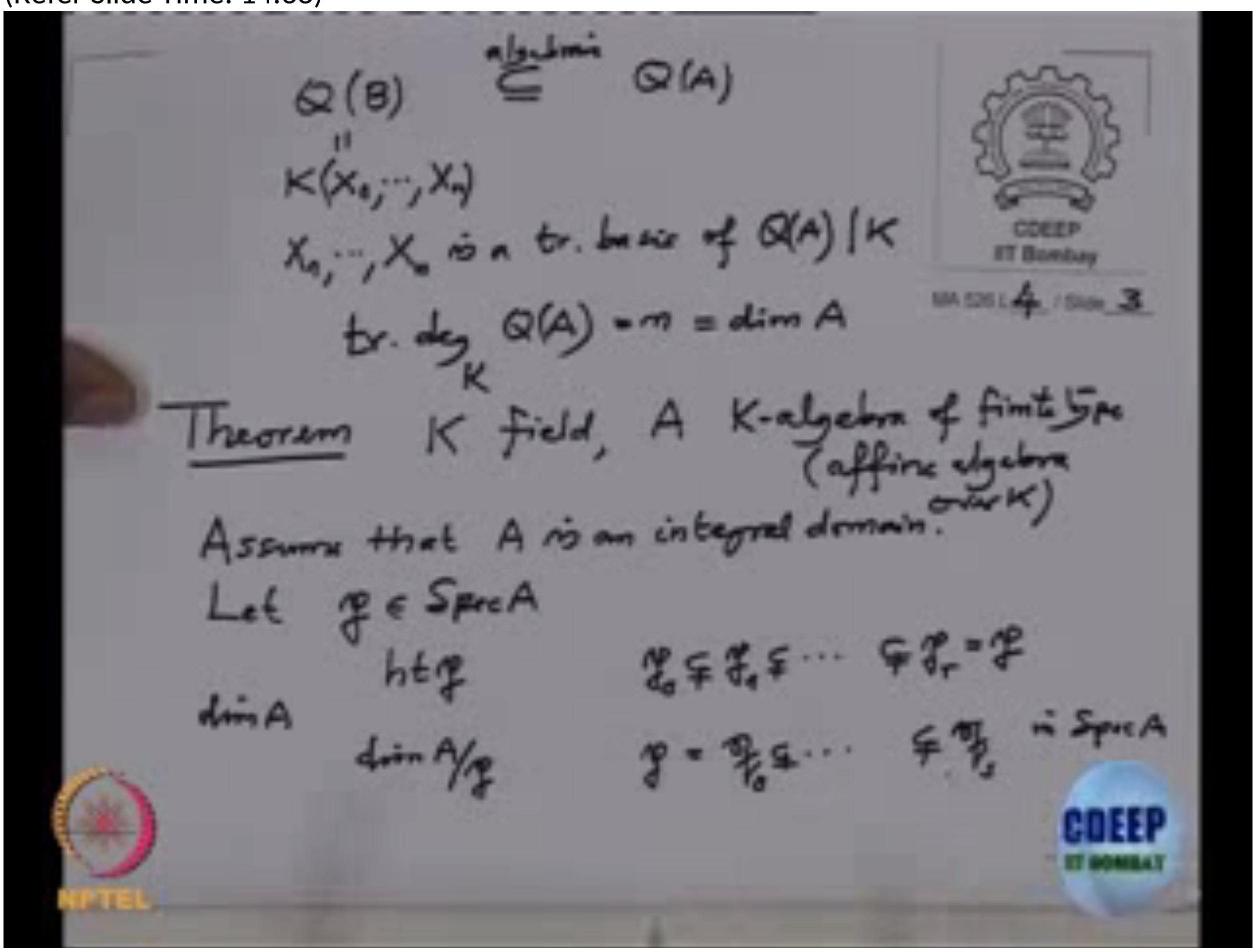

But the assertion here we want to prove is actually it is equal. So this is for, so the assertion is, so then, dimension of A = height P + dimension of the residue classing $\frac{A}{P}$, proof. As I said that it is clear that dimension of A is bigger equal to height P + dimension $\frac{A}{P}$. So let us... let me give this height P as name H. I want to put the reverse in any quality. So I am going to apply Nagata's version of NNL, so I apply Nagata's version of NNL to A... to the ring A, so you need affine algebra A, that is A, and I need a chain also, so but the chain I just take P. There is only 1 element, that is the length 0 chain. So Nagata's version will say, there exist a polynomial ring inside this, so that is, let's call it B is $K[X_1,\ldots,X_n]$ and this is integral. Such that this P, when I contact to B it is generated by variables. So P contracted to B is generated by X_1 to X_n. This is a polynomial ring, polynomial algebra, that means X_1 to X_n are algebraically independent, this is for some r, bigger equal to 0. First of all note that the dimension of A is n, because this is integral and now the next thing to note is this B is polynomial algebra, therefore note that B is an integrally closed domain. Polynomial ring is a UFD and UFDs are integrally closed, that is may be the shortest, right. So its integrally closed and integral homomorphism, then I want to say, the going down holds, going down theorem holds for this extension.

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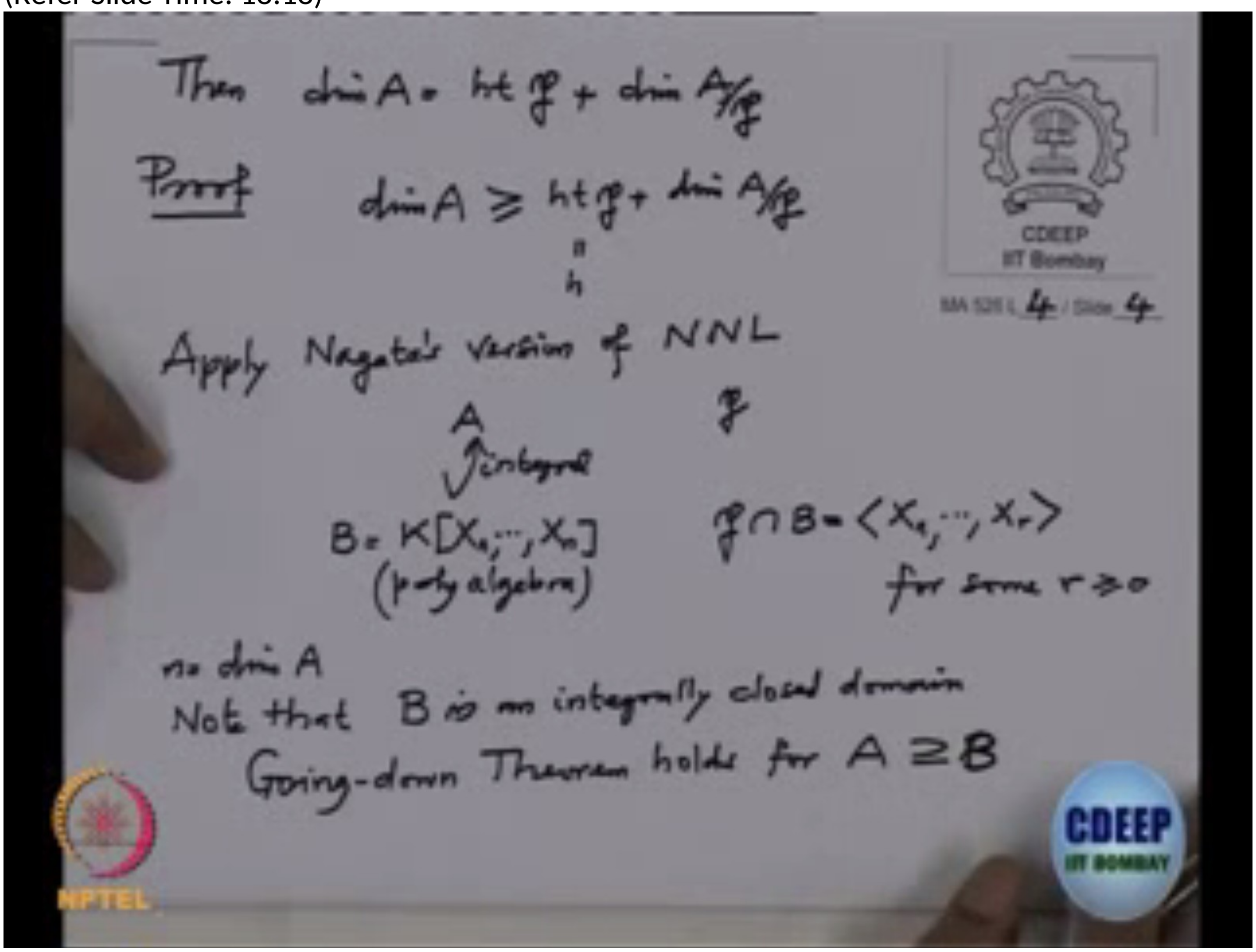

See last time I recalled what is going down, right. So the going down one of the consequence there is, if I take the prime ideal P in A and contact to B, then the height will not chain. So in particular if P is a prime ideal in A and if I take Q to be = the contraction of P to B then height of P = height of Q.

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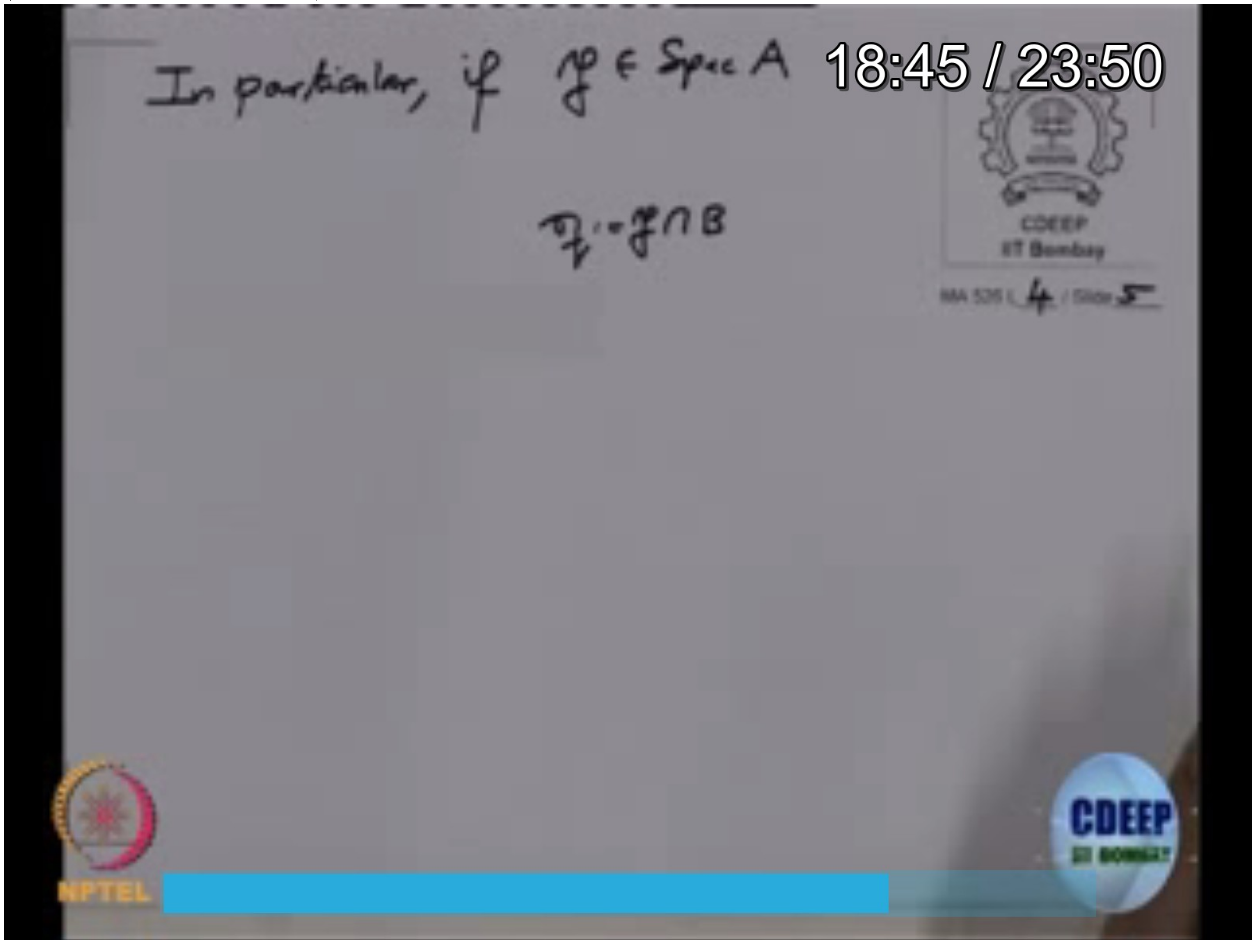

This is very simple to check from going down, but height of Q is height of Q is generated by X_1 to X_r, because when we have chosen this extension, so that contraction of P generated by X_1 to X_r. So this is height of ideal generated by X_1 to X_r, which is Right, that is because you definitely have a chain, so this one definitely have a chain, which starts at 0, ideal generated by X_1, ideal generated X_1, X_2, this is a proper chain, all the way to ideal generated by X_1 to X_r. So therefore height of this Q is at least r, but on the other hand, I want to check that, the height is exactly r. So what we need to check now is height of Q cannot be bigger than r, if it is bigger than r, then we should get a contradiction, okay. So if height is bigger than r, then look at dimension of B, which is bigger than equal to height of Q + dimension of the residue classing $\frac{B}{Q}$, but we are assuming height of Q is the least r, so this is strictly bigger than r and this residue class ring is a polynomial ring in n variables, more the ideal generated by polynomials up to X_1 to X_r. So it will disappear. The variables X_1 to X_r will disappear and the remaining variables will be $X_{r+1} X_{r+2}$ up to X_n, and those are n-r in number. So this dimension is n-r. We approved that the dimension of the polynomial ring or a field is the number of variables, so this is exactly n-r, which is n. So this shows that the dimension of B is big... strictly bigger than n, but that is not possible, because dimension B is n, so not possible. So and what is the dimension of the residue class ring, dimension of $\frac{A}{P}$, this is dimension of $\frac{B}{P}\cap B$, because when I go mod, I get an integral extent, this is because $\frac{A}{P}$ and this A... $\frac{B}{\langle X_1, \ldots, X_r \rangle}$, this is an integral extension, this is integral, because original extension A to B was integral. So I just gone, passed down to the residue class rings. So this is integral, therefore dimension of this = to dimension of this, dimension of this is exactly n-r because this B_1 to... the residue class ring is nothing but $K[X_{r+1}, \ldots, X_n]$.

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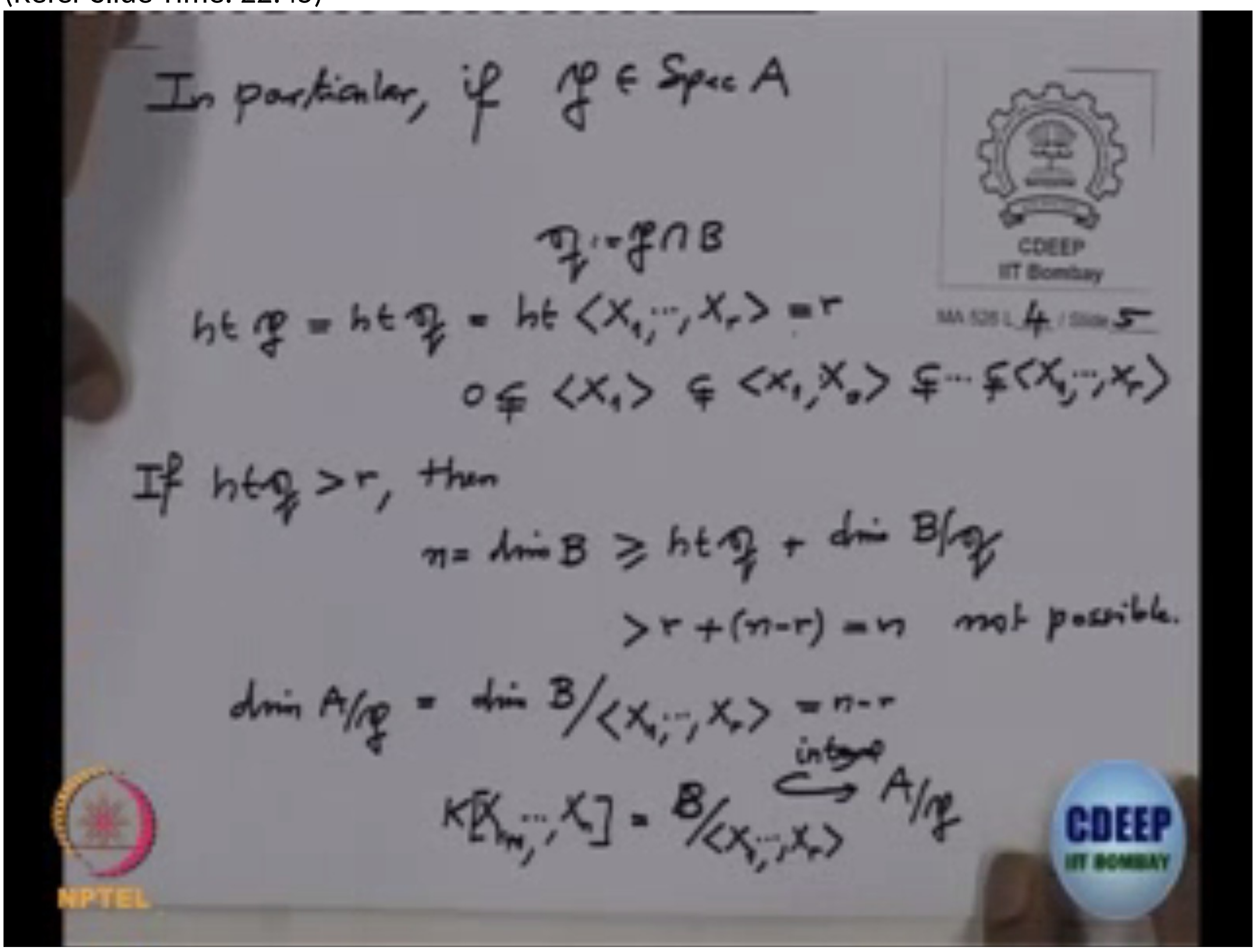

So therefore when you add up, height of P + dimension of $\frac{A}{P}$, height of P is r, oh I mixed it up, I called it H sometime, so let's correct here, this is not H, this is r letter, okay. So therefore this is r + this is n-r, which is n, and which was nothing, but the dimension of A, because again integralness, so that proves the theorem, right, okay.

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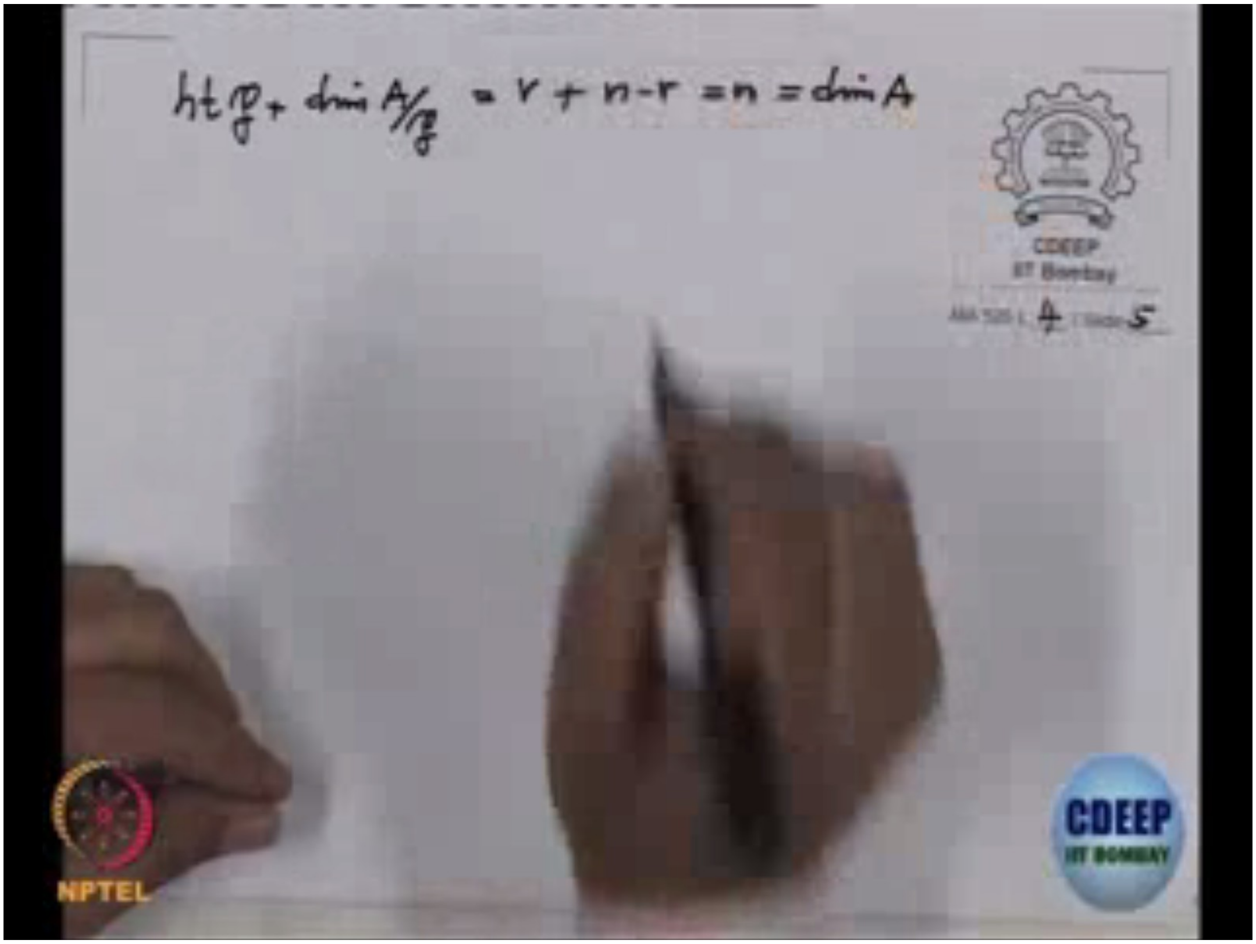

$\mathrm{ht}\,\mathfrak{p} + \dim A/\mathfrak{p} = r + n - r = n = \dim A$
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