INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

IIT BOMBAY

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

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Lecture No. – 15
Nil Radical and Jacobson Radical Of Finite type Algebras over a Field

Last lecture we have seen normalization lemma and some of its consequences. Today we will have more consequences of normalization lemma,

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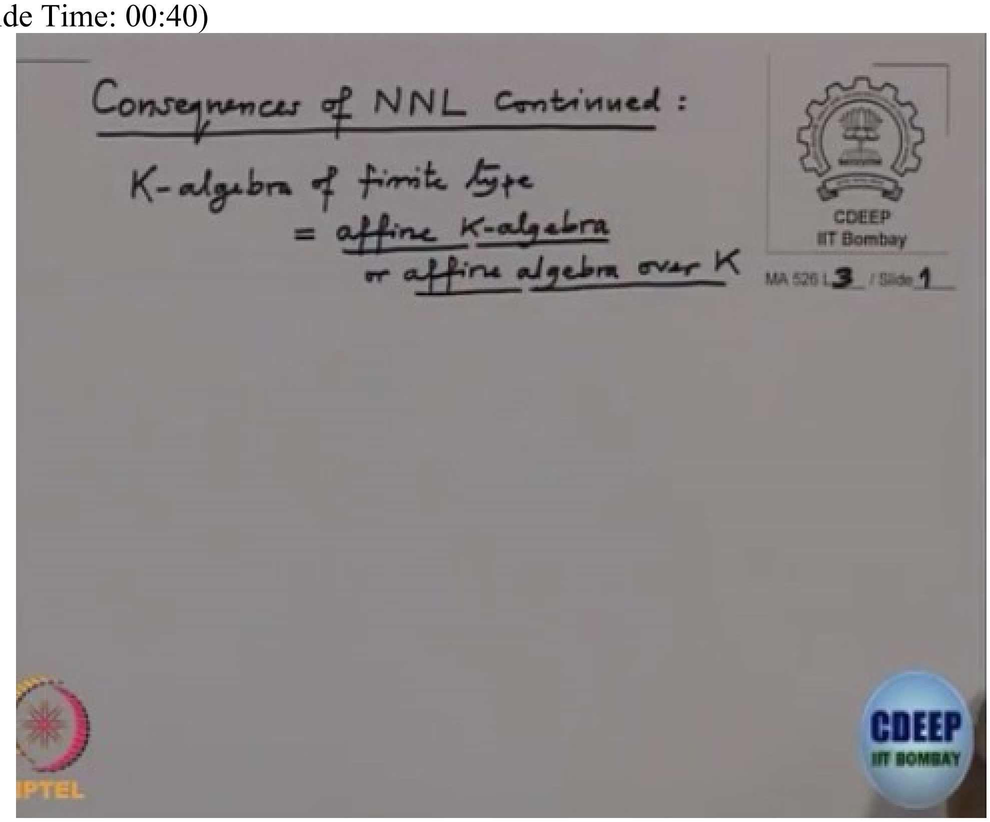

and if there is a time then we will go to more refined version of normalization lemma.

So as usual K is the field and we consider K algebra the finite type, they are also some times called affine K algebras or affine algebra over K.

So the next consequence if you have,
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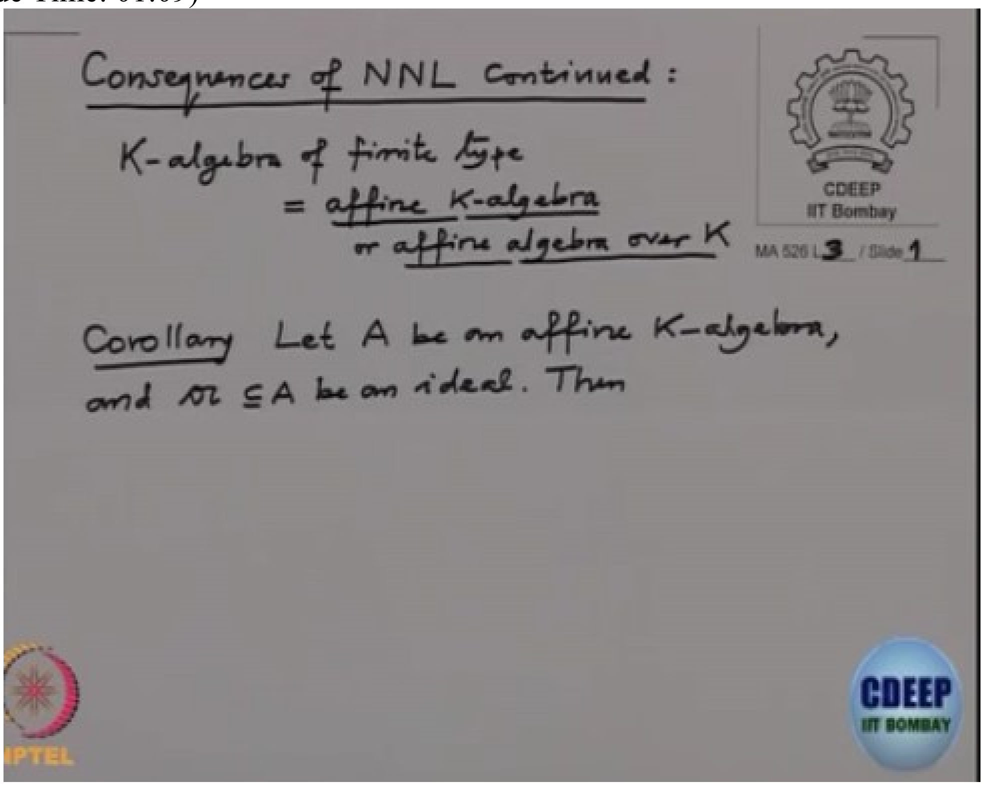

let A be an affine K algebra, and A be an ideal in, both is an ideal in A, then the radical of A is intersection of all maximal ideals in A which contains A, so in particular nil radical of A equal to Jacobson radical of A, we call Nil Radical of A if the ideal of nil important elements of A which is also intersection of all prime ideals of A,
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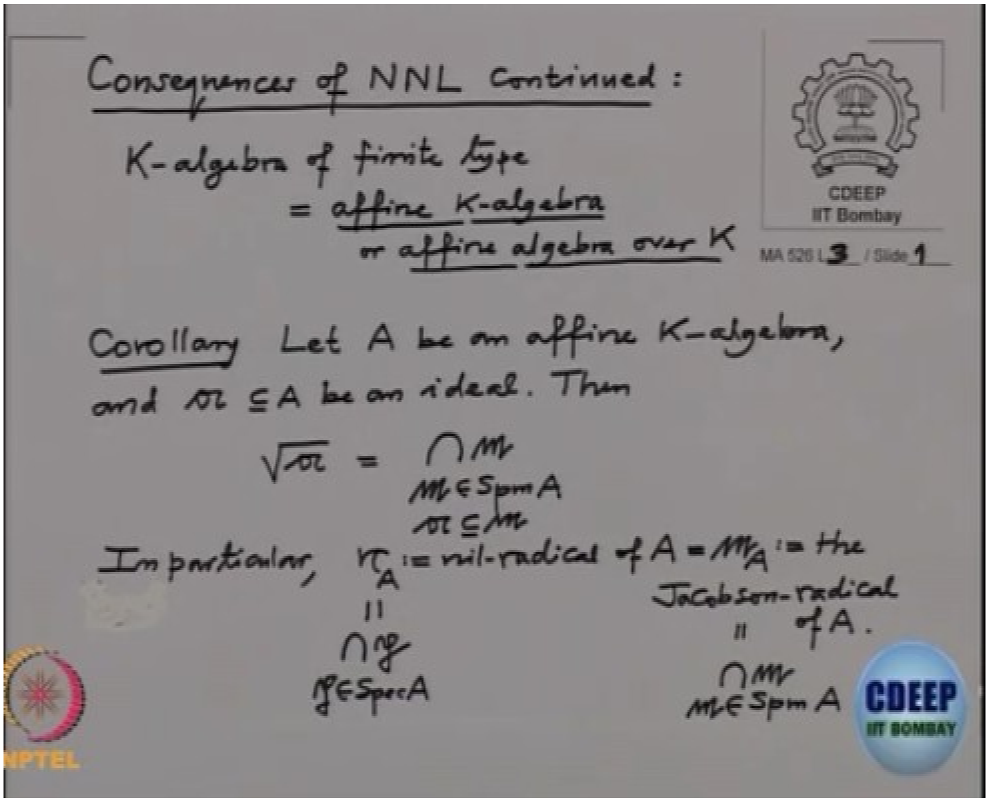

and the Jacobson radical intersection of all maximal ideals of A, so one containment is always clear that nil radical is contained in the maximal, the Jacobson radical.

However the other inclusion is may not be correct, because for example even for ring of integers nil radical is 0, no not for ring of integers but $\mathbb{Z}_p$ for example, so in general nil radical could be a strictly smaller than the Jacobson radical, so affine algebra is very important in this assumption.

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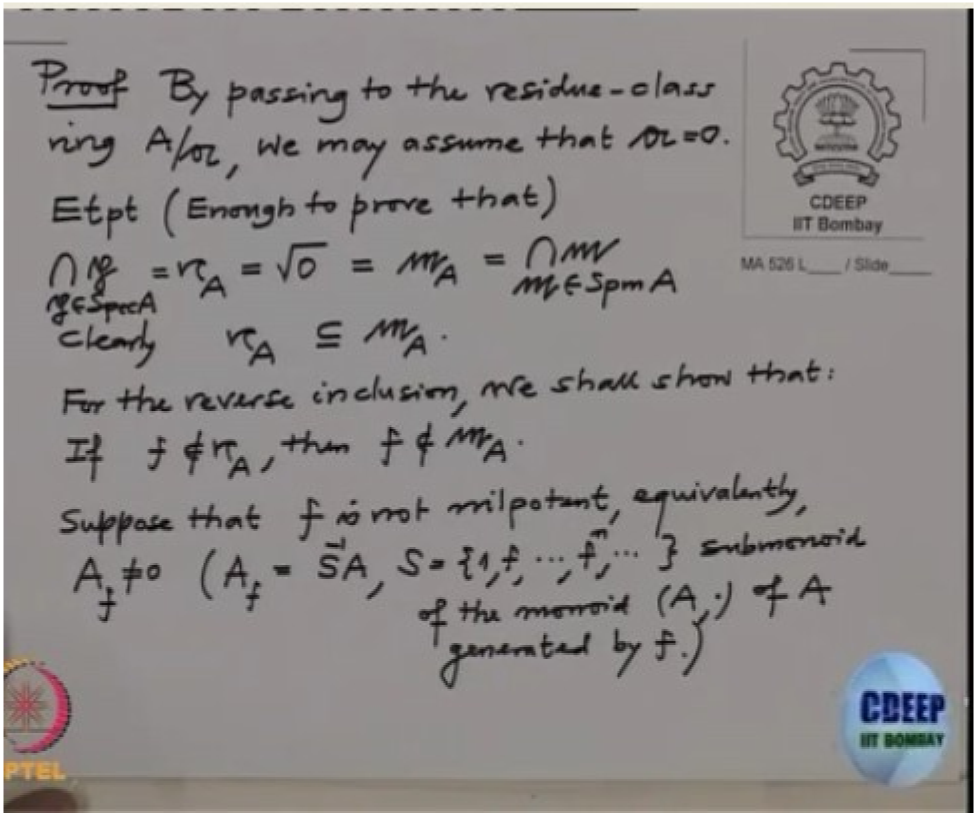

So proof, by passing to the residue class ring $\frac{A}{a}$ we may assume that the ideal is 0, so we want to prove that the nil radical hold to Jacobson radical right, so by going to the residue class ring replace A by $\frac{A}{a}$, and then again A becomes 0 and then we want to prove that the nil radical of this ring equal to the Jacobson radical.

Now one inclusion is obvious, nil radical is contained in the Jacobson radical because nil radical is intersection over a bigger set, and Jacobson radical is a intersection over smaller set.

Further reverse inclusion we will show that if some element is not nil important then it is not here, it's not in the Jacobson radical, then we mean they are no more new elements in the Jacobson radical. So suppose F is not nil important, then I want to show that F is not in the Jacobson radical that means I want to show that there is a maximal ideal in the ring so that F is not in that maximal ideal. So F is not nil important is equivalent to saying that the localization at F is an nonzero ring, where localization at F means S inverse A, where S is the multiplicative submonoid generated by F.
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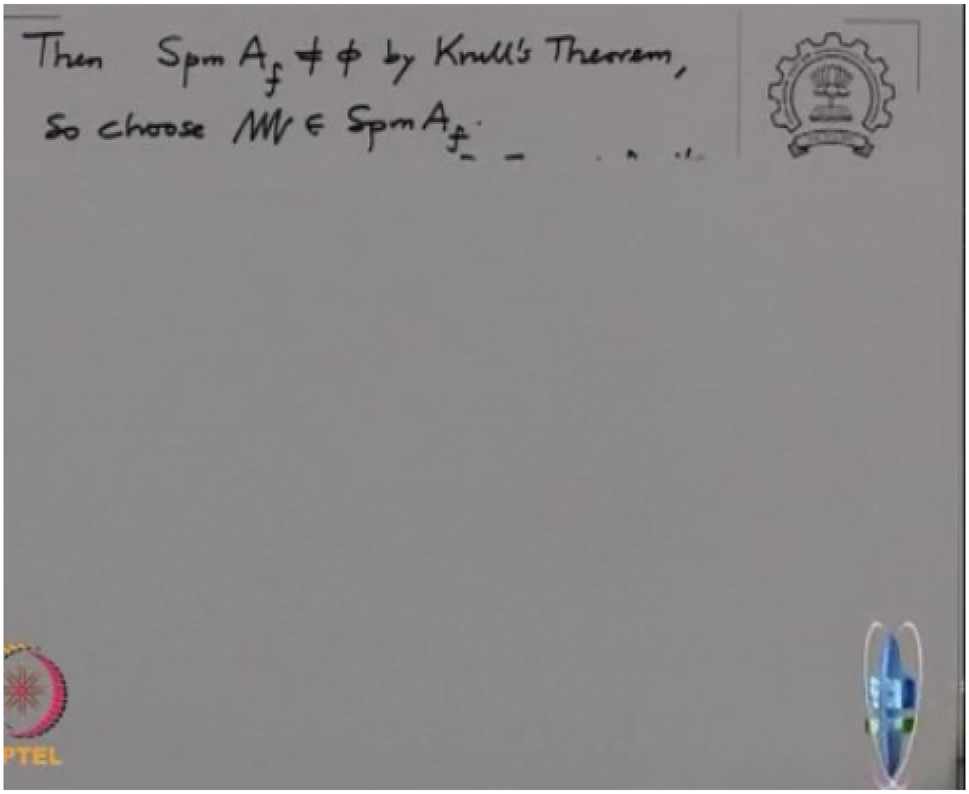

Now because this ring A localized at F is nonzero ring we know by Krull's theorem there is definitely a maximal ideal, that means there is a capital this gothic M which is in the SPM of A localize at F, remember I'm denoting maximal ideals by SPM, but I'm not using the notation max here, to get that confusing we did the maximum, anyway so the K algebra A localize at F this is actually a finite type K algebra, because it is generated by, along with the generators of the algebra A which are finitely many along with that 1 over F generates that algebra, so it's a finite type K algebra, so I can apply the earlier consequence which I had call it Hilbert's Nullstellensatz, if I have a field extension which is finite type then it should be algebraic, so AF modulo gothic, capital gothic M, this is an algebraic extension.

Now let us contract this maximal ideal to A, A to A localized at F natural homomorphism, and this capital M which is gothic M, it's a maximal ideal in A localized at F so I contact it to A, that means consider this small gothic m, capital gothic M intersection A,
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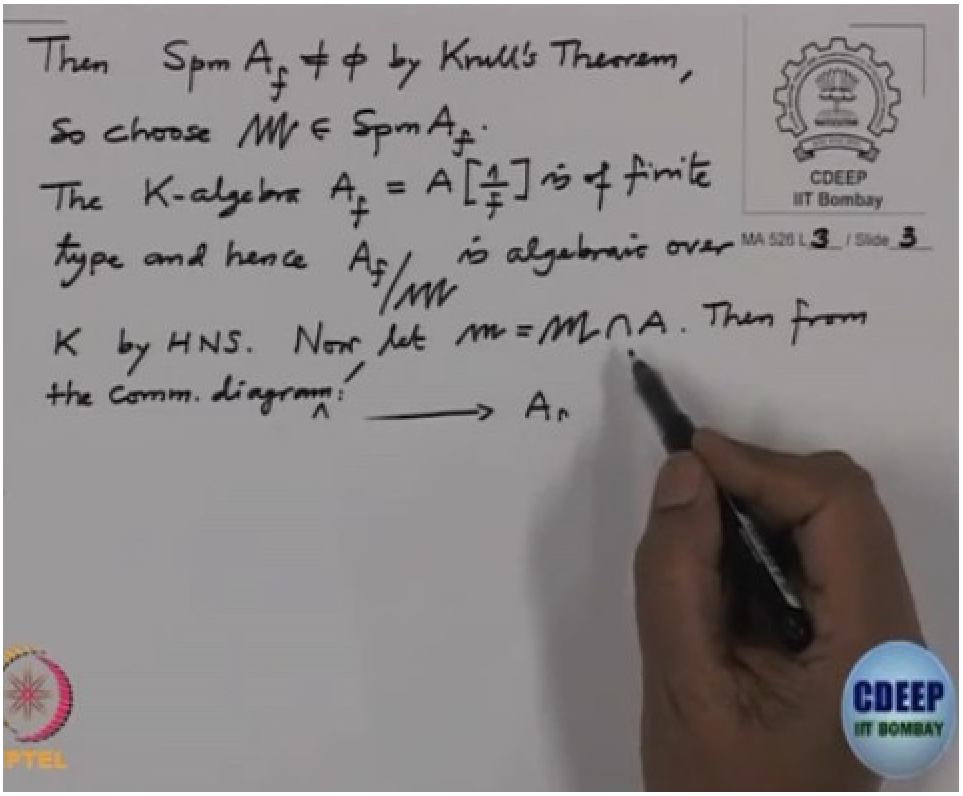

this is a contraction of capital M to A, so look at this commutative diagram, so A to A localized at F this is a natural localization map, $\frac{A_F}{M}$ this is the natural subjective map, and A to A by small gothic m this is also natural gothic M, and K is contenders, so we have this commutative diagram.

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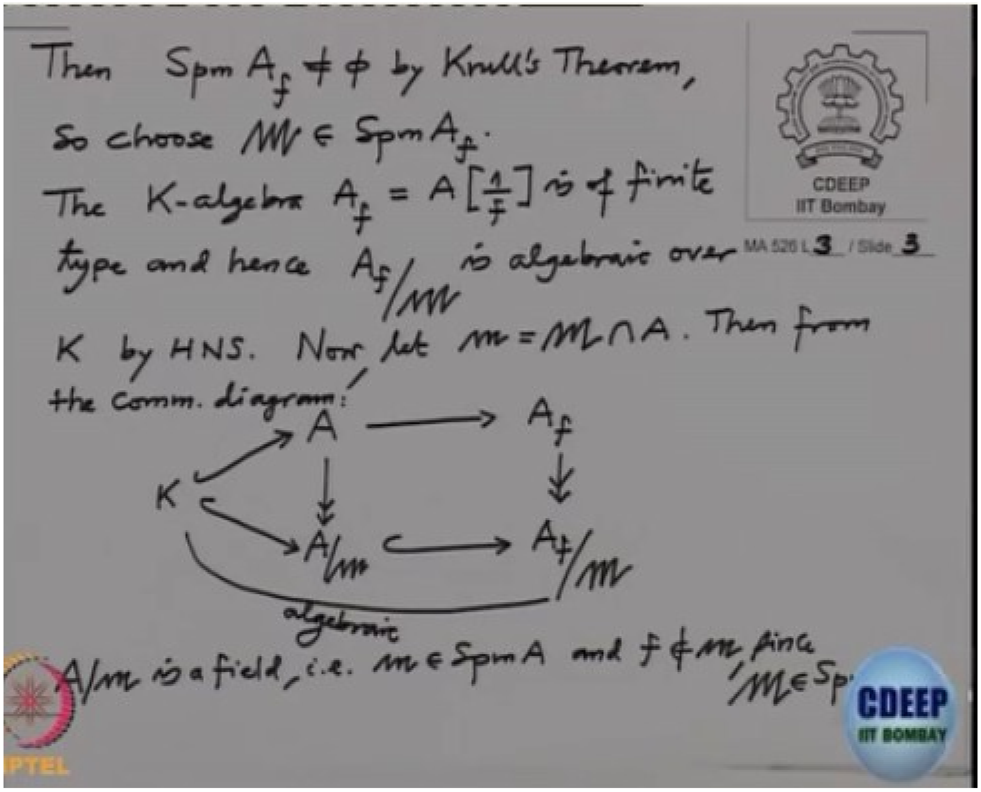

In this commutative diagram this K to this extension because this is the finite type over K, and it's a field therefore by HNS this is algebraic extension, and this is A/small gothic m this is a ring in between, so if I could prove that this is a field, if $\frac{A}{m}$ if I proof it is a field then that will mean that this small gothic m is a maximal ideal in A, and that maximal ideal cannot contain F because if it contain F then capital M is also contain F and that is not possible, so I have to justify that this A, the subring in between the algebraic extension of fields is also a field, this is what I need to justify.

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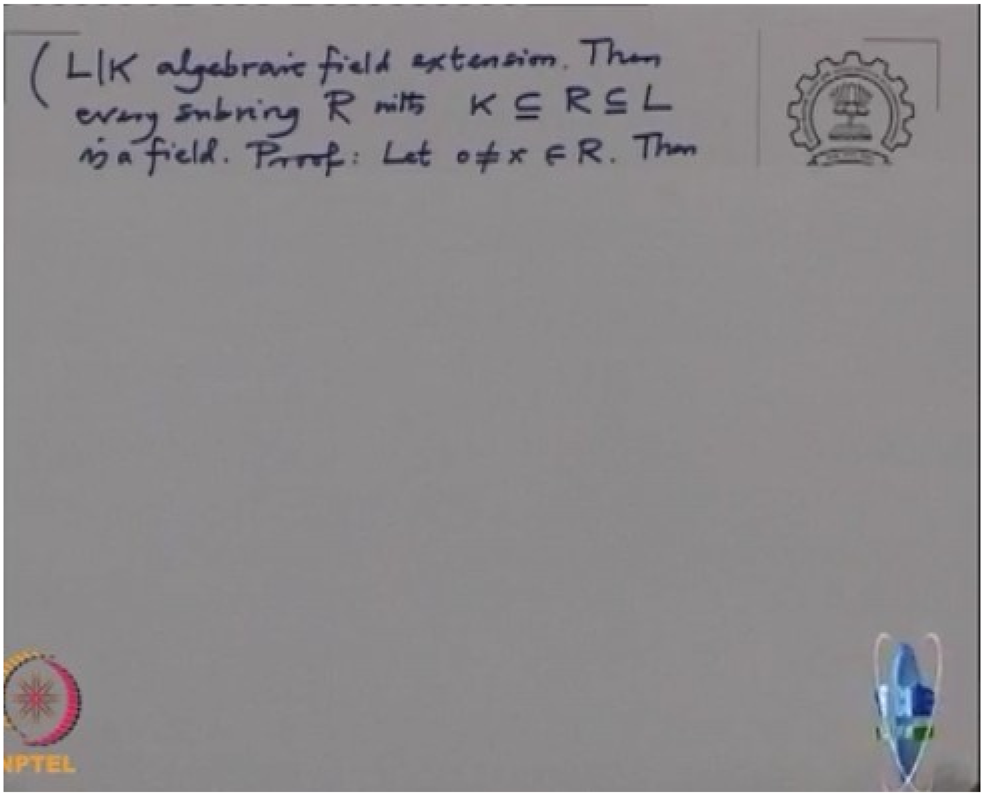

So let us take an algebraic extension of fields and a subring in between, I want to check that this is also field, to check that I have to check that every nonzero relevant of R is invertible, so if X is contained in R, and K is contained in R then the smallest of algebra of R which contain X will also be contained L, so this K square bracket X is contained in R, but X is algebraic over K, so therefore this K is small x, this is nothing but it's a residue class of the polynomial ring in N variable, modulo the ideal generated by the minimal polynomial of X, so therefore we just have to note that the minimal polynomial is irreducible polynomial, because if it is irreducible it's a polynomial in one variable over a field it will generate a prime ideal, prime ideal on nonzero, prime ideal in a PID is maximal, therefore this is maximal, so modulo that it will be a field and we will be done.

So in particular X inverse will belong to this sub algebra $K[X]$ therefore it is contained in R and then we have finished our problem, but you can do little bit simpler than this, so that's what I have written here variant. So look at the sub algebra generated by X, because X is algebraic this sub algebra is finite dimensional over K, and because X is algebraic.

Then look at the multiplication map by x on this sub algebra $K[X]$, so it's y going to $x \times y$, right, this is obviously K linear map and which is injective because x is nonzero, so we have a K linear map and a finite dimensional vector space which is injective, therefore it has to be bijective, in particular subjective and hence once it is subjective, one is in the image therefore they are either y so that x times y equal to $\lambda_X(Y)$ which is 1, and then Y is the inverse, okay.
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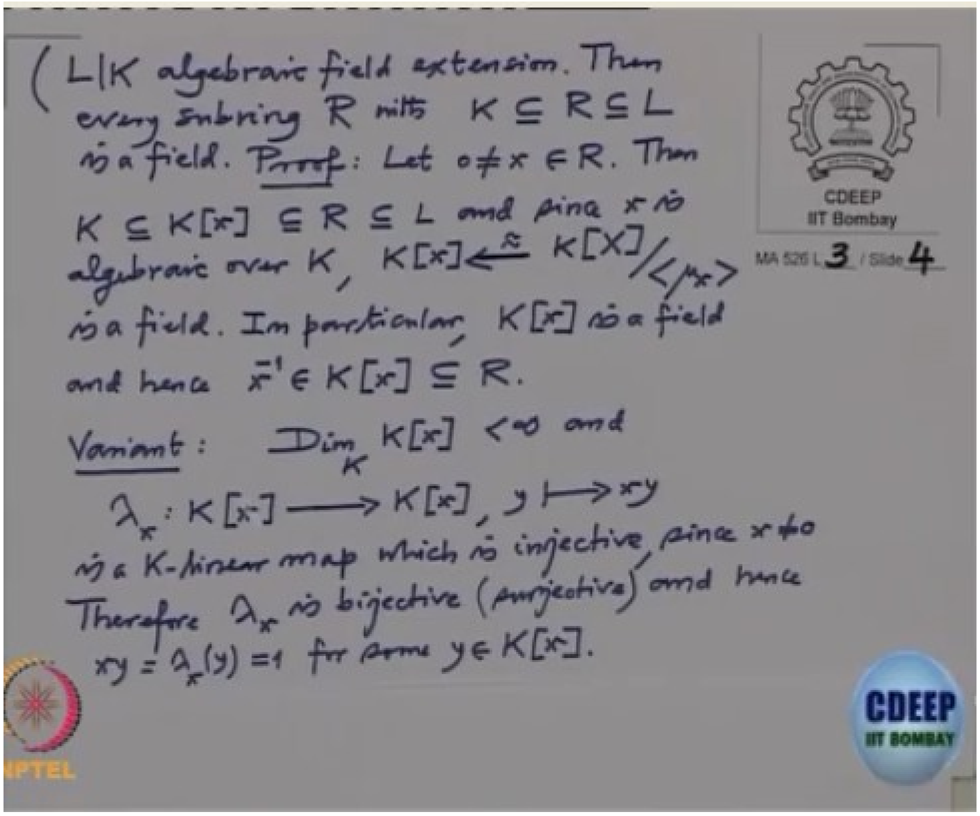

Next one, this also very important consequence, in general if you have a ring homomorphism then you must have seen contraction of a maximal ideal may not be maximal, for example if you take integers embedded in a rational numbers, 0 is the maximal ideal in rational numbers but it is not a, contraction is also 0, 0 is not a maximal ideal in the ring of integers, but if you assume that it's a K algebra homomorphism of affine algebras then the contraction of a maximal is also maximal, so that's what this next consequence is, so for every capital M which is a maximal ideal in B, M intersection A is also maximal ideal in A.

Proof, again look at this diagram, A to B given K algebra homomorphism then go mod,
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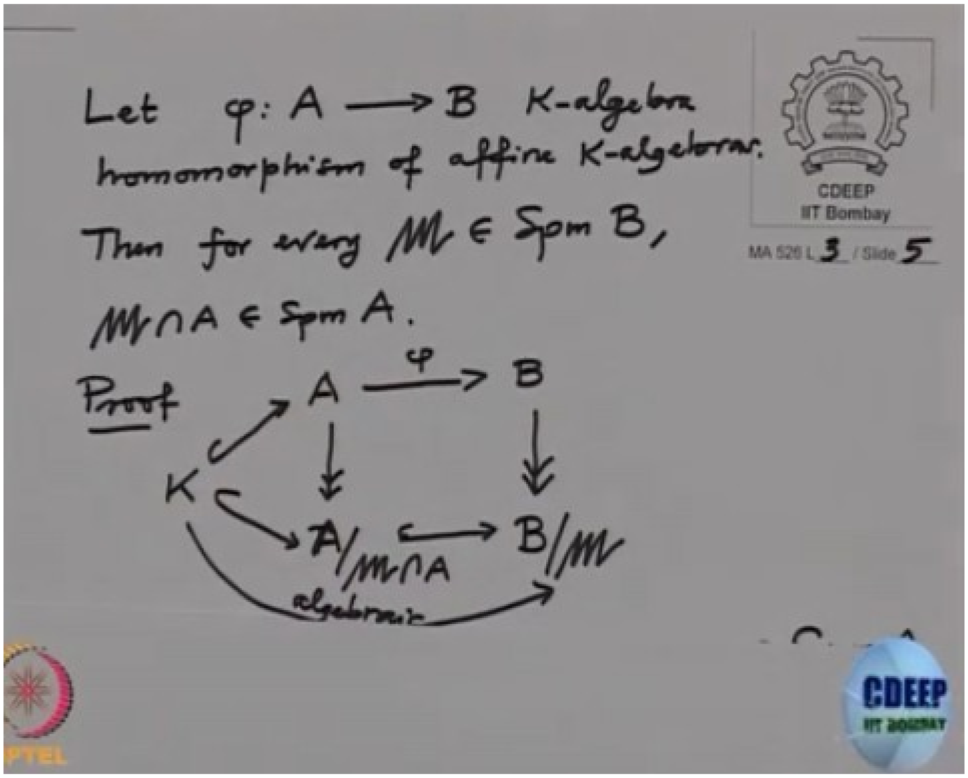

go mod capital M, go mod the contraction of M and then we have this vertical arrows are subjective, this double arrow means subjective, so K is contained there and again by Nullstellensatz $\frac{B}{M}$ which is again finite type K algebra which is a field, and therefore it is algebraic over this K, and this one is in between ring, and just now we saw if you have an algebraic extension in between ring is also a field, so therefore it is a field and that simply means that the contraction is a maximal ideal.

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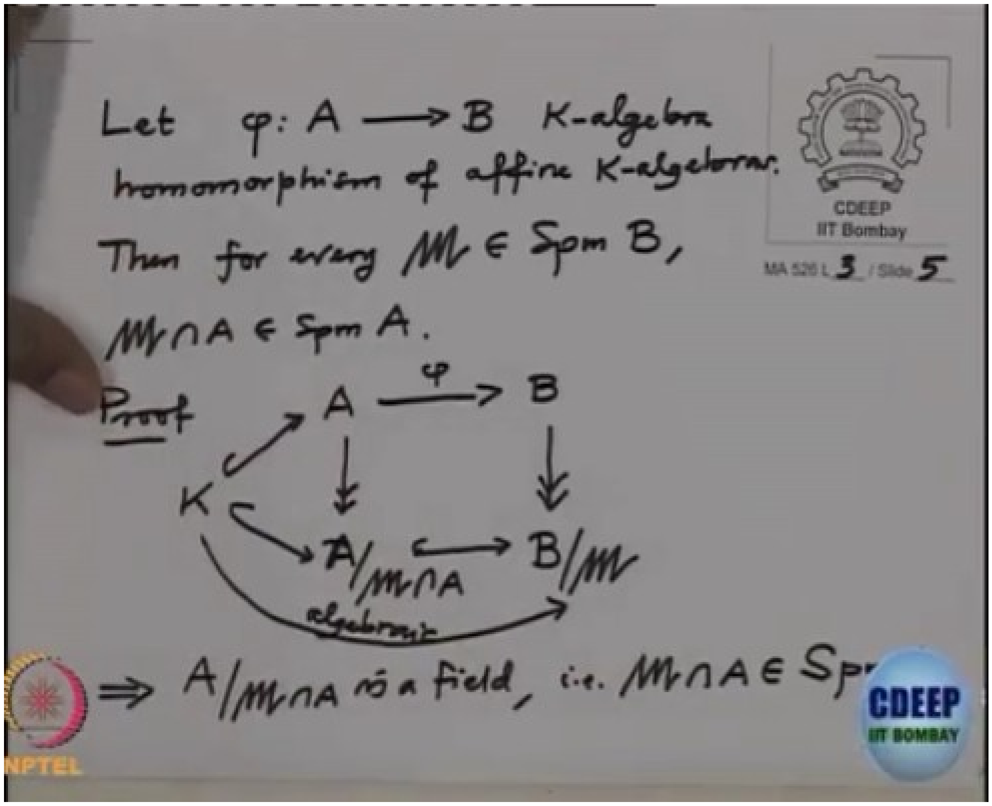

Okay, now one more consequence, this I wanted anyway to recall this thing, so I will take this opportunity to recall the proof of this, if I have an integral extension of rings then there dimensions are equal, so dimension of a ring is by definition, supremum of R such that they have a chain of prime ideals of length R in A in spec A, and in one of our tutorial section I'll show to you that why this is a correct dimension, I'll start with definition with the, from a algebraic geometry and we'll show you this forces this 2 way dimension that is what Krull did it,
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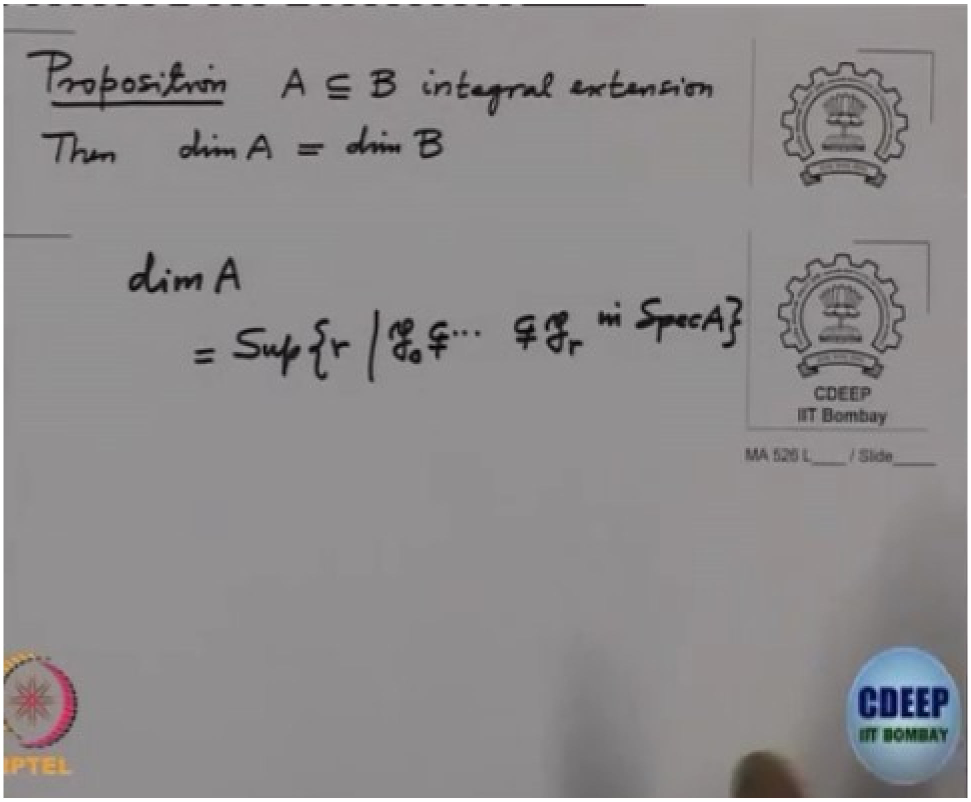

so to me it looks like this strange definition where did it come from? Right, because this supremum doesn't may not exist and how do you compute etcetera, that is a whole, this course is whole in the first part we revolve around, I'm saying how do you compute this, so various techniques from local algebra and affine algebra is specially designed to compute this dimension, and when is it finite etcetera.

Okay, so now I assume that you are familiar with integral extension but to be more, I'll just recall what it is, so some basic tracks I'll recall about integral extensions without proof, A contained in B, and then once it's a ring extension,

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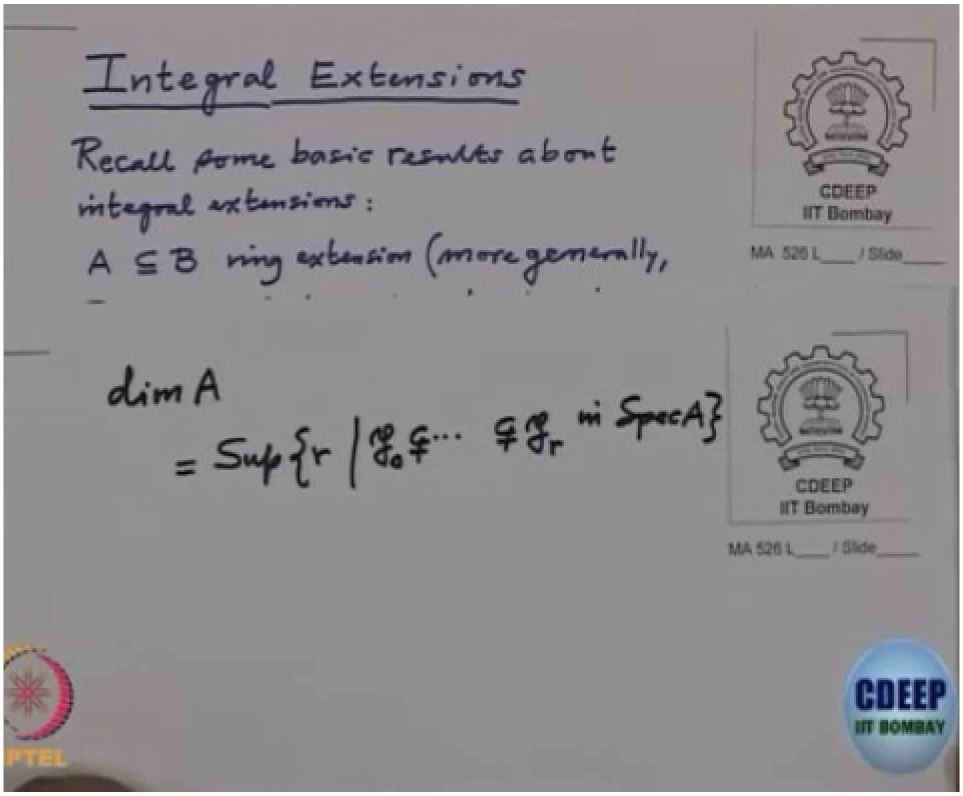

also sometimes you don't need any inclusion, you can only look at the ring homomorphism and instead of saying integral extension, one can say that the homomorphism is integral, and this is actually a generalization from the algebraic field extensions, so where the rings are not fields but arbitrating, but we still have to assume they're commutative.

So an element x is integral over A, if it satisfy equation of integral dependence, equation of integral dependence is a monic equation in x, and when you substitute the variable equal to the small x it becomes 0,

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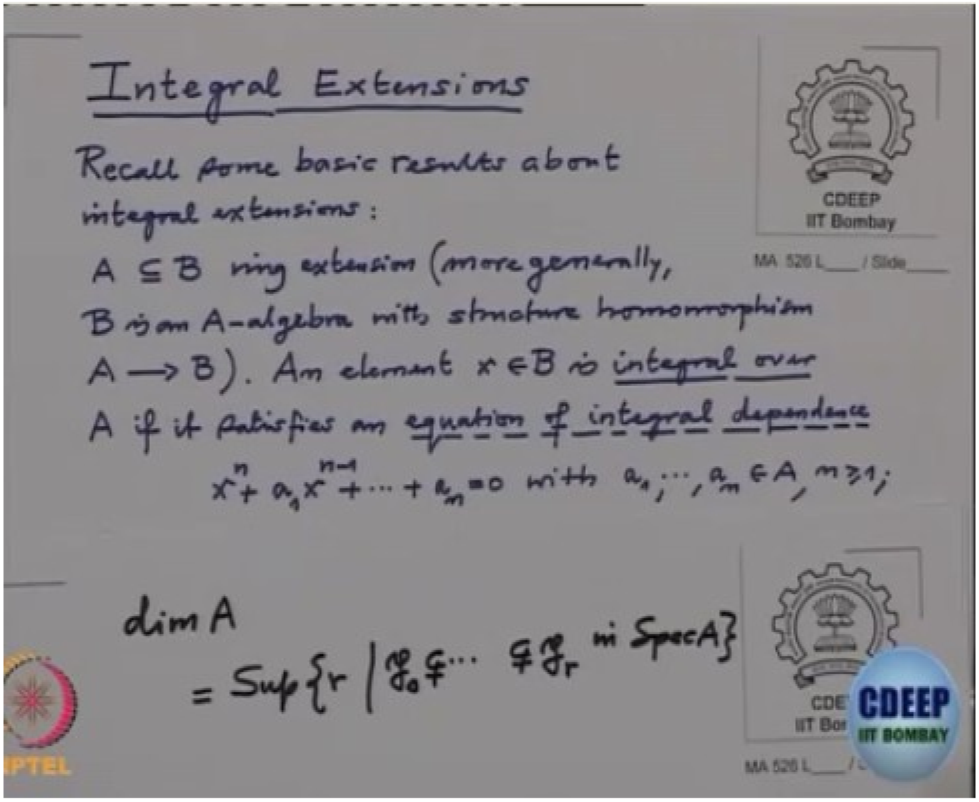

coefficients are in the ring A where in the case of homomorphism this obviously this means that you take A, the coefficients are coming from A means there images of A under this homomorphism, this is equivalent to saying that A sub algebra Ax of B, this is a finite mod M, it is generated by $x,\ldots,x^{n-1}$, in fact it is a free A algebra.

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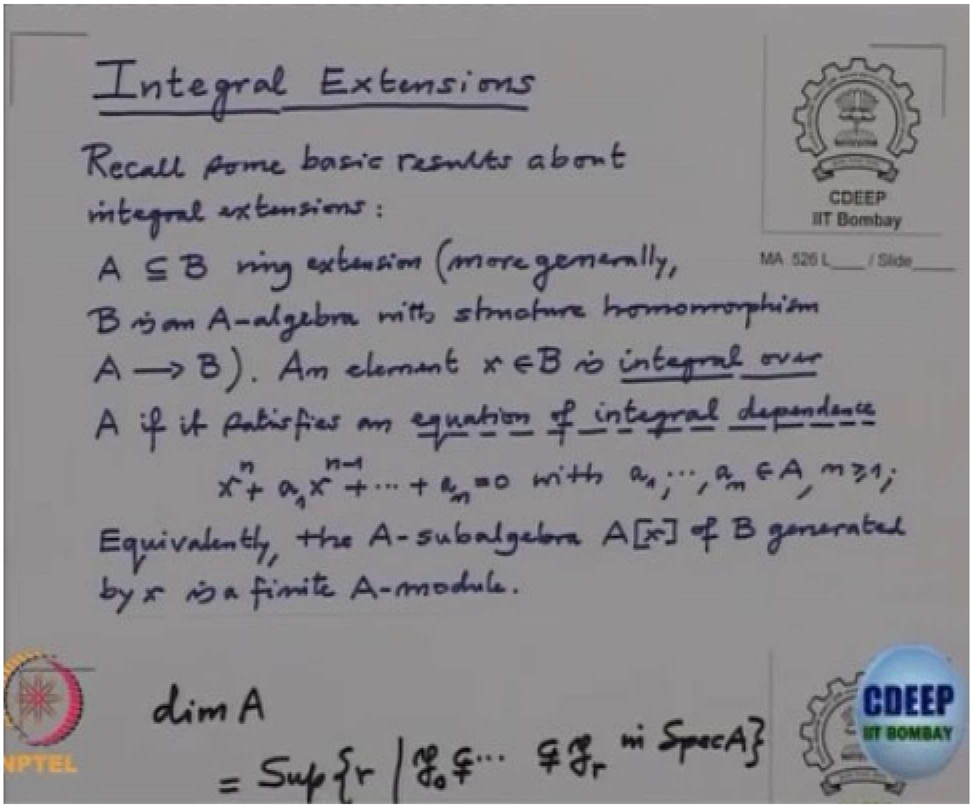

In this case when we say, when B is every element of B is integral over A, then one says that B is an integral extension of A, so this, in case of integral extension finiteness, a finite type and finite are these two concepts are equivalent, so I mean to be more precise integral extension which is finite type, this is equivalent to saying it's a finite extension, finite means it's a finite as a module, okay.

Now some 2, 3 very important results which we'll keep using in this course and hence after I will not explicitly state them,

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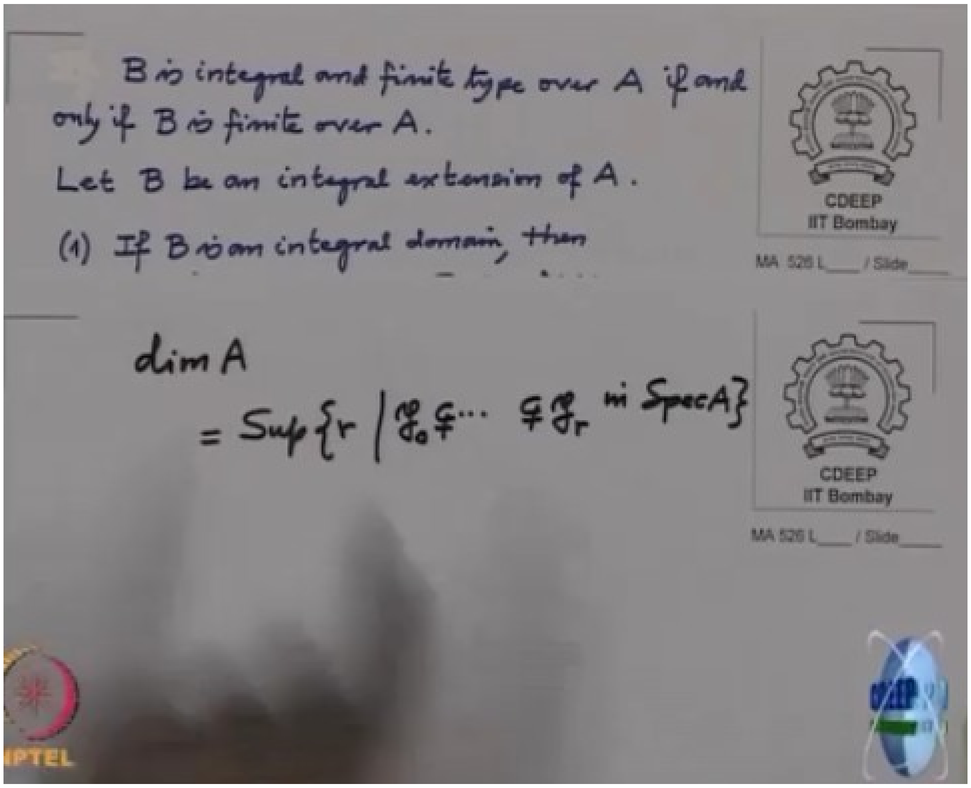

so for example if I have an integral extension where upper ring is, upper ring is an integral domain then it's a field if an only the base ring is the field, if I have prime ideal Q in B, then the contraction is also prime ideal that is always true, but when upper ideal is maximal then the contraction is also maximal. If I have two prime ideals in B, one contained in the other and after contraction to A if they are equal then they are equal, these are very basic facts which you would have learnt in your earlier courses.

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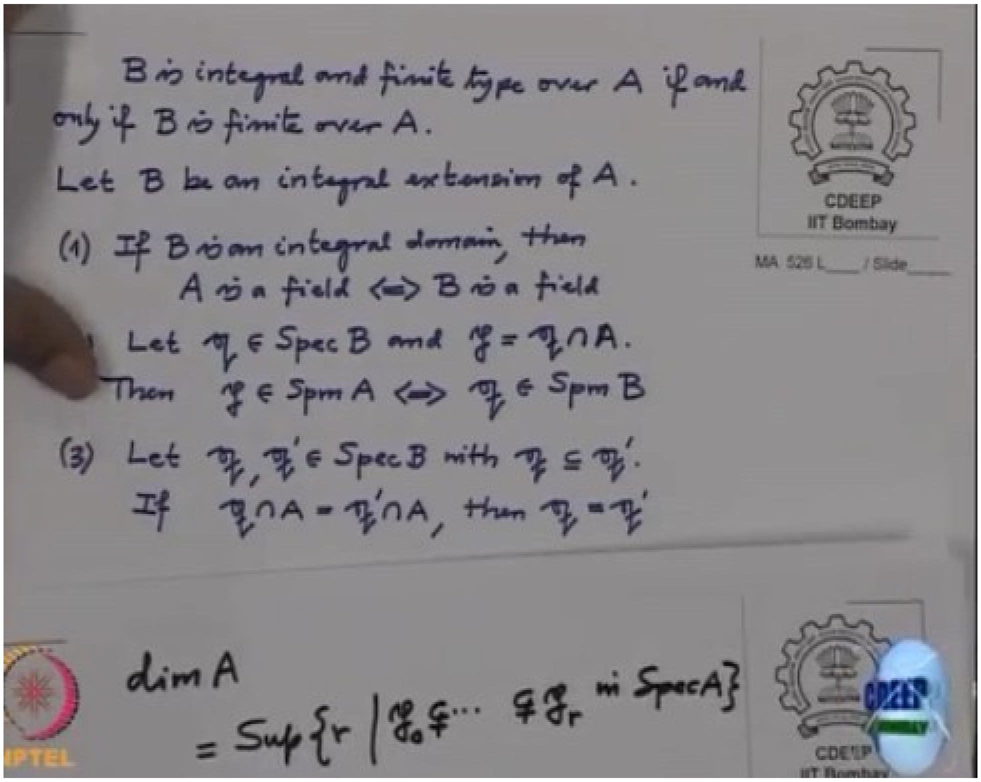

And now the two most important theorem which are used often,

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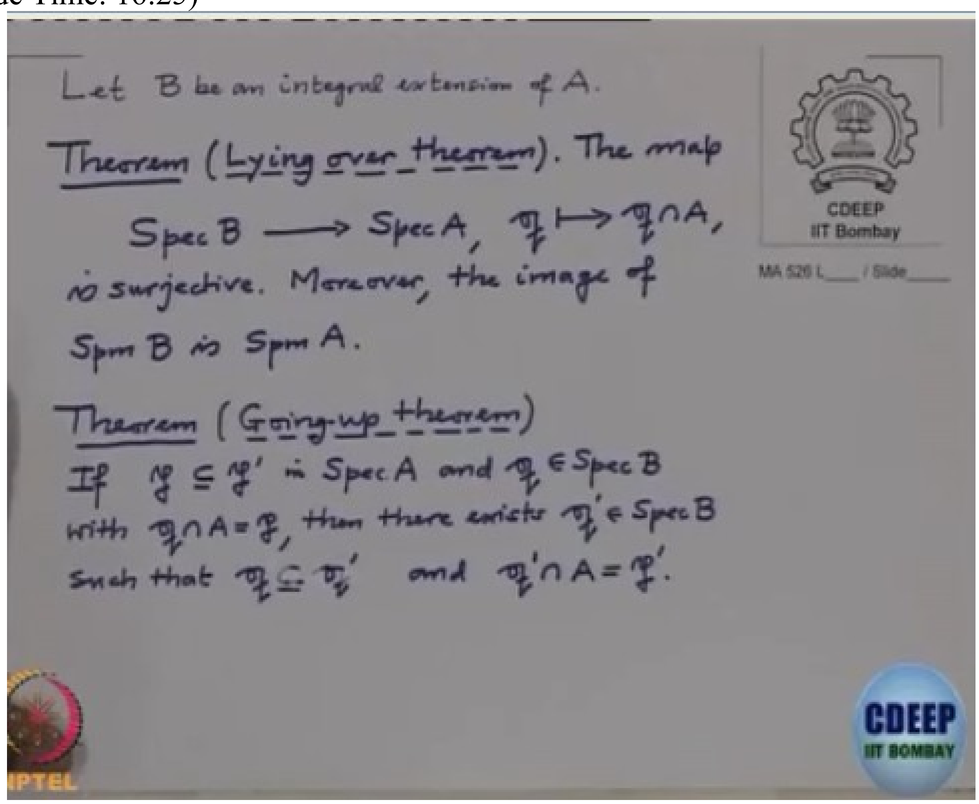

this is called a line over theorem for integral extension, this is a map contraction is a map, think of contraction a map from spec B to spec A,

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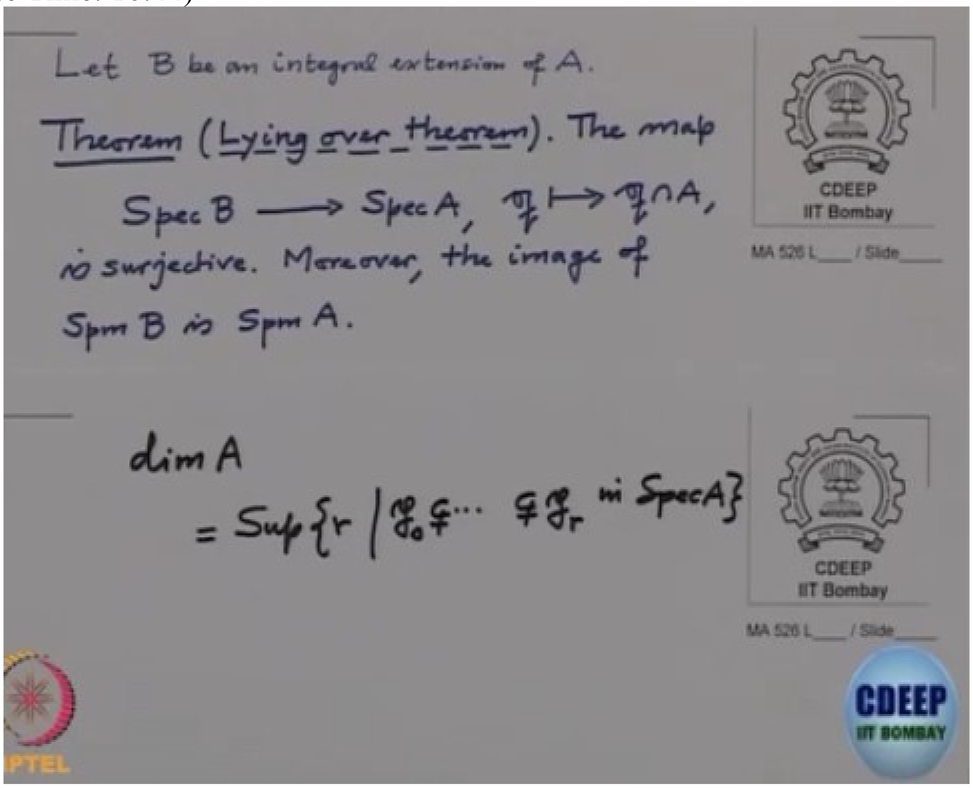

this map is subjective, this is what the integral will imply, this is just, that means given a prime ideal in A there exists a prime ideal in B so that the contraction of this is precisely the given prime ideal in it.

The next one is what is called as a going up theorem, if I have two prime ideals in A, one contained in another, and I have a prime ideal in B which lies over the first one, then I can find a prime ideal Q in B which lies over the next one and also it contain the given one which is, you can always find this subjectivity will tell you can always find a prime ideal which lies over there, but it should also respect the inclusion, so this is called going up theorem, okay.

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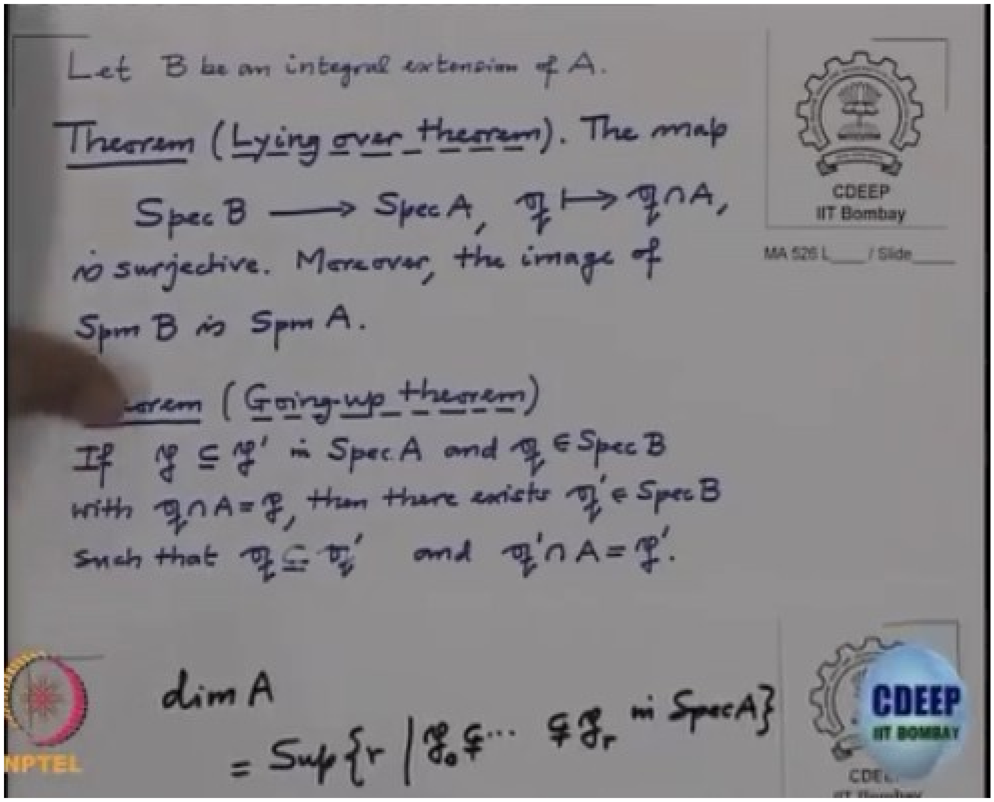

Now with this we are ready to prove that we have integral domain, integral extension of rings, then there is two dimensions are equal, so what do I have to prove? We have to prove that this is supremum of the chains for the prime ideals in A, this is supremum of the chains of prime ideals in B, so I will show you one, first I will show you the dimension of B is less equal to dimension of A, so you start with the chain in B, Q_0 to Q_r a chain in spec B and you just contract it to A, and only we have to check that these are prime ideals which is clear, and also we have to check that the inclusion remains, the properness of inclusion remains that follows from the, one of the result what we stated, if you have two prime ideals, if they are not contained up, they are not equal up then there contraction also not equal. So that shows, now taking over the supremum that show that the supremum, so that is the dimension of B less equal to dimension of A.

Conversely now I have to show that if I have a chain in the spectrum of A, P_0 contained in not equal to P_1 etcetera, P_r ,

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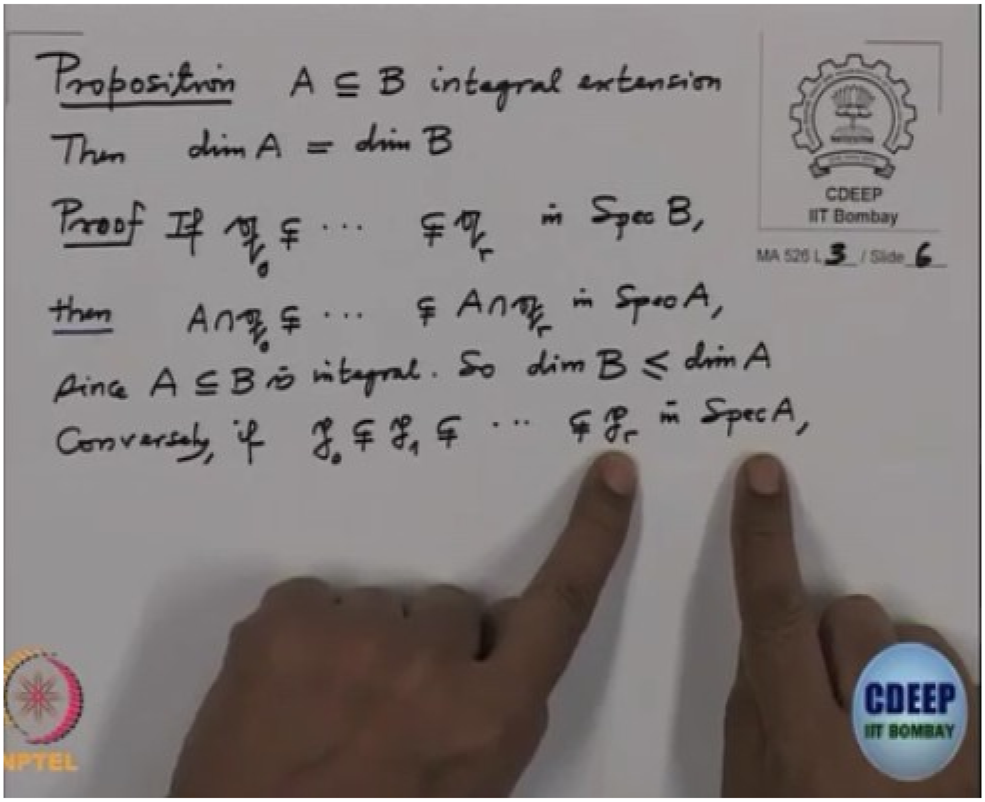

then I want to produce a chain of the same length R in spec B, but that I will use, first with subjectivity of the spectrum, so given this P_0 I choose Q_0 which is lying over P_0 , and then now I use a going up theorem to choose Q_1 which lies over P_1 and which contains Q_0 , this is going up, and keep going up precisely for this reason this theorem is known as going up theorem, so with this we have a chain in spec B, and therefore dimension is less equal to dimension B altogether dimension A equal to dimension B, for integral extension dimensions are equal.

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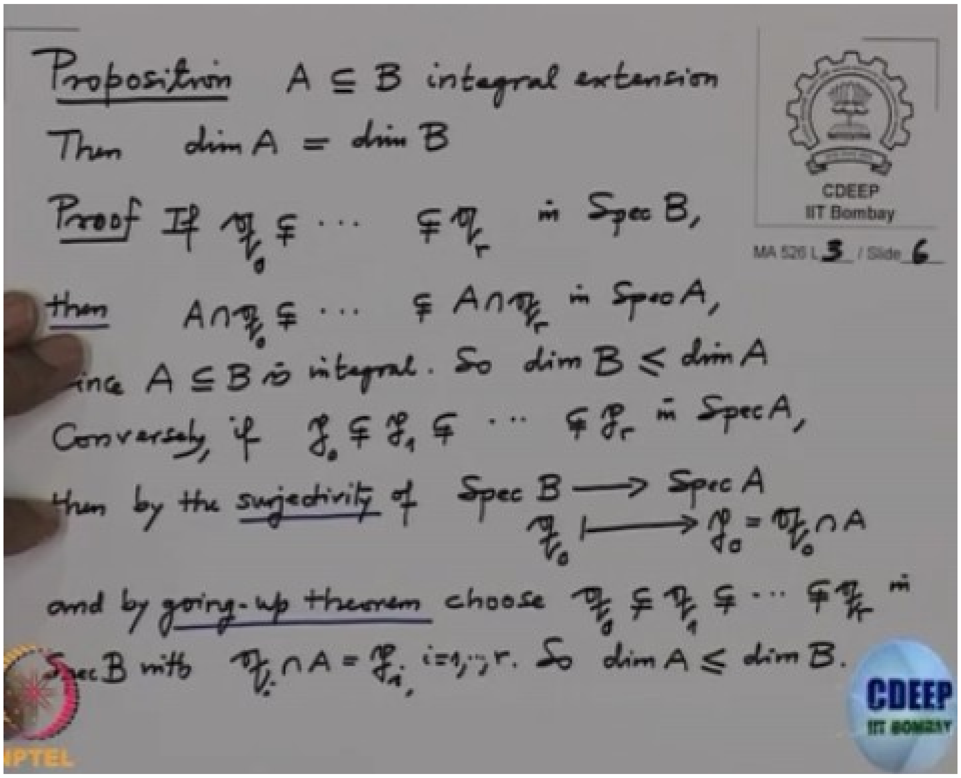

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