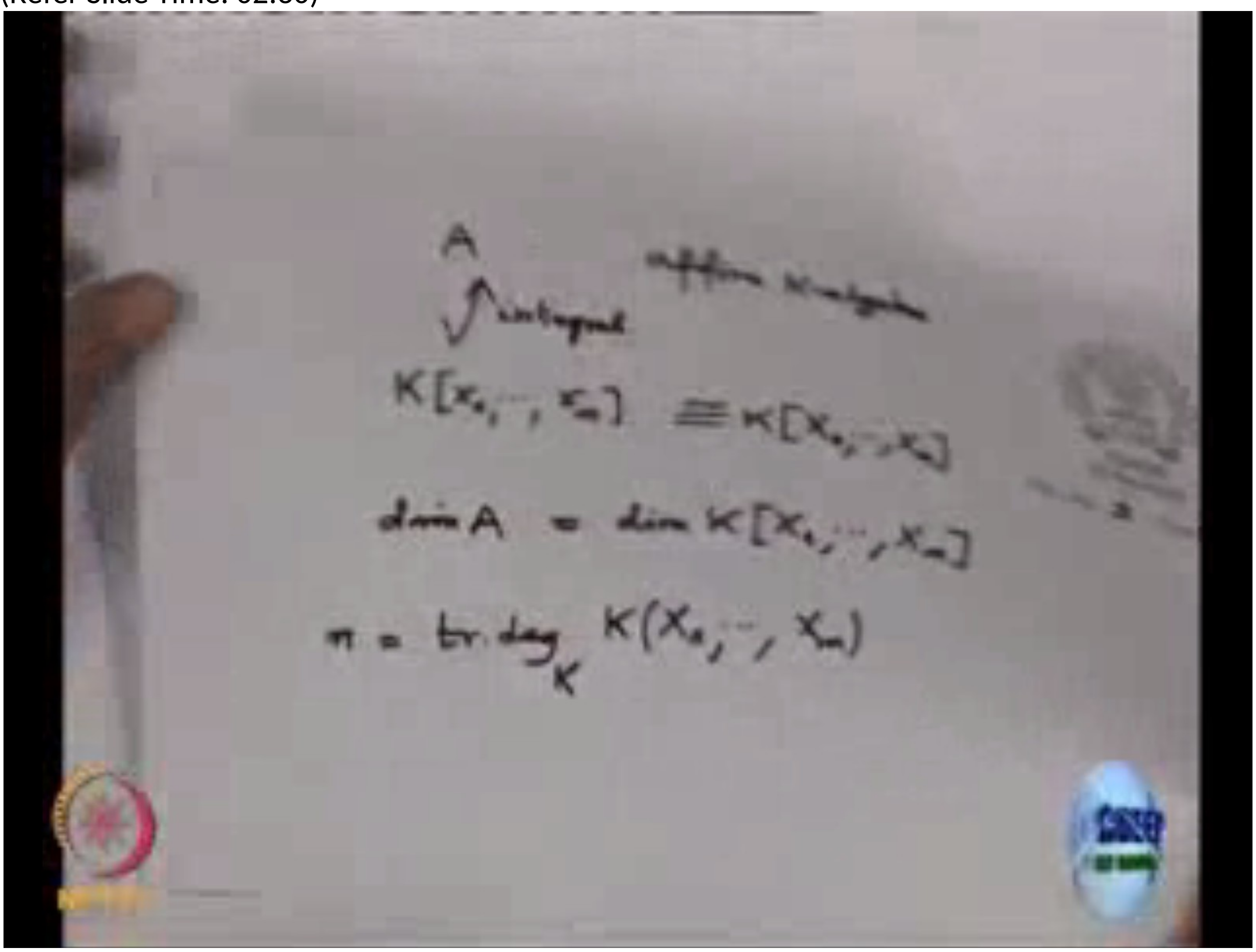
Speaking (foreign language). Knowledge is supreme.

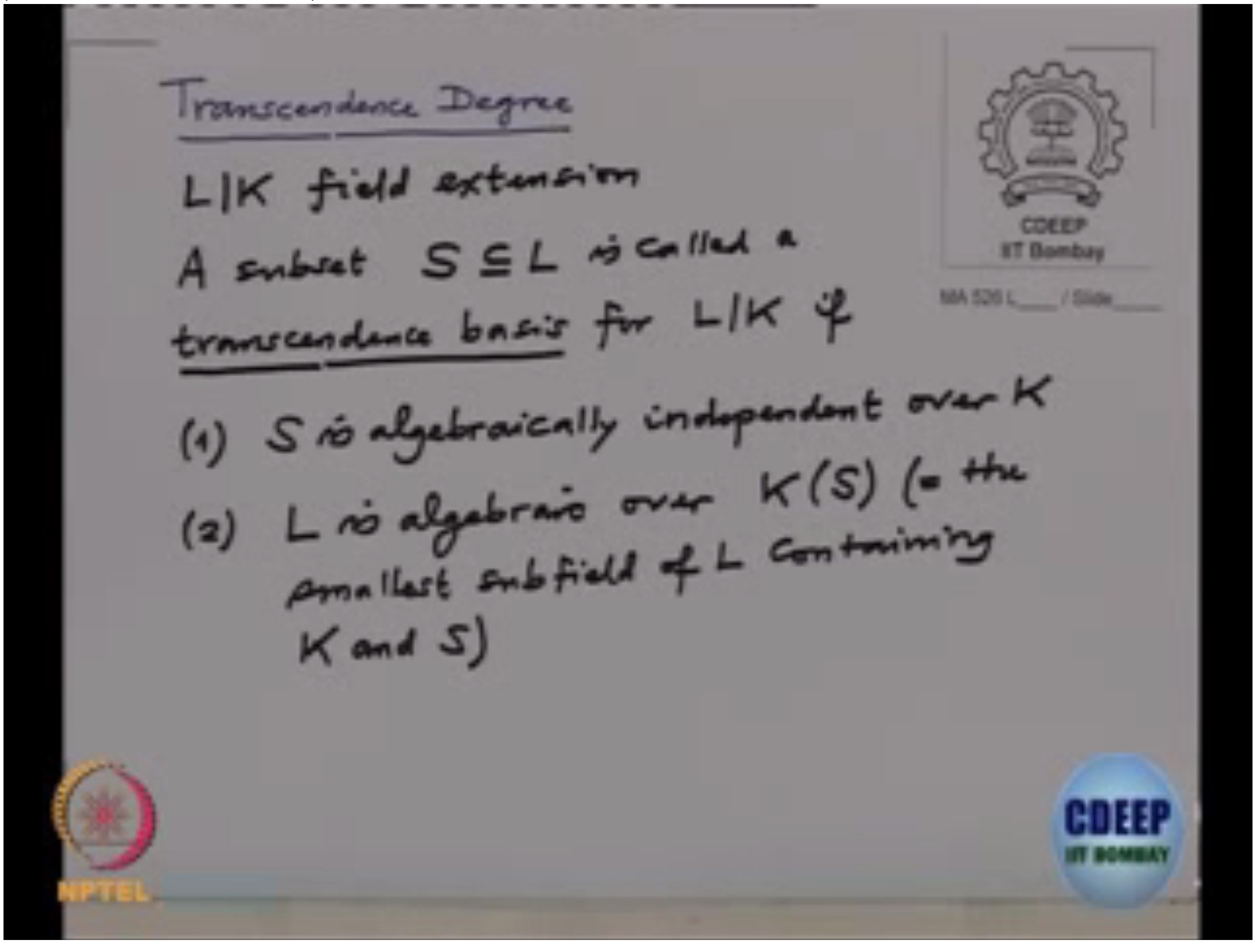
Now remind… let me remind you the normalization Lemma, what did we prove, we proved that if I have a finite type K algebra A, then I can find an elements there which are algebraically independent and this extension… this extension is integral and this is the polynomial algebra, because they are algebraically independent. So therefore just now what we have proved is dimension of this… this is a fine K algebra, dimension of A is equal to dimension of polynomial algebra in n variables. And now this integer n… this integer n is nothing, but… this n the number of variables, that also is nothing, but the transcendent degree of the rational function field in n variables. So I will recall little bit about transcendent degree, okay. So that is what I was saying that this is one way to compute dimension in case of affine algebra and which is finite, because we are dealing with finite type algebra, so you all that involve finitely many variables, polynomial rings and therefore it is finite.

(Refer Slide Time: 02:30)



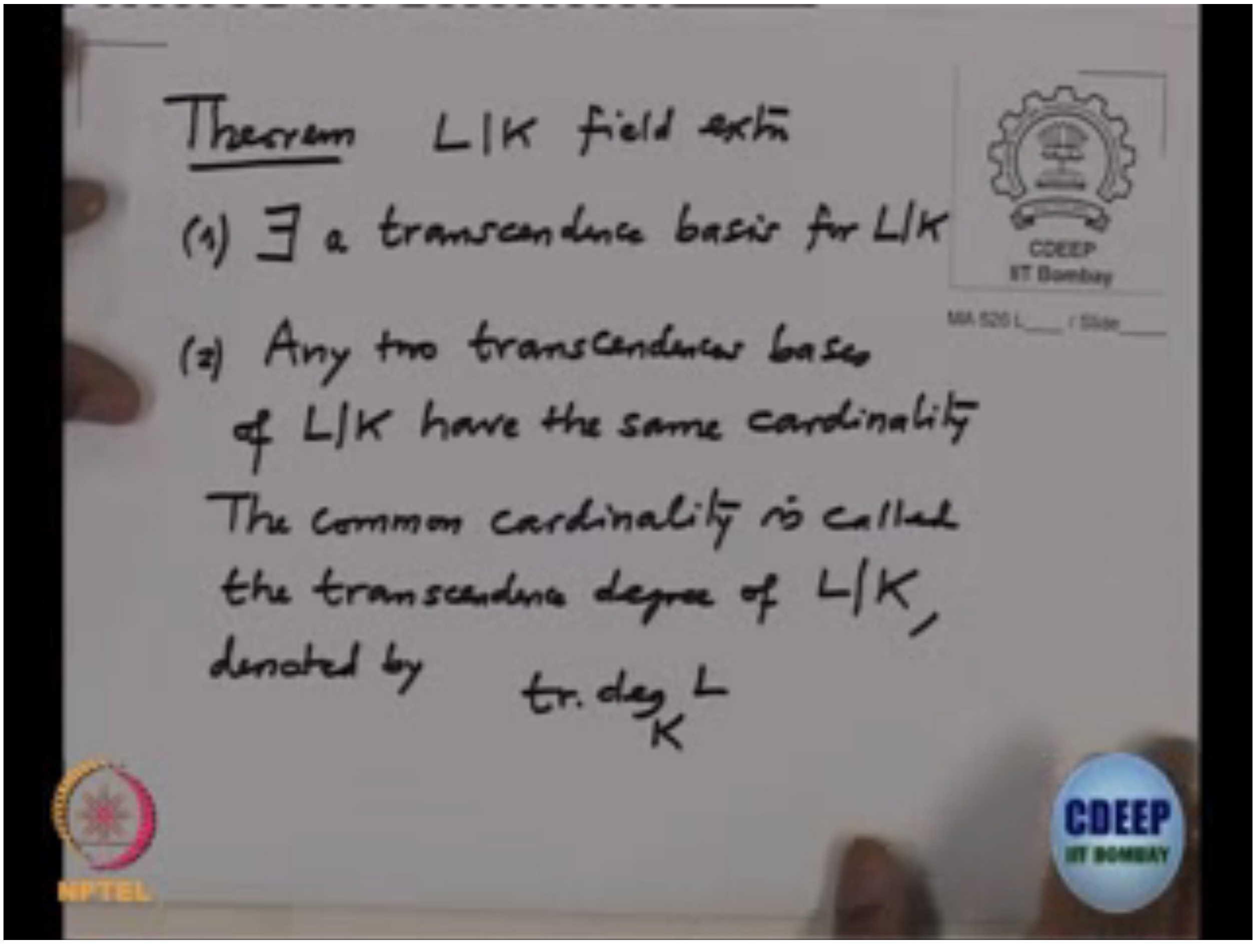
Okay so let us recall some basic things about transcendent degree. So if I have a field extension $L|K$ and subset S of L is called a transcendent basis of $L|K$ and subset S of L is called a transcendent basis of $L|K$, if two things, number one S is algebraically independent over K and second L is algebraically over $K(S)$. This is the smallest field of L continue K and S or as a field it is generated over K by S.

(Refer Slide Time: 03:16)



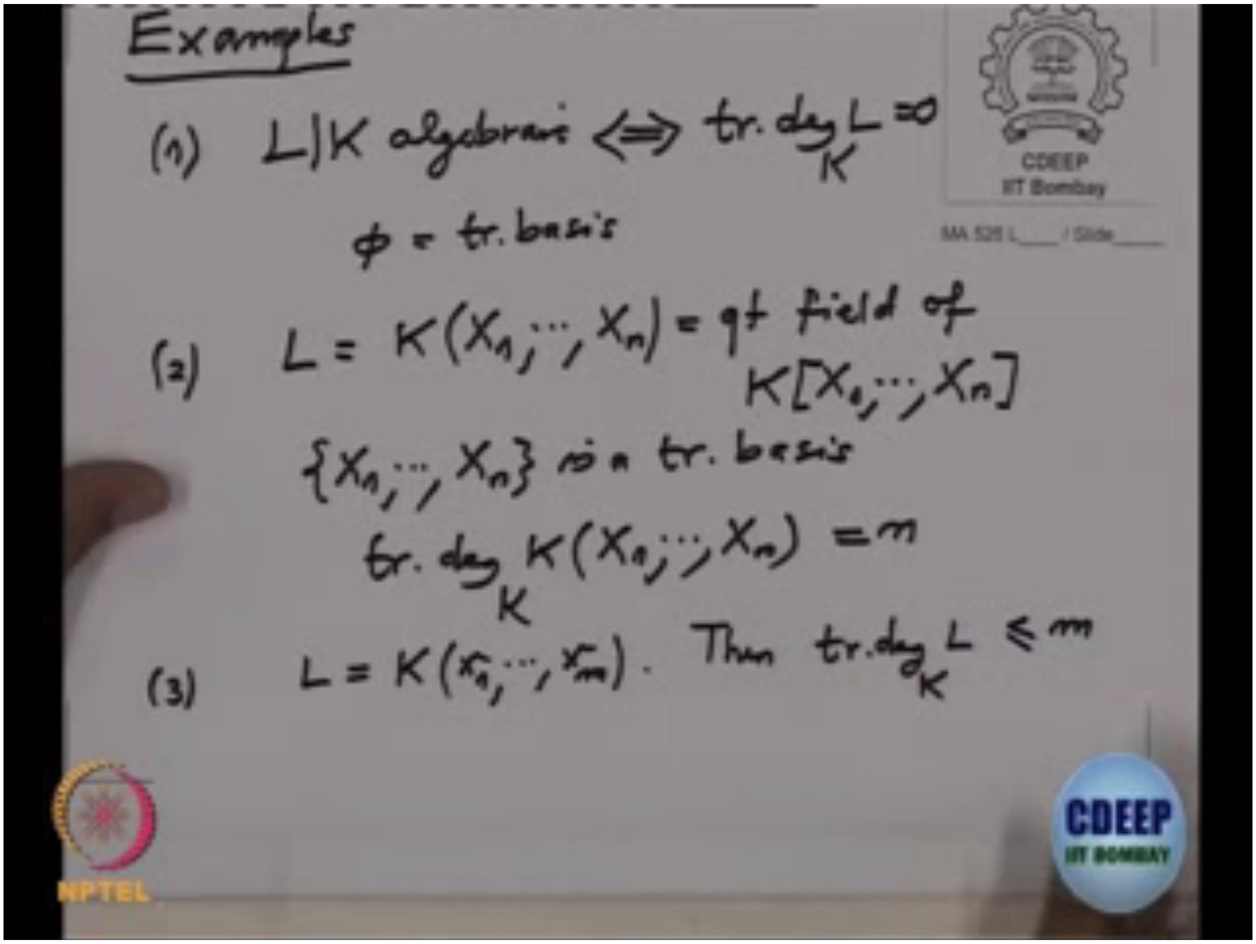
And then what proves that, I will state it as a theorem. It says that if I have a field extension $L|K$ then one, there exist a transcendent basis for $L|K$ and two any two transcendent basis of $L|K$ have the same cardinality. And this common cardinality is called a transcendent degree of $L|K$. The common cardinality, the transcendence degree of $L|K$ denoted by $tr.deg_K L$.

(Refer Slide Time: 05:10)



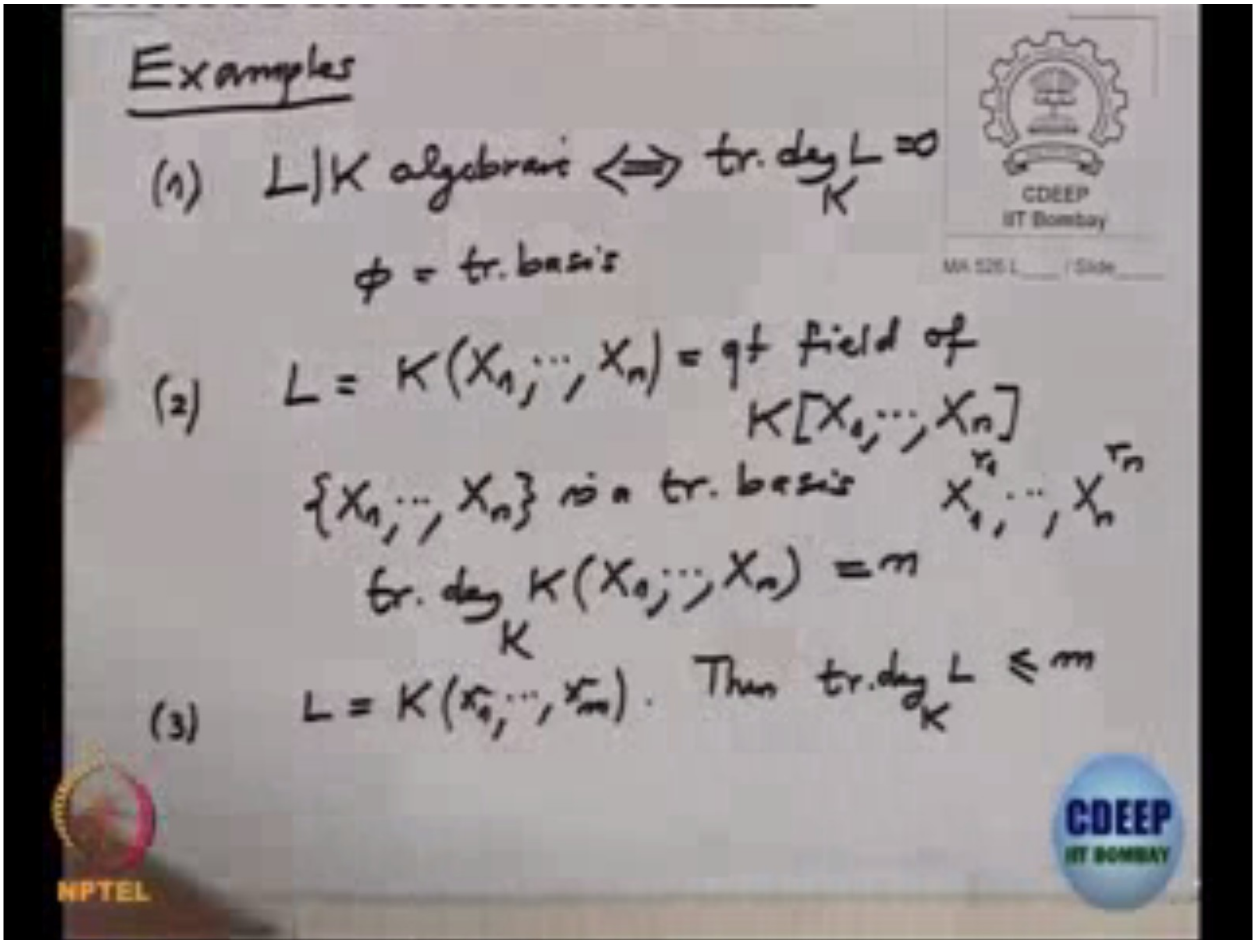
This one has obvious properties. Okay, before… before I go on to the properties, we should say some examples… see some examples. $L|K$ is algebraic, if and only the transcendent degree of $L|K$ is 0, in fact in this case, empty set is a transcendent basis. You don't need any… any… So second, if I have rational function field, L is the rational function field, this means that quotient of A, quotient filed of the polynomial algebra, then obviously X_1 to X_n is a transcendent basis. So in this case the transcendent degree of $K(X_1,\ldots,X_n)|K$ is n. Third, if we have a field, which is generated over… generated over some $K(x_1,\ldots,x_m)$, some elements X_1 to X_m , then the transcendent degree is small or equal to m. Equality holds if and only if this X_1 to X_m are algebraically independent.

(Refer Slide Time: 07:25)



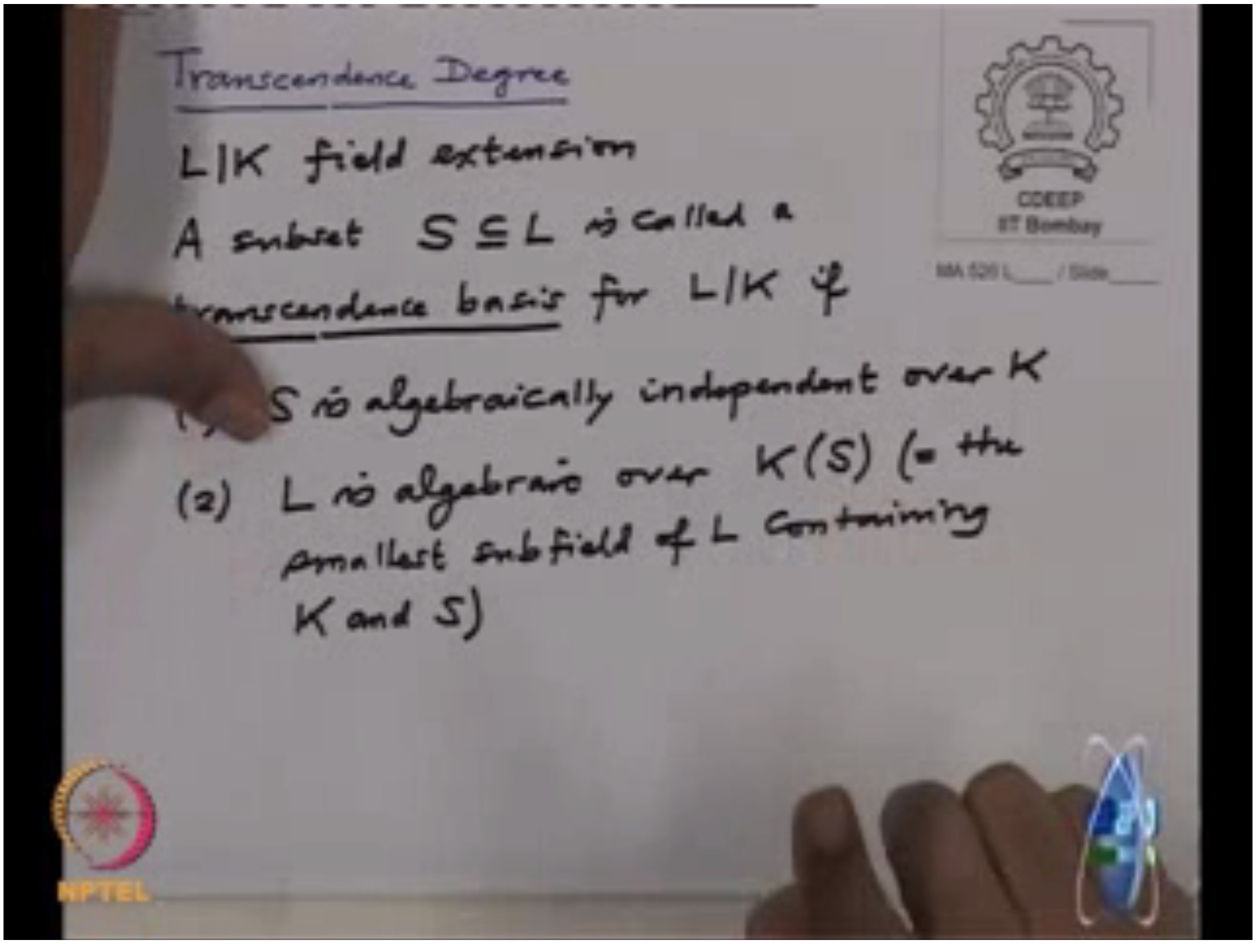
Okay this is one transcendent base, but you can give another transcendent base also, because you can simply take the powers. Suppose I take integers $X_1^{r_1}$, $X_n^{r_n}$, this is also transcendent base. That's also easy to check.

(Refer Slide Time: 07:53)



So remember here this… this… proof of this theorem about transcendent basis and cardinalities are equal etc., this goes similar to the theorem, where you prove every vector space as a basis and any two basis of the same cardinality, either finite or infinite, same… same ideas work here also. The only difference is very important difference, is well in the vector space scale you say that K=L right. Here in the definition, you might wonder, why do I put the second condition like this and not $L=K(S)$, its point to think about it.

(Refer Slide Time: 08:39)



Okay one more example, which also will be used sometime later, which I prepared here only. So for… if you take polynomial ring in N variables and if you take what are called elementary symmetric polynomials, in these variables, that simply means you take S_1 , which is sum of these variables, S_2 , which is product 2 at the time, that is $X_1 X_2 + X_1 X_3 + \ldots$ and so on, summation $X_i X_j$ and so on and Sn is product of all. S_3 is a product, sum of the product of 3 at a time. So these symmetric elementary… these are called elementary symmetric polynomials in X_1 to X_n . They arose in the study of equations, mainly Galois, Galois theory. They arose because there is a relation between the roots and the coefficients of the polynomial and the roots, these are the precisely the coefficients of the polynomials, right. So what I want to say is, this $S_{1,} S_{2,\ldots}, S_n$ is a transcendent basis of the rational functions in n variable over K. They are also n in number. So this is a different transcendent. Sometime it is useful to work with this transcendent basis than the variables, okay.

(Refer Slide Time: 11:05).

(4) $K[X_1, \cdots, X_n]$

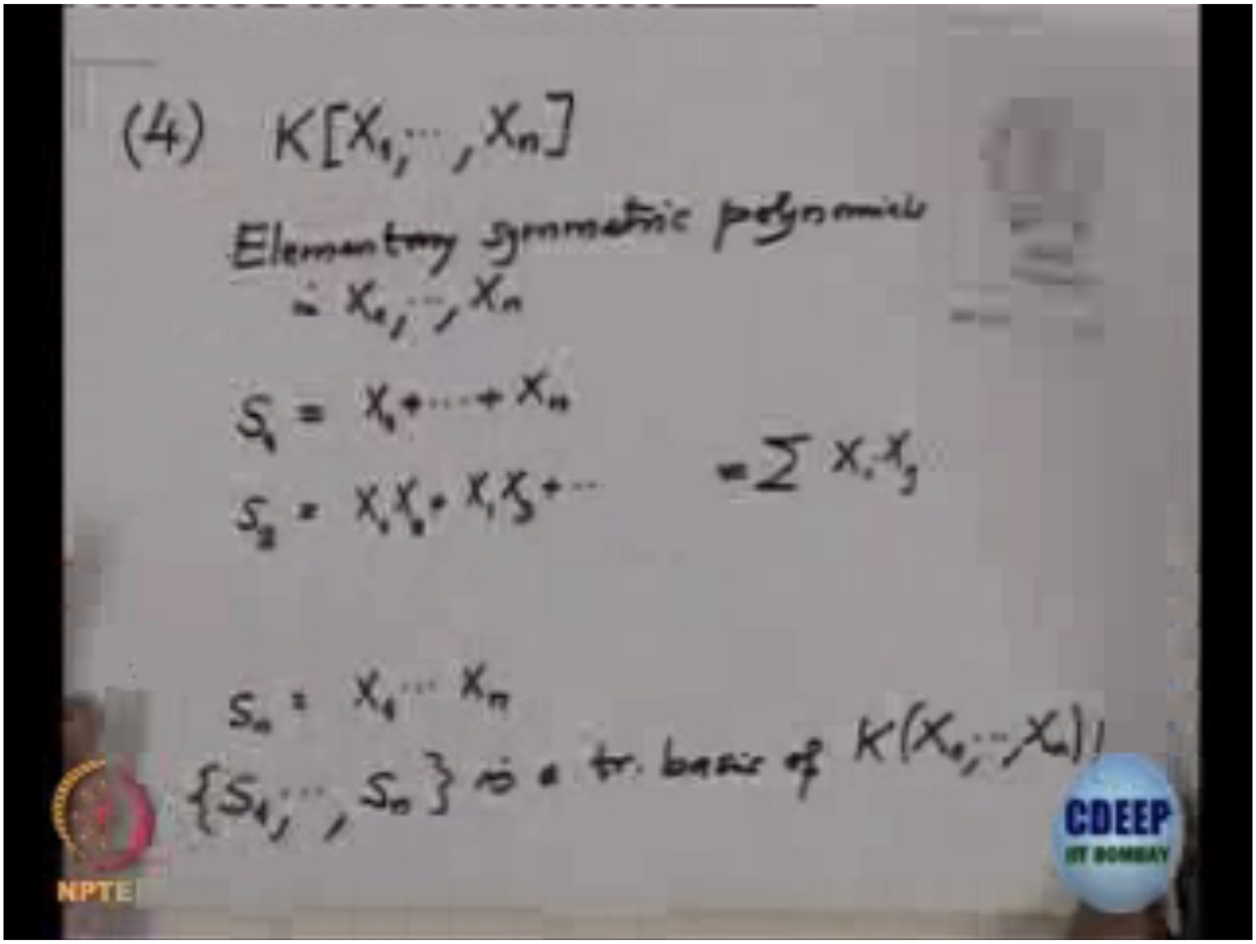
Elementary symmetric polynomials in $X_1, \cdots, X_n$

$s_1 = X_1 + \cdots + X_n$

$s_2 = X_1X_2 + X_1X_3 + \cdots \quad = \sum X_iX_j$

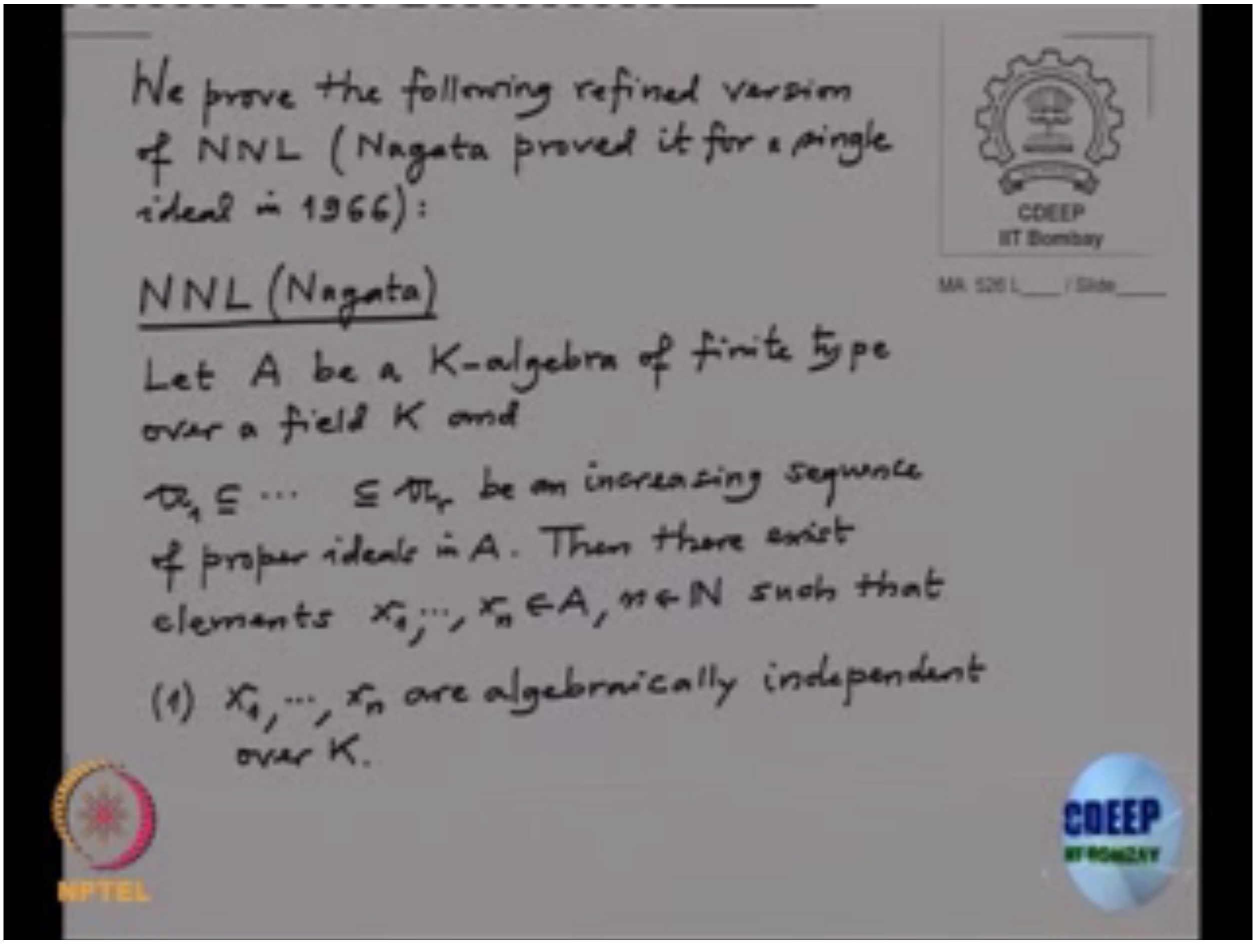
$s_n = X_1 \cdots X_n$

$\{s_1, \cdots, s_n\}$ is a tr. basis of $K(X_1, \cdots, X_n)$



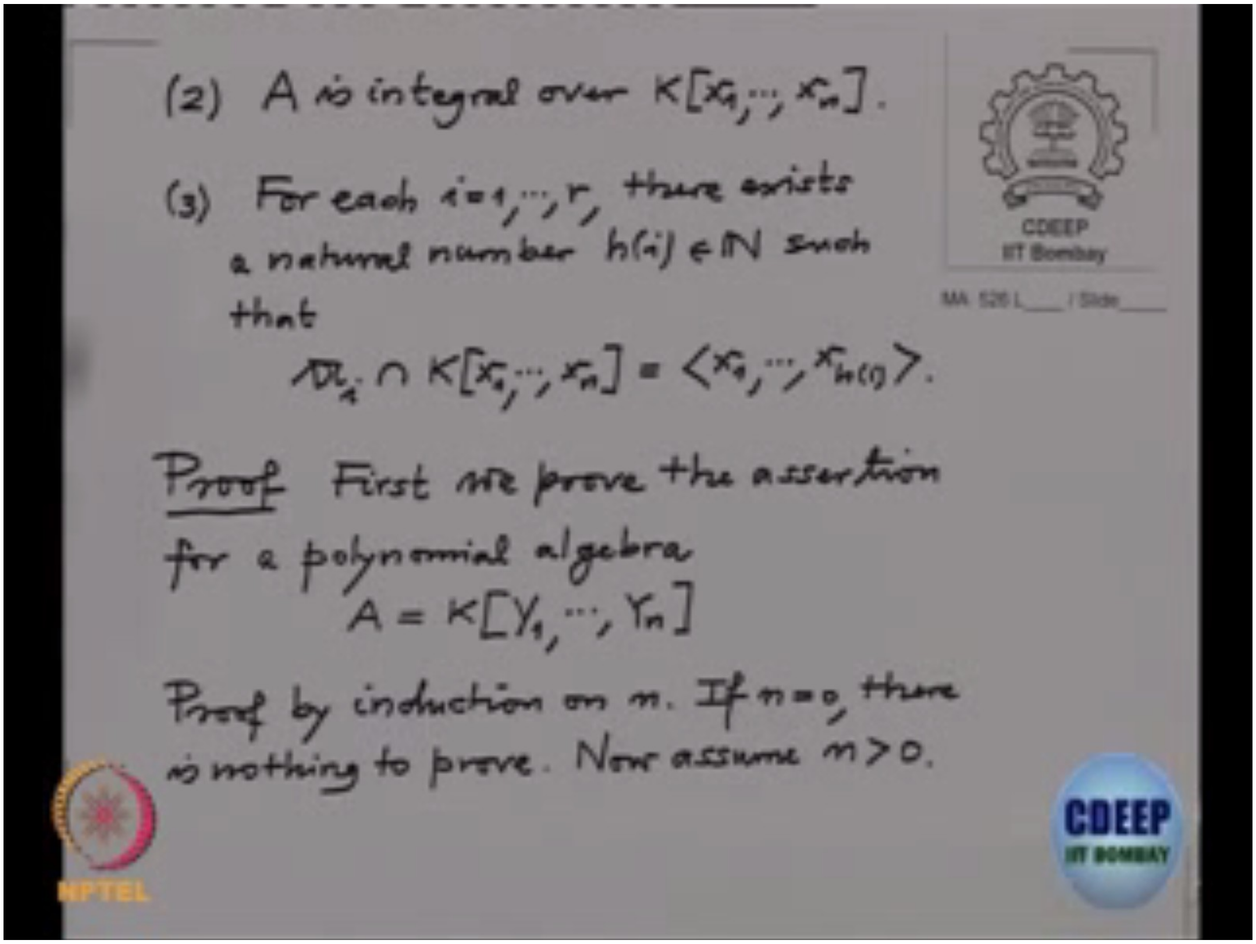
So with this, I will... I will, for today's thing I will stop the consequences, but now I want to talk about the refined version of the normalization lemma. So that is, this is a refined version of Noether's normalization Lemma. It was proved by Nagata in 1966 in the famous, in his famous book Local Rings. It was published in around the same time. So I will state for more ideals. So if you have a A algebra K... A is a K algebra of finite type over a field K and suppose I have a increasing sequence of ideals in A, then there exist elements x_1 to x_n ch that one x_1 to a x_n re algebraically independent over A.

(Refer Slide Time: 12:16)



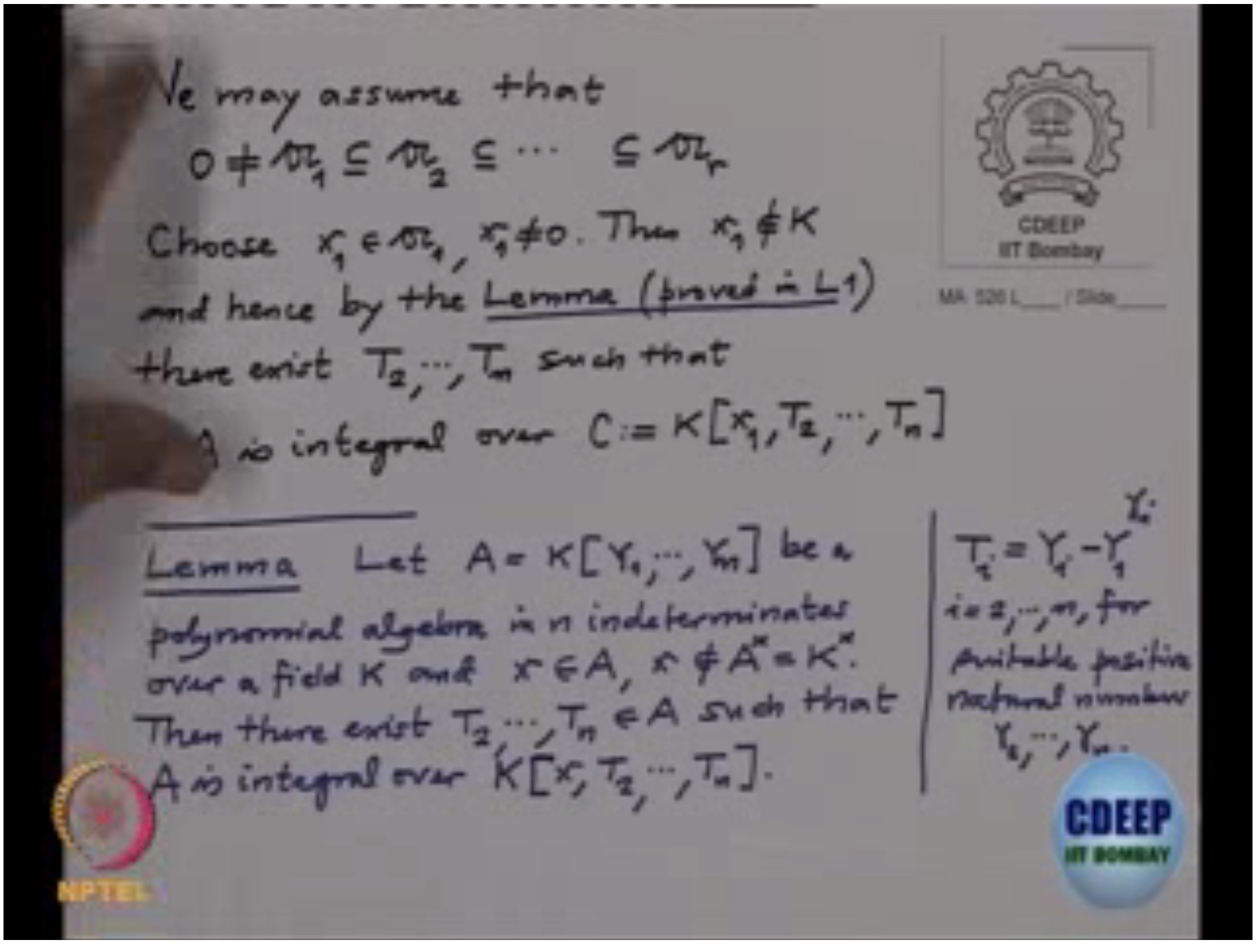
And second A is integral over x_1 to x_n, this is what we had in classical version as well, but the third condition is now new. So it says that for each i= 1 to r, there exist a natural number $h(i)$, such that when I contract this ideal A_i to this polynomial algebra $K[x_1,\ldots,x_n]$, it is generated by the variables x_1 to $x_{h(i)}$. It could happen that $h(1)=h(2)$. It could happen. So they are just integers, they are non decreasing. Okay, so I will... I will be little slow to prove this, because otherwise, it's too fast, no. So first I will prove this assertion to the polynomial algebra and then we will worry about finite type later. So A is our polynomial algebra in $K[Y_1,\ldots,Y_n]$ and I am going to prove this assertions by induction on n. n is the number of variables. If n=0 there is nothing to prove, because if n=0 then A is the field and K and nothing to prove. So I will assume n is positive.

(Refer Slide Time: 13:45)



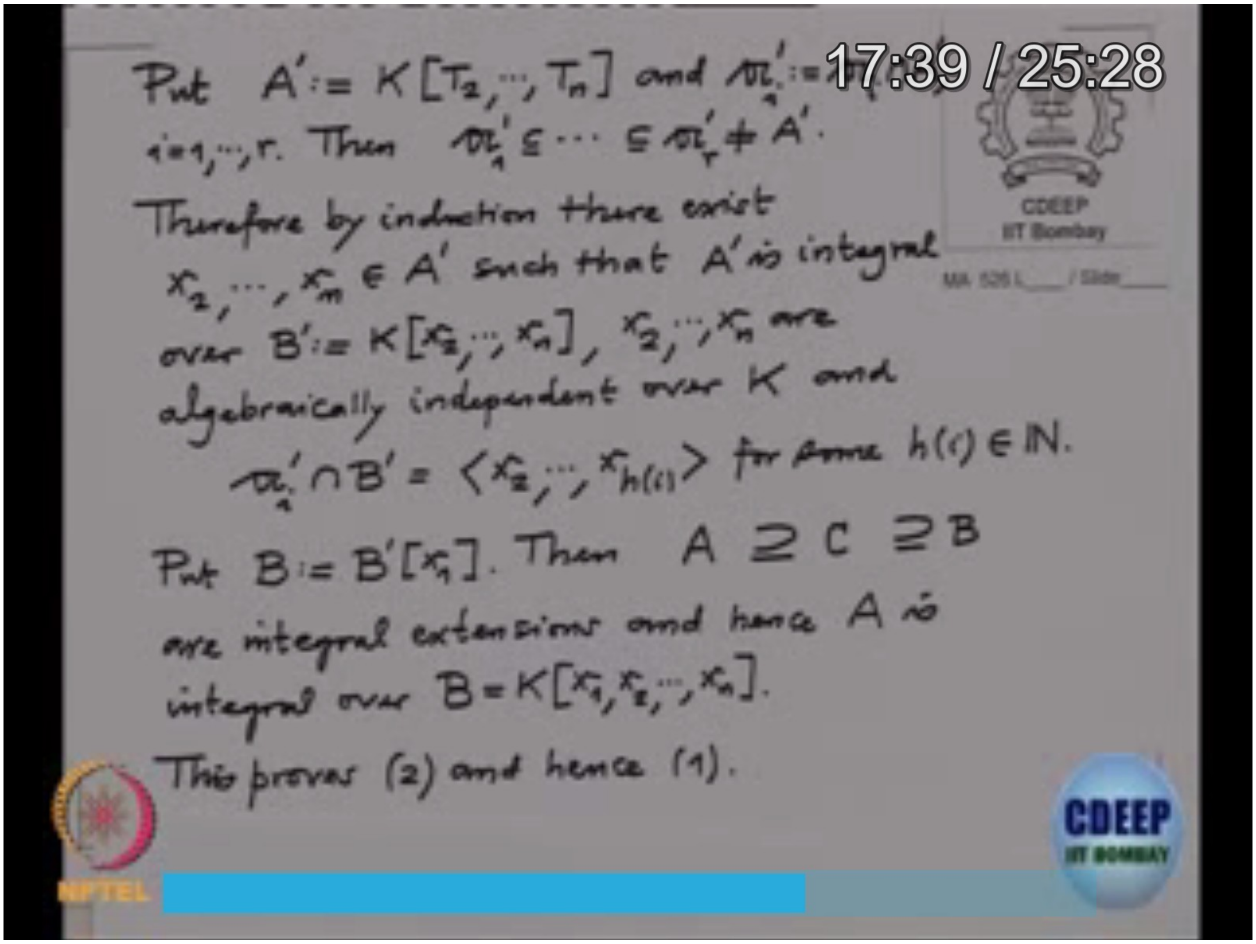
And we can also assume that the last... the first ideal is non zero, if it is 0, then you just.. so once it is non 0, I can chose a non 0 element there, call it x_1, then x_1 is not in the field, because we have considered the chain of proper ideals. So the bigger ideal, the last ideal is non unit ideal, so earlier ideal is also non unit ideals, so this x_1 cannot be in K, because otherwise, it will be a unit. And hence by, remember the Lemma, we have proved in the first... second lecture that if I have a polynomial in n variables, then I can make a change of variables, so that it becomes more in one of the variable. So I want to use that Lemma. So that will also show that I can find T_2 to T_n, so that think of this x_1 is a polynomial. And by changing the variables, we have seen that that one is integral or the remaining one, right. So therefore T_2 to T_n, there exists T_2 to T_n, so in that, this algebra A is integral over this algebra C, which is generated by x_1 T_2 T_n. So here I recall the Lemma for you. The Lemma... either polynomial algebra and determinants X is an element in A, X is not a unit, that is not a constant polynomial, then you can find T_2 to T_n, such that this is integral or this... Remember what we did, T_i was $(Y_i - Y_1)^{\gamma}$ As and we have chosen suitable gammas.

(Refer Slide Time: 15:35)



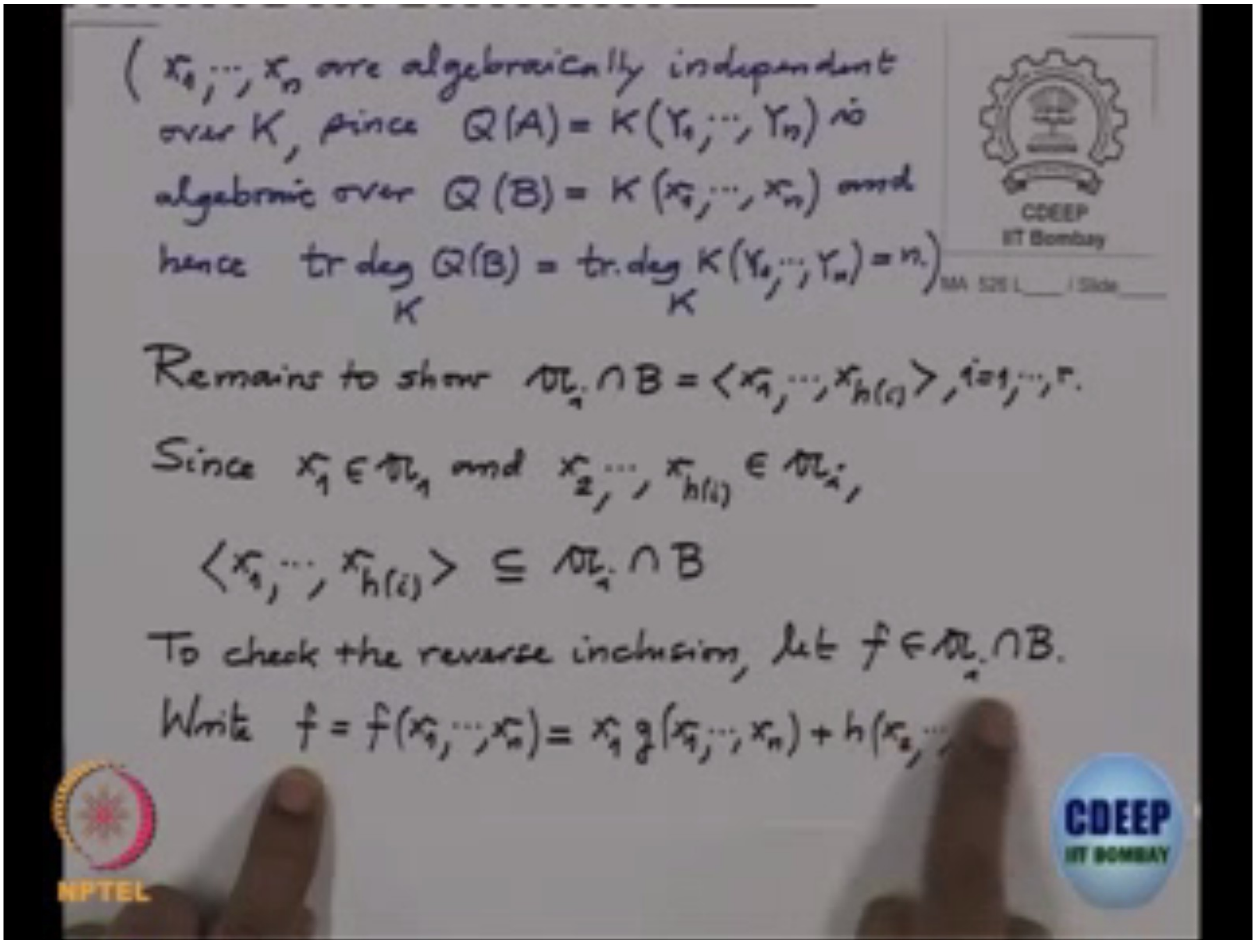
And then that, okay, so that was the proof. So I… we found x… x_1 so that this x is… A is integral over this ring C x_1 … generated by $x_1 T_2$ and T_n. Now let us put $A^{'}$ = $A[T_2, \ldots, T_n]$ and contract these chain of ideal, given chain of ideal study is $A^{'}$. Then we get a chain, some of them might become equal, doesn't matter, and the last, see all proper ideals. Therefore by induction, because now A prime is generated by lesser number of elements, I can apply induction, conclude that there exist x_2 to x_n in $A^{'}$, such that, A prime is integral over the sub algebra generated by x_2 to x_n. $x_{1,} x_{2,} \ldots x_n$ are algebraically independent and this contacted ideal to this $B^{'}$ are generated by the variables, that was the third condition in the… Now you put B= $B^{'}$ and x_1 generated by… $B^{'}$ algebra generated by x_1. So we have a chain like this A containing… contains C and contains B. These are integral action chains. Well this is integral, this integral by because $A^{'}$ is integral over $B^{'}$ and this is just a base change, because I have adjoined only the x_1. So these are integral extensions, so therefore composed with integrals. So that means A is integral over this, this proves 2, 2 is… 2 is what integral and we still have to prove 1, that algebraically independent, we still have to justify.

(Refer Slide Time: 17:42)



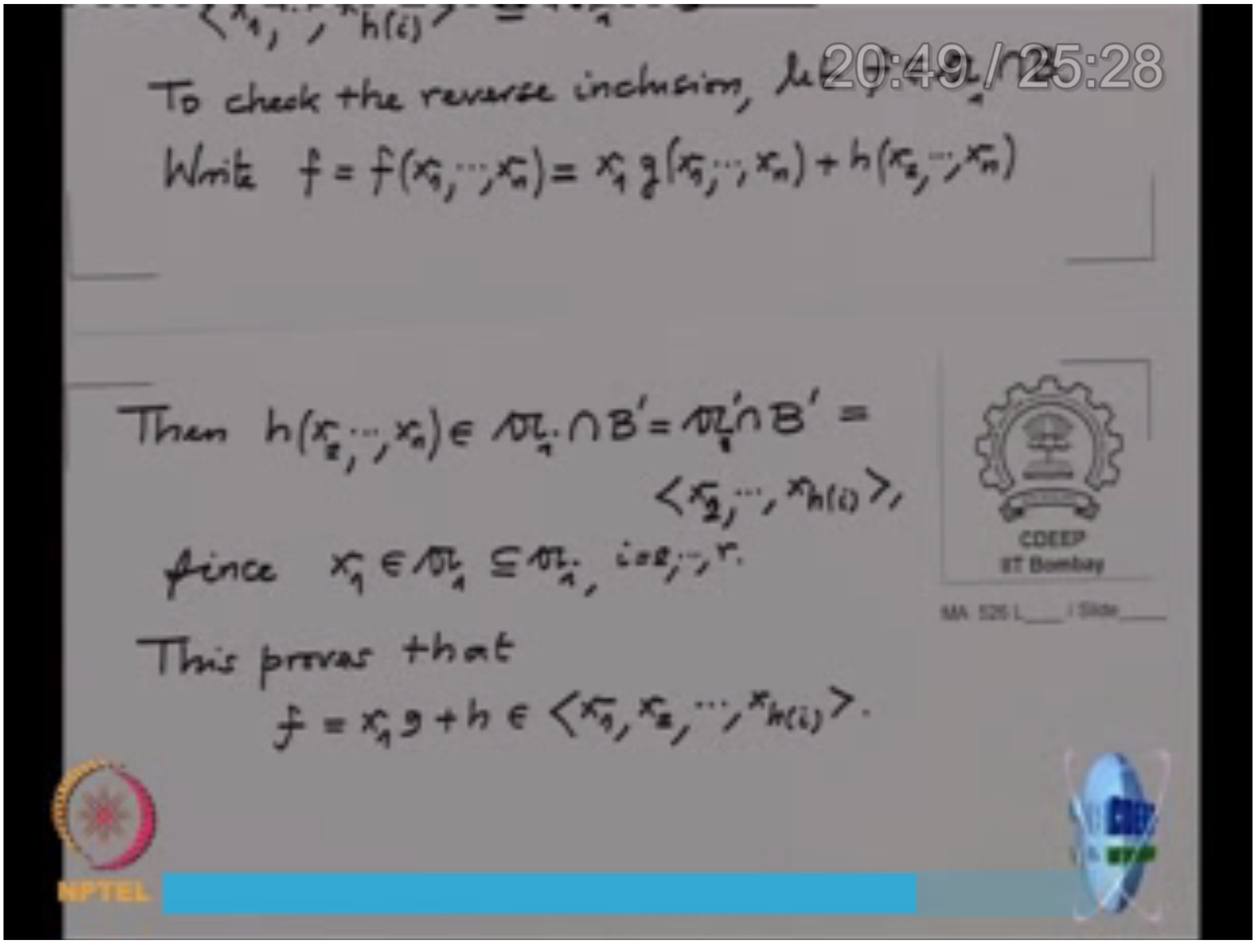
But that is obvious again, because you see $x_1, \dots, x_n$, I want to show that they are algebraically independent. Now look at the quotient fields. Quotient field of A is a rational function, and N variables, quotient field of B is rational functions in $x_1, \dots, x_n$ and the transcendent degree of quotient field of A is n, but this is integral, therefore the transcendent degree will not change. So it is n. So that means they must be algebraically independent, otherwise, transcendent degree will be dropped, right. So we have proved 1 and 2, and now we want to just prove 3, that means the contractions are generated by $x_1, \dots, x_{h(i)}$. Now X1 we have chose… x_1 what chose in A1, the ideal A1, and x_2 to $x_{h(i)}$ are in A_i, and I want to prove this actually equally there. So 1 inclusion is obvious by the setup, to prove the other inclusion, I have to prove that every element which is here is also here. So start with an element here F and write this F, F is in B, so F is a polynomial in $x_1, \dots, x_n$. So this polynomial, I want to split into two parts, the… the first part is the one, which involves x_1 and the other part is free from x_1. So collect all the monomials, which involve x_1 and the remaining one, which is free from x_1, you put it in the other part. So therefore any F, I can write it as a x_1 times some polynomial in $x_1, \dots, x_n$ plus some polynomial in $x_2, \dots, x_n$.

(Refer Slide Time: 19:36)



Now you see here x_1 was already in ideal A_1, so it is in every ideal. So this x_1 is there and by assumption F is there, we started with F in this ideal, x_1 is already there and F is there, therefore H is there. So h of $x_2, \ldots, x_n$ belong to this contraction ideal, but this ideal is same as A prime AI prime B… B prime and by induction hypothesis, this was generated by x_2 to $x_{h(i)}$ so therefore h $x_2, \ldots, x_n$ belongs to x_2 $h(i)$, because x_1 is already there. So this proves that $F = x_1 g + h$, which belongs to this and we are done. So that proves this refined version of normalization Lemma for the polynomial case. I would like to take a little bit gap here to ask me if it was okay or shall I repeat some part?

(Refer Slide Time: 20:50)

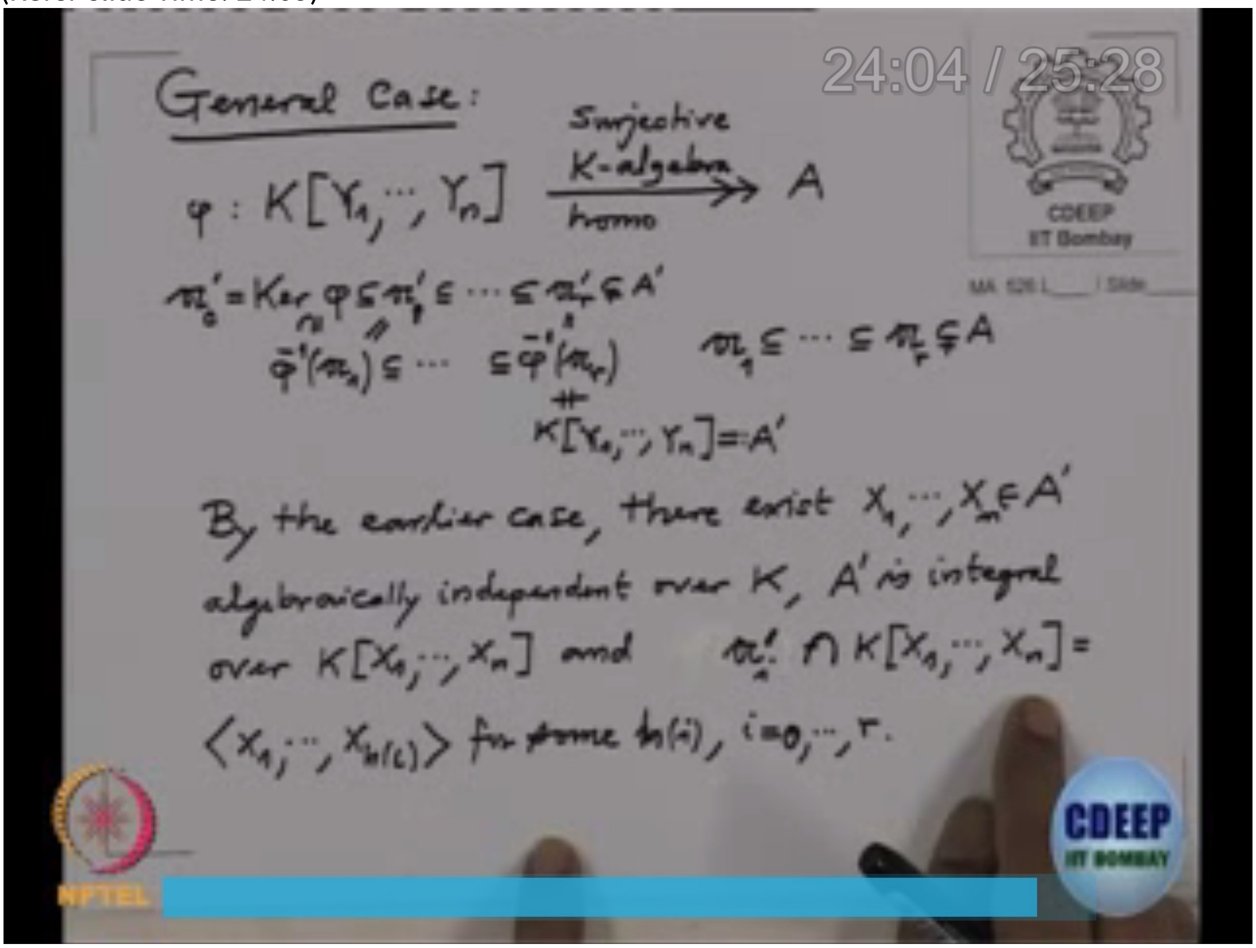


Sorry…

Why do we need to refine?

Because you see, it is always better to have a description of ideal, which is generated by the variables. See instead of studying ideals in a polynomial ring where you don't know what the generators, the structure of the generators, this way you can actually deal with the ideal generated by variables, which are more easier to deal with, for calculation purpose. Also some of the earlier observations, also you use a more general version, it will become easier. For example, when we prove the dimensions are equal, if I directly apply the third part to the ideals, prime ideals, then it will also tell you that dimensions are equal. Of course we… we use the integral extensions etc., but this will also tell that… tell that the dimensions are equal. Okay so shall we continue this? So now I have to prove that general case. The general case is not so difficult. So general case means, we are assuming now that we have proved the statements 1 2 3 for the polynomial case and we want to reduce it for arbitrary of n algebra. So anyway, any… every fine algebra, there is a subjective map from the polynomial algebra to the given algebra. And now given chain of ideals, I am going to contract it. So the chain of ideals were in A, so I am going to pull it back to the polynomial algebra. So I will get a chain and also I will get one more… one more element in the chain, the first one that… namely the kernel, because once I pull it back, all of them will contain the kernel. So Kernel will be another ideal here, which I call it $A_0^{'}$. So the pull back chain now is $A_0^{'}$ contained in $A_1^{'}$ contained in $A_r^{'}$. And now I apply the earlier case to this chain, so that means we can find the variables X_1 to X_n algebraically independent, so that A prime… A… A prime is this one is integral over this element $K[X_1,\ldots,X_n]$ and $A_i^{'}$ contracted to this is generated by this part of the variables. And now I just have to push it. Remember here A not prime, this starts with 0. A not prime contracted to this $K[X_1,\ldots,X_n]$ is generated by X_1 up to X_h not and the $A_i^{'}$ for example will contract it to H_1 . H_1 is bigger = H not. So it will be later than this, right.

(Refer Slide Time: 24:05)



So therefore, you just look at this cumulative diagram, A not prime is here, this is Y_1 to Y_n , this is the subjective map, and then all this are integral extensions. This is ideally generated by Hospital not here, so when I go more in this polynomial ring, you can start with H not plus 1 and so on. This is just… this… this is generated by this and this… this ring is what we found and this is just pushing up, this is just A is nothing but this quotient, this… this one is just nothing but this quotient this, and you just pushed on the problem here and this is easy to check that is $A_i \cap B$. Now B is this, that is… now the variable will start numbering from $H_0,\ldots$ etc. There is no new ideal involved here, you just have to check.

(Refer Slide Time: 25:04).

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}'_0 & \longrightarrow & K[Y_1,\cdots,Y_n] & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\uparrow & & \big\uparrow \text{integral} & & \big\uparrow \text{integral} & & \\
0 & \longrightarrow & \langle X_1,\cdots,X_{h(0)}\rangle & \longrightarrow & K[X_1,\cdots,X_n] & \longrightarrow & K[X_{h(0)+1},\cdots,X_n] & &
\end{array}
$$

Now we check that $X_{h(0)+1}, \cdots, X_n$ are the required elements (satisfy the (1), (2), (3) of NNL). For this, it is enough to check that $\mathfrak{m}_i \cap B = \langle X_{h(0)+1}, \cdots, X_{h(i)} \rangle$ for all $i = 1, \cdots, r$, which is clear from the equalities: $\mathfrak{m}'_i \cap B' = \langle X_1, \cdots, X_{h(i)} \rangle$ and $\mathfrak{m}_i \cap B = \mathfrak{m}'_i \cap B' / \langle X_1, \cdots, X$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a}'_0 & \longrightarrow & K[Y_1,\dots,Y_n] & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\uparrow & & \big\uparrow \text{integral} & & \big\uparrow \text{integral} & & \\
0 & \longrightarrow & \langle X_1,\dots,X_{h(0)}\rangle & \longrightarrow & K[X_1,\dots,X_n] & \longrightarrow & K[X_{h(0)+1},\dots,X_n] & &
\end{array}
$$

Now we check that $X_{h(0)+1},\dots,X_n$ are the required elements (satisfy the (1), (2), (3) of NNL). For this, it is enough to check that $\mathfrak{a}_i \cap B = \langle X_{h(0)+1},\dots,X_{h(i)}\rangle$ for all $i=1,\dots,r$, which is clear from the equalities: $\mathfrak{a}'_i \cap B' = \langle X_1,\dots,X_{h(i)}\rangle$ and $\mathfrak{a}_i \cap B = \mathfrak{a}'_i \cap B' / \langle X_1,\dots,X_{h(0)}\rangle$.