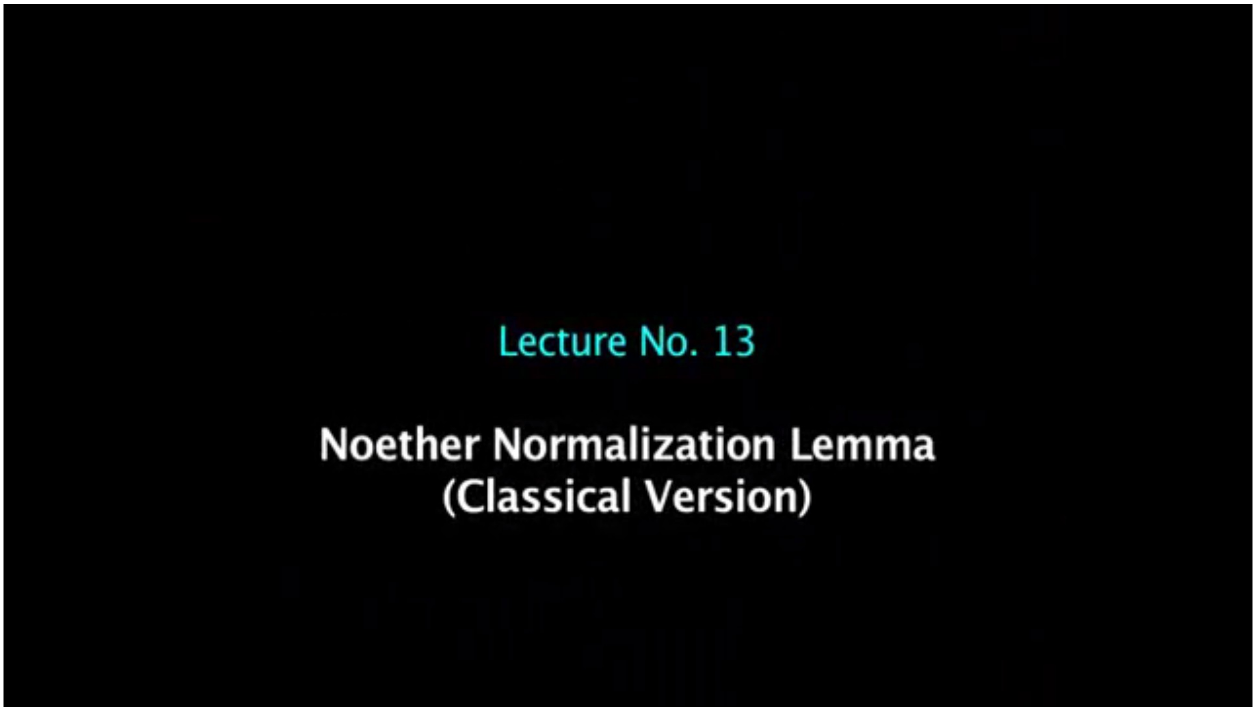
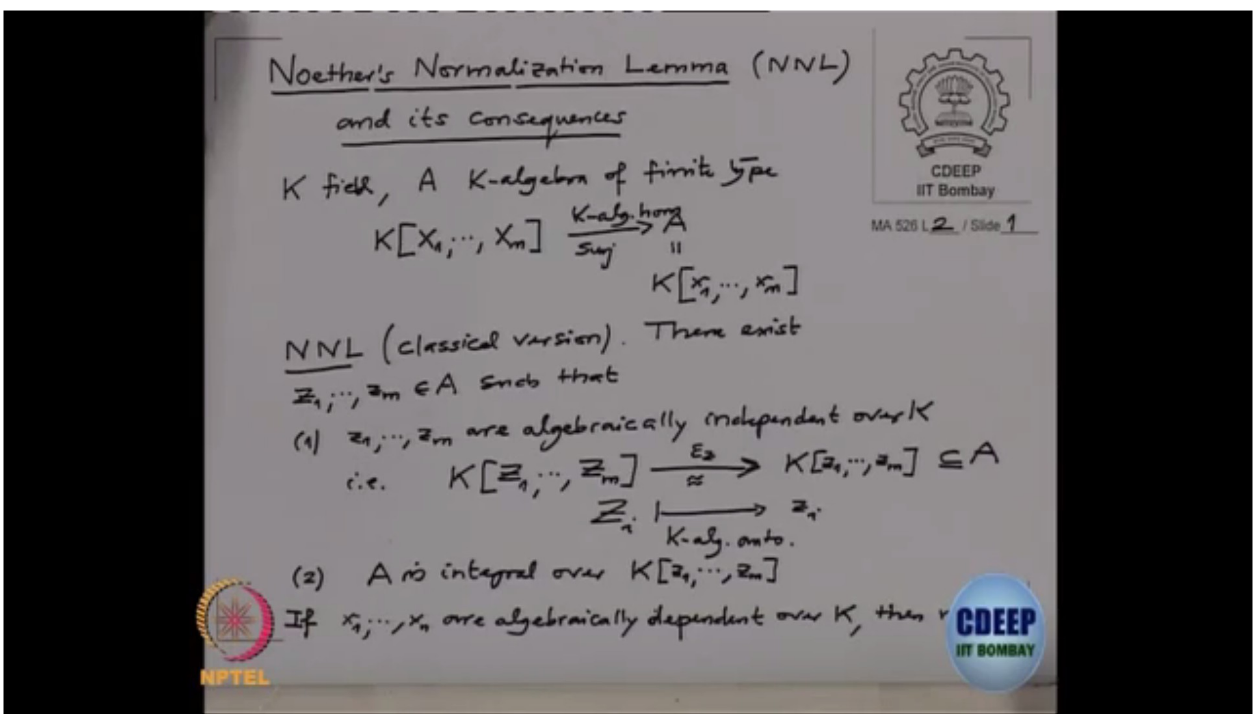


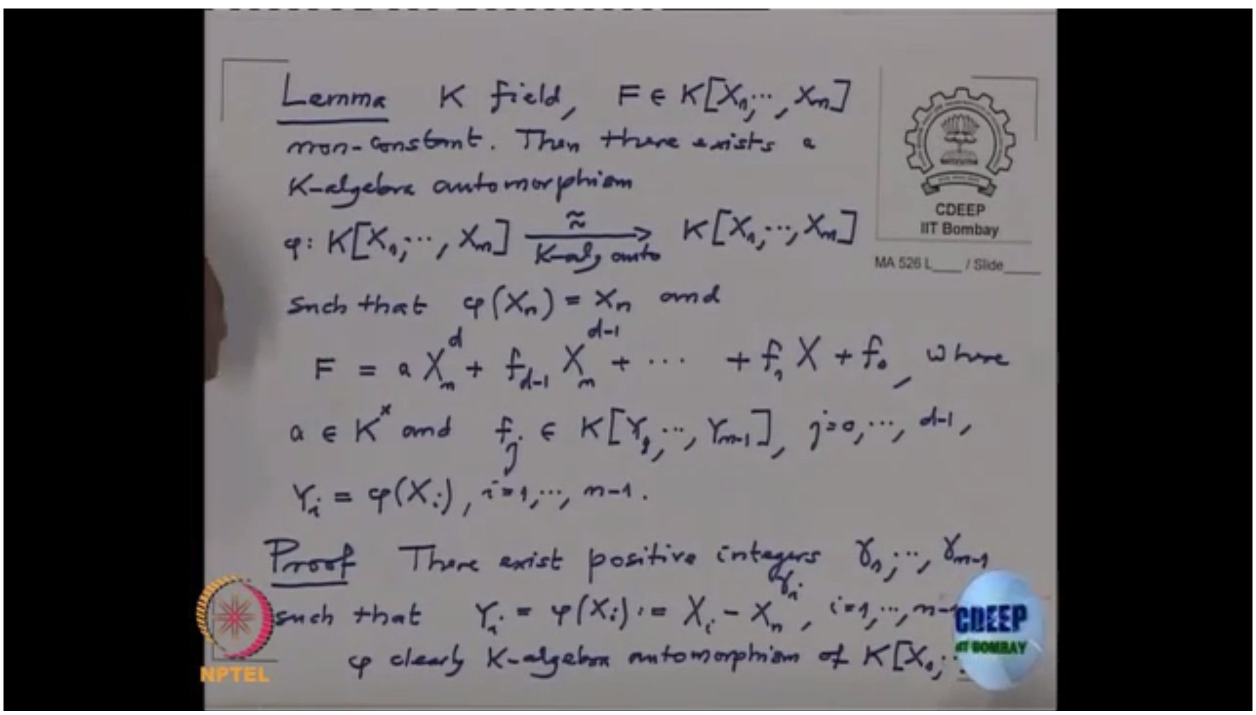
Prof. Dilip. P. Patil: Today's module is Noether's Normalization Lemma and its consequences. This is one of the most fundamental theorem in finite algebras, and we will see it has many consequences. So in today's lecture, I will finish the proof of Noether's Normalization Lemma, which I will abbreviate NNL, and today I will prove classical version of normalization lemma.



So the setup is like that, so K is always a field and a K-algebra of finite type. Last time I told K-algebra of finite type simply means that there is a subjective K-algebra homomorphism from the polynomial ring in finitely many variables to A, subjective, K-algebra homomorphism subjective. So this means that this A is generated by -- the K-algebra generated by the images of X, so those I will denote by x. So the normalization there exists -- this is a classical version, and then next time we will more general than this.

So with the same notation, there exists elements $z_1, \dots, z_m \in A$ such that, 1) $z_1, \dots, z_m$ are algebraically independent over K. It simply means if you take the map from the polynomial algebra in m variables to the sub algebra generated by z_i, this is a notation for -- this is the K sub algebra of A generated by $z_1, \dots, z_m$. That simply means this is the smallest sub algebra of A which contains $z_1, \dots, z_m$, and this map is an evaluation map. ϵ_z, which is Z_i goes to z_i. Algebraically independent means this map is an isomorphism, K-algebra automorphism.

So that is one. The second is, A is integral over $K[z_1, \dots, z_m]$. Integral means, even element of A satisfies the monic equation over this sub algebra. Moreover, if $x_1, \dots, x_n$ are algebraically dependent over K, then this m is strictly less than n. So this is what we will approve today.

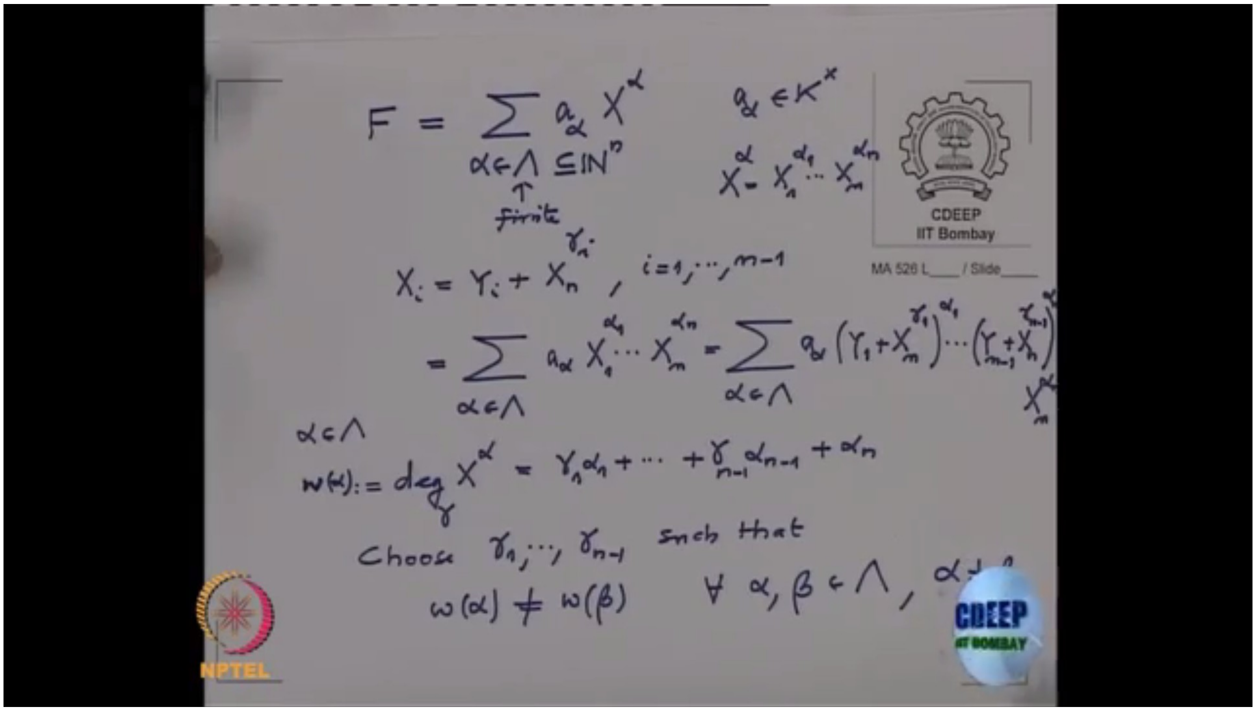


To prove this, I will need one key lemma, which has also independent interest. So let us -- first, me prove a lemma. We will prove this after we finish the proof of the lemma. So the lemma we need is K is a field and suppose I have a non- constant polynomial F in several variables, $K[X_1, \dots, X_n]$ non-constant. Then there exists a K-algebra automorphism of the polynomial algebra $K[X_1,\ldots,X_n]$ to itself, K-algebra automorphism, such that it fixes the last variable. Let us give the name for this, which is $\phi, \phi(X_n)=X_n$, and this given polynomial F becomes monic in X_n, F becomes looks like a subcontinent $X_n^d+f_{d-1}X_n^{d-1}+\ldots+f_1X_n+f_0$ where this a is a non-zero constant.

I will use this notation for non-zero elements of the field, K^x and this f_j are in $K[Y_1,\ldots,Y_{n-1}]$ where this Y_j is -- this is for any $j, j=0,\ldots,d-1$ and Y_j are nothing but images of $\phi(X_j)$. This is for $i=1,\ldots,n-1$. So it's very simple that given any non-constant polynomial in several variables we can make it monic in one of the variables by changing the variables, and when one say change of variable, that simply means under automorphism. Sorry. No, but F is here, see the same F here. Okay, so after proving the lemma, we will see something more about the automorphism and this one. So let us prove the lemma first.

So proof, proof of the lemma is very simple. Let's see. So I claim that there exists natural number or positive integers $\gamma_{1,}\ldots,\gamma_{n-1}$, such that -- yeah, I want to give an automorphism, so I just have to give values on X_i such that they are subjective -- such that $Y_i=\phi(X_i)$, this I want to define it as $X_i-X_n^{\gamma_i}, i=1,\ldots,n-1$. So if define ϕ like this, $\phi(X_i)$ is this, and of course X_n we wanted fixed. First, observe that this φ is an automorphism. That is simple, because once -- you only have to prove it is subjective, because subjective K-algebra automorphism will also be injective, because both of these are Noetherian rings, and subjective automorphism from a Noetherian ring to itself is always injective. That you must have seen in the first course of this variable. If you have not, simply check that.

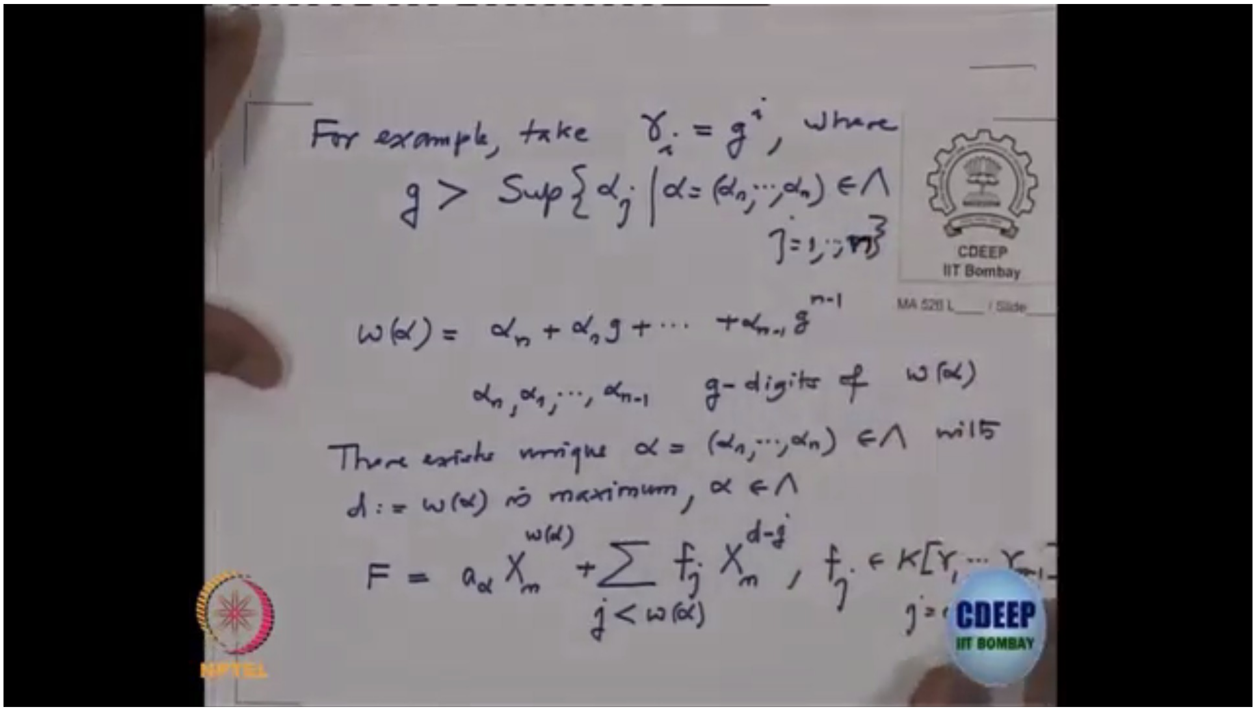
So it's clear that X_n is in the image and X_n is in the image, each X_i is in the image. Therefore, it is clearly automorphism. So φ is clearly K-algebra automorphism of $K[X_1,\ldots,X_n]$. And now we just have to check that if I plug it in in this F, if I plug this Y_i, right X_i-Y_i plus this and just plug it in F, and we will check that then it becomes monic in X_n.



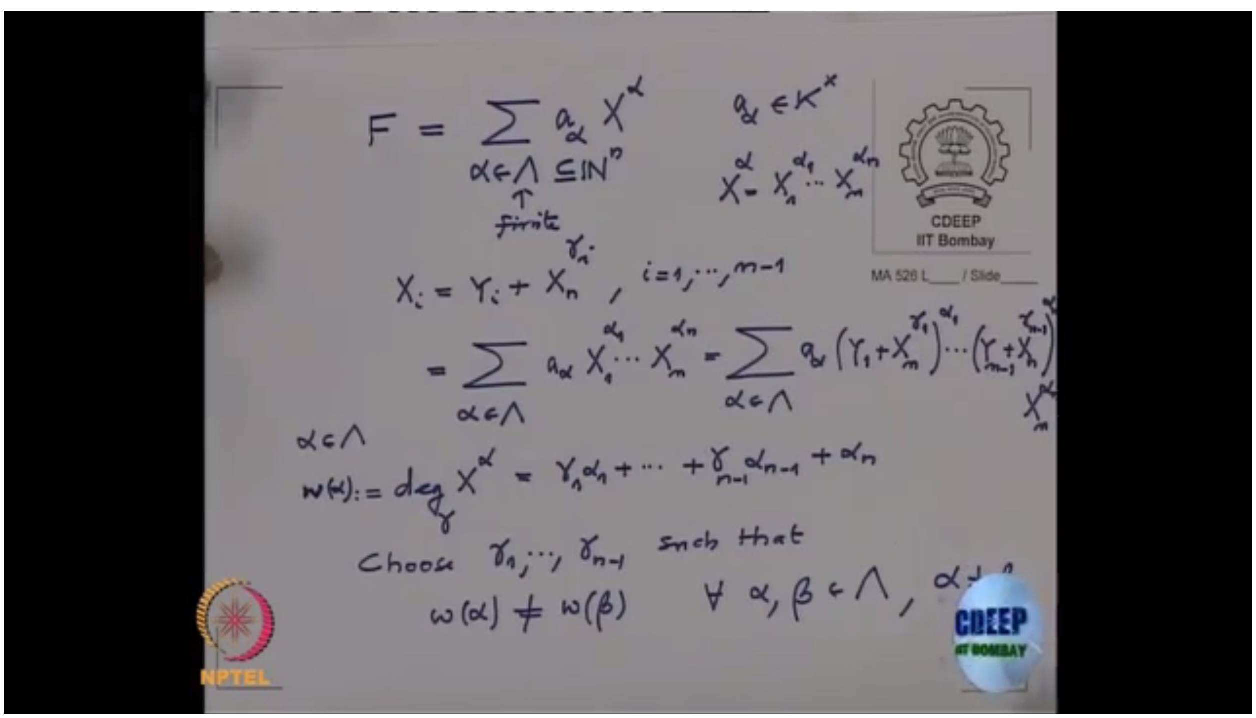
So let us it do it quickly. So that is very easy. So F is a polynomial, so F is a finite sum of monomials. So F is therefore summation, summation is ranging over finite monomials, so $a_\alpha X^\alpha$ and α is finitely many indices, right, so α is in this, right, this is a finite subset of $\mathbb{N}^n$, this is a finite subset, and that a_α are elements in K^x, and standard notation we use, X^α means $X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, that is just standard calculus notation, okay.

So now we want to put $X_i = Y_i + X_n^{\gamma_i}$ for $i = 1, \ldots, n-1$. So this will become, this X will become $\sum_{\alpha \in \lambda} a_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, so this will become $\sum_{\alpha \in \lambda} a_\alpha (Y_1 + X_m)^{\alpha_1} \ldots (Y_{n-1} + X_m)^{\alpha_{n-1}} X_n^{\alpha_n}$. This is what just plug it in.

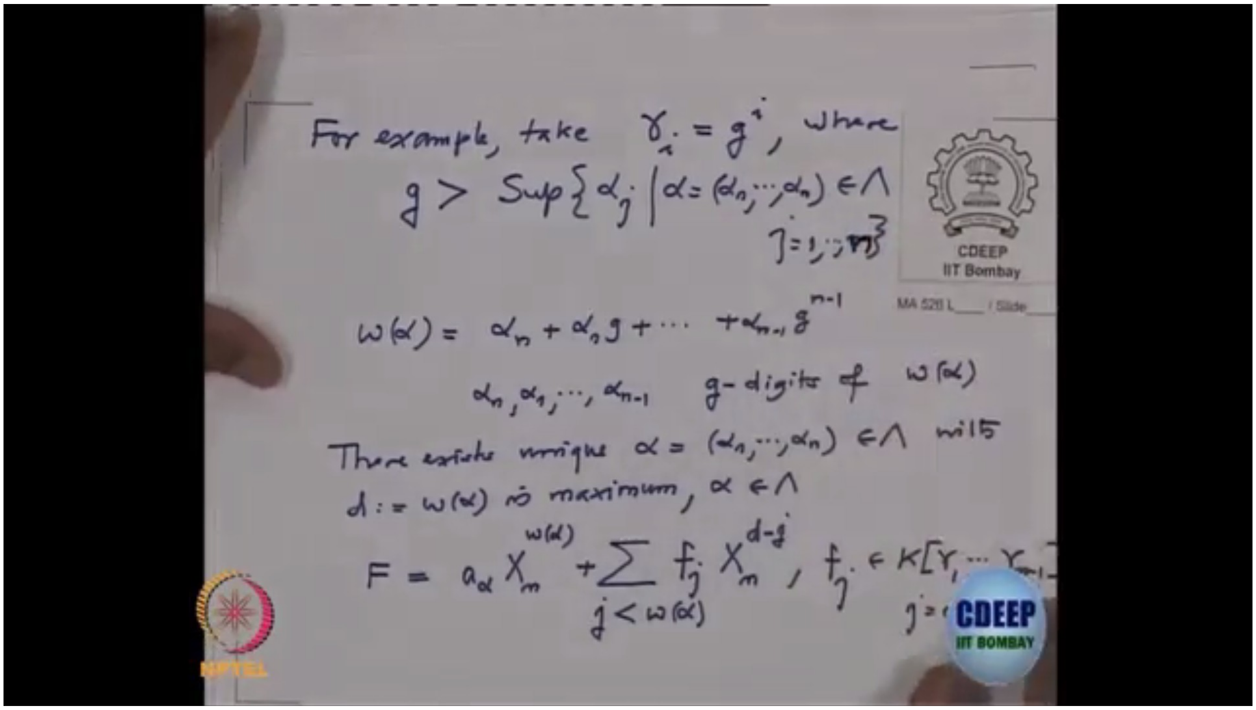
So when we expand by binomial what will you get the coefficient here? So let us put $\deg_\gamma X^\alpha$, this is the weighted degree where you give weights to $X_1^{\gamma_1}$ and $X_{n-1}^{\gamma_{n-1}}$ and last one is 1, X_n is weight 1. So this is $\gamma_1 \alpha_1 + \ldots + \gamma_{n-1} \alpha_{n-1} + \alpha_n$. This is the way the weighted degree. So if you do that -- then if you expand it -- so I want to choose. See we want to show that there exists $\gamma_1, \ldots, \gamma_{n-1}$. So I want to choose, so choose $\gamma_1, \ldots, \gamma_{n-1}$, let us -- I want to give this abbreviated, let us call it $\omega(\alpha)$. So for each $\alpha \in A$ we put $\omega(\alpha)$ equal to this expression if you like. That is called deg_γ of the monomial X^α. And we choose $\gamma_1, \ldots, \gamma_{n-1}$ such that all these $\omega(\alpha)$ are different for α, β which appear in this polynomial and different α. So this is what we have to justify.



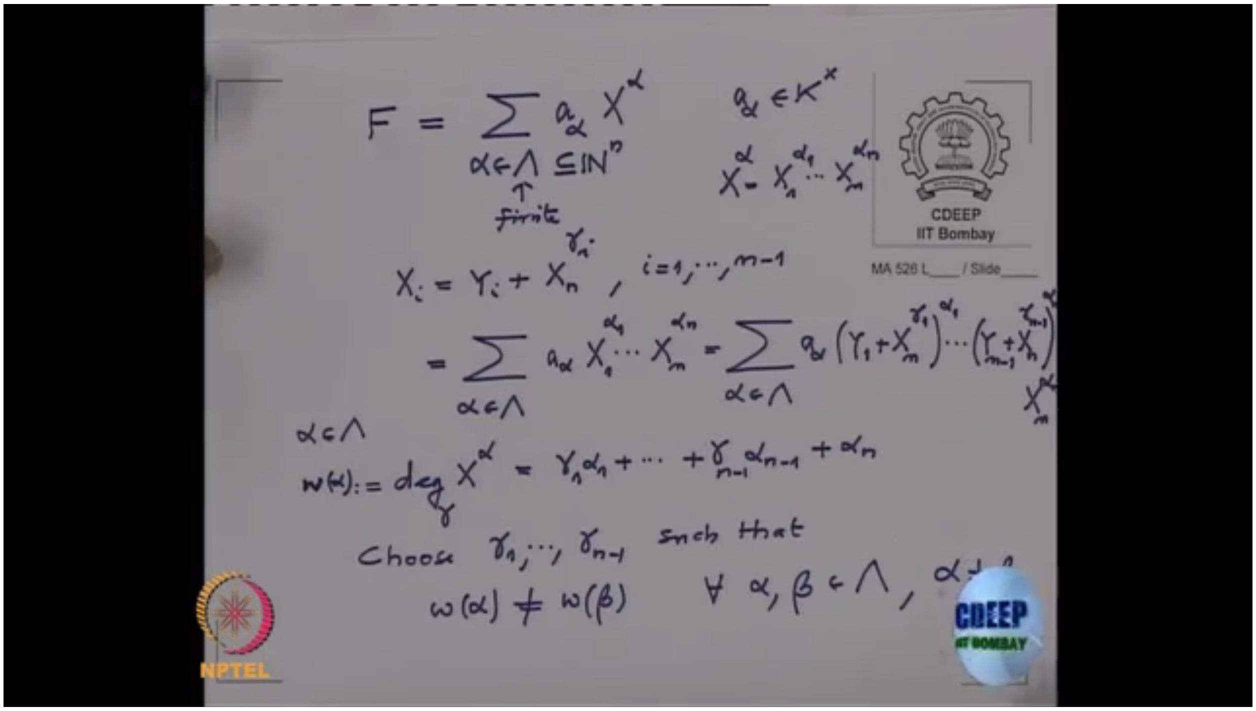
So this is possible is because what we do is, for example, take γ_i to be g^i, where g is bigger than all the components of all the α which are appearing here. So this is where g > supremum of, this is a finite { $\alpha_j; \alpha=(\alpha_1,\ldots,\alpha_n)\in A_j$ }. So in this Γ, only finitely many tuples appear and they have finitely many components, and I choose a natural number which is bigger than everybody.



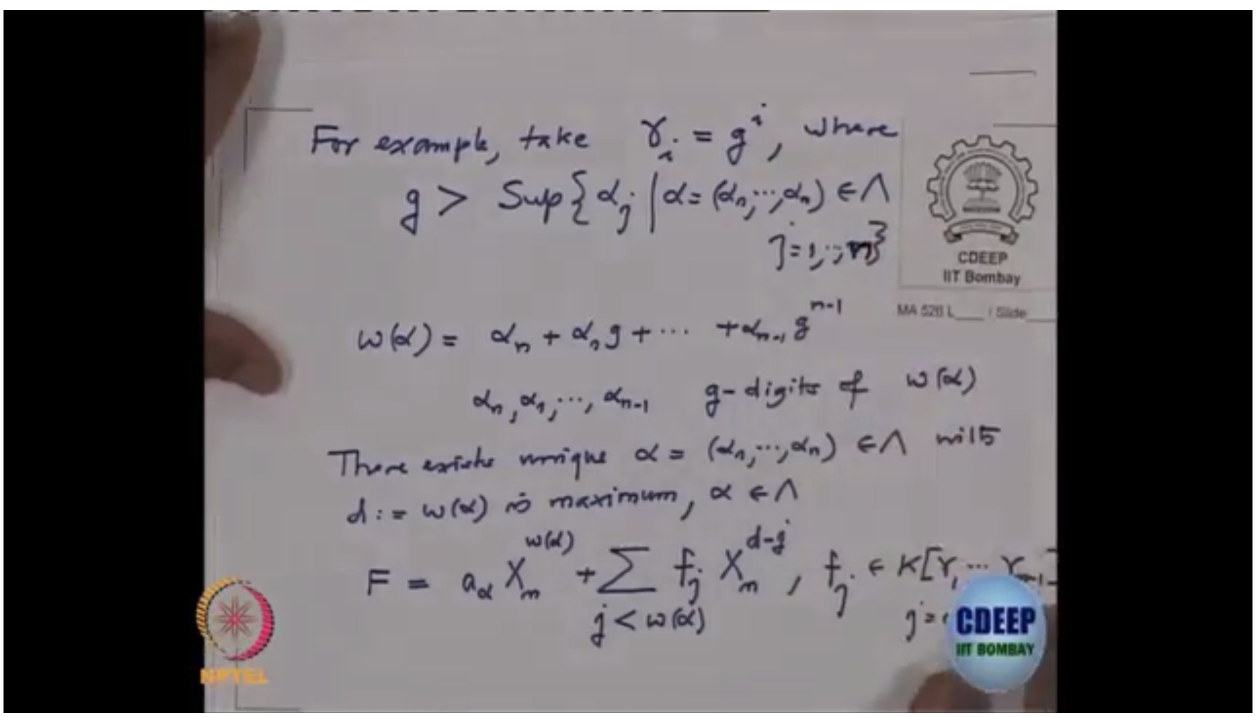
So now if you look at this expression, this will be a geodic expansion.



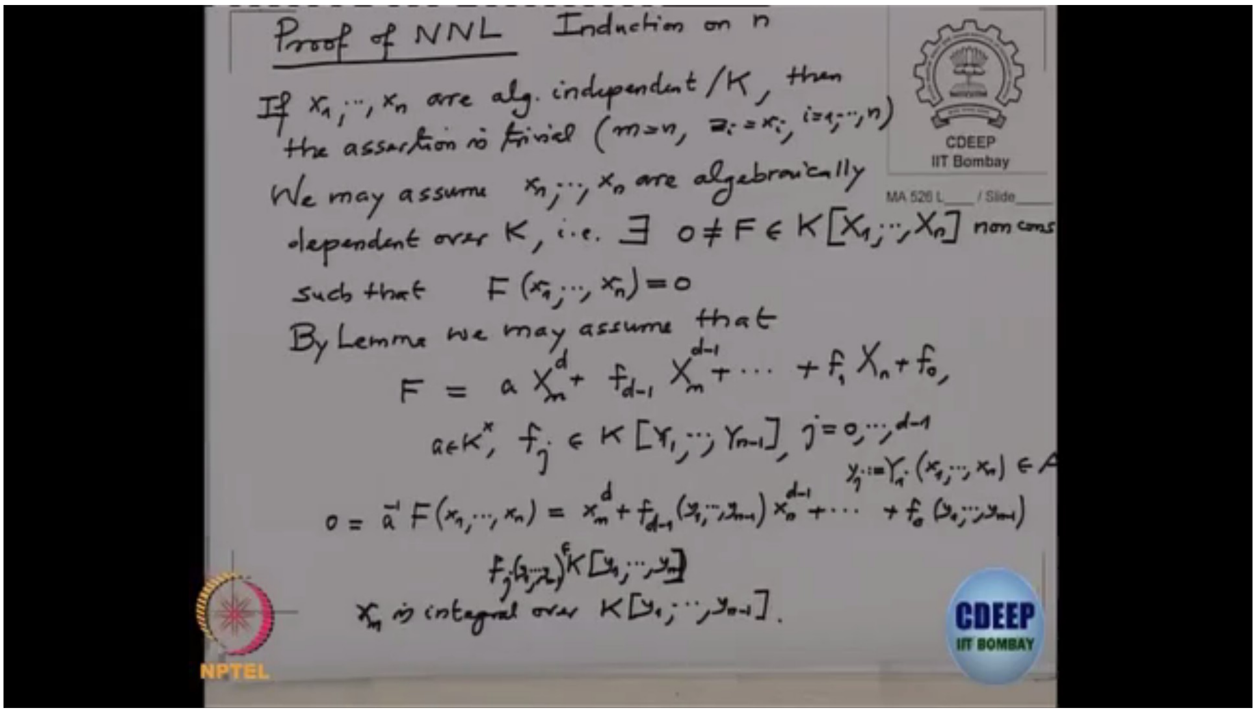
So when I take $\gamma = g^i$ that $\omega(\alpha)$ looks like -- I will write this term first. $\alpha_n + \alpha_1 g + \ldots + \alpha_{n-1} g^{n-1}$, so these α are much smaller compared to g, so they are nothing but the geodic g-digits, so $\alpha_n \alpha_1 \ldots \alpha_{n-1}$, they are g-digits of $\omega(\alpha)$. So when to $\omega(\alpha) = \omega(\beta)$, then the digits will be equal, but the $\alpha = \beta$, right. So we can always choose, because there are finitely many components.



So therefore, in this expression, only one maximum will come, because all the monomials have different degrees, this γ degree. Therefore, only one is finite set and all elements are different. So when you compare only one will be maximum.



So that means there exists unique $\alpha=(\alpha_1,\ldots,\alpha_n)\in A$ with $d:=\omega(\alpha)$ is maximum. Where $\alpha\in A$. So therefore, when I plug it in, you will write F will look like coefficient of that maximum a_α and then $X_n^{\omega(\alpha)}+\ldots$ now the lower degree terms, so that will be $\sum_{j<\omega(\alpha)}$ and some coefficients that I will call $f_j X_n^{d-j}$, where f_j are now -- if you see this expression, when you expand it, you will get Y and those f_j are the polynomials in Y, $j=0,\ldots,d-1$. So That is what we wanted to prove, we can monic in X_n with -- coefficients have changed, but again those Y are again polynomials in X_i. So that proves the lemma. These one -- all α_j, all the components of α where α is wearing that finite set. No g is a fixed, bigger, a very big natural number. Oh here, yeah, n, thank you. Okay, so that proves lemma.

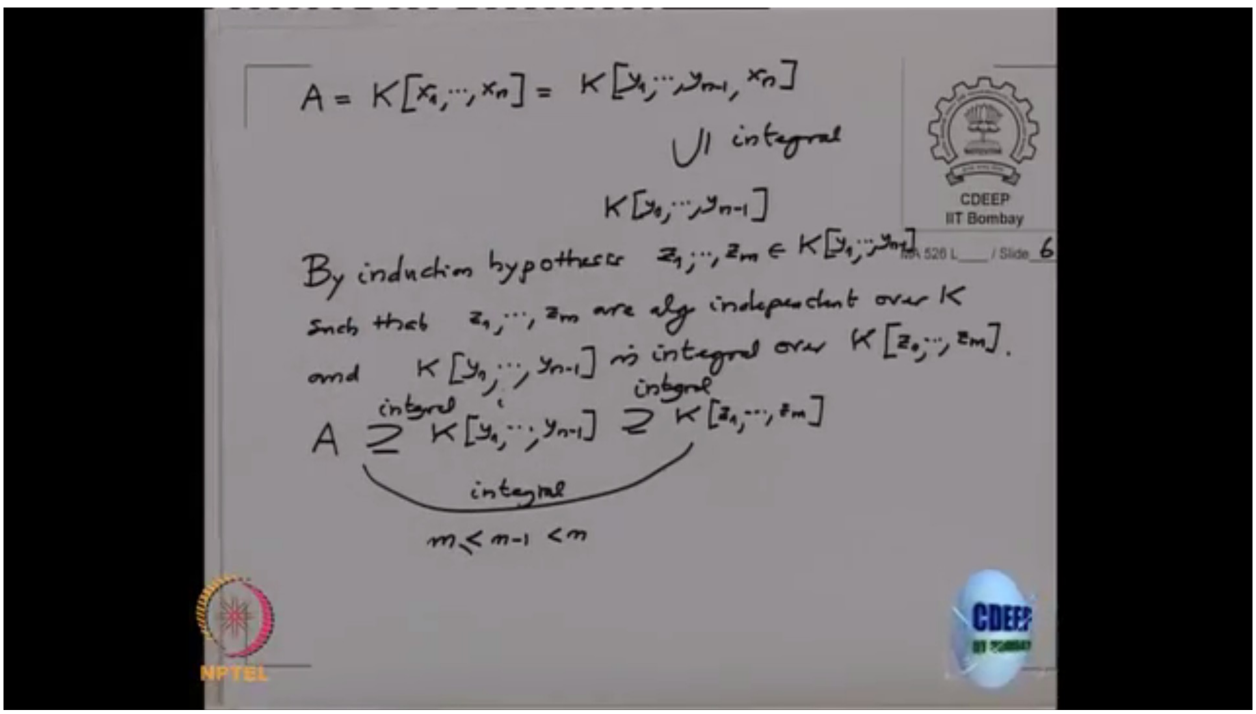


Now let us come back to the proof of the normalization lemma, classical so proof of NNL. So what do we want? We want to find elements $z_1,\ldots,z_m$ such that they are algebraically independent and A is integral lower than sub algebra generation by $z_1,\ldots,z_m$. So this, I am going to do it by, prove by induction on n. Remember n is number of algebra generated for the finite table algebra A. So if $x_1,\ldots,x_n$ are algebraically independent over K, then we have nothing to prove, because I will take m = n and $z_1,\ldots,z_m$ as $x_1,\ldots,x_n$, they are algebraically independent and when a is integral over A that is obvious. So if $x_1,\ldots,x_n$ are algebraically over K, then the assertion is trivial. Simply take (m=n, $z_i=x_i$).

So we may assume, they are not algebraically independent, that means they are algebraically dependent. So we may assume $x_1,\ldots,x_n$ are algebraically dependent over K. That means there is a non-zero polynomial in n variables, so when I plug it in the variables equal to this element, when I substitute, then it becomes a 0 polynomial. So that is there exists non-zero polynomial F in n variables $K[X_1,\ldots,X_n]$ such that $F(x_1,\ldots x_n)=0$. Okay, so I have these non-zero polynomials and non-constant actually. This will be actually non-constant, so non-constant. So this polynomial by using lemma I will -- by using the automorphism, I will change it, I will make it monic in one of the variables, monic in the last variable.

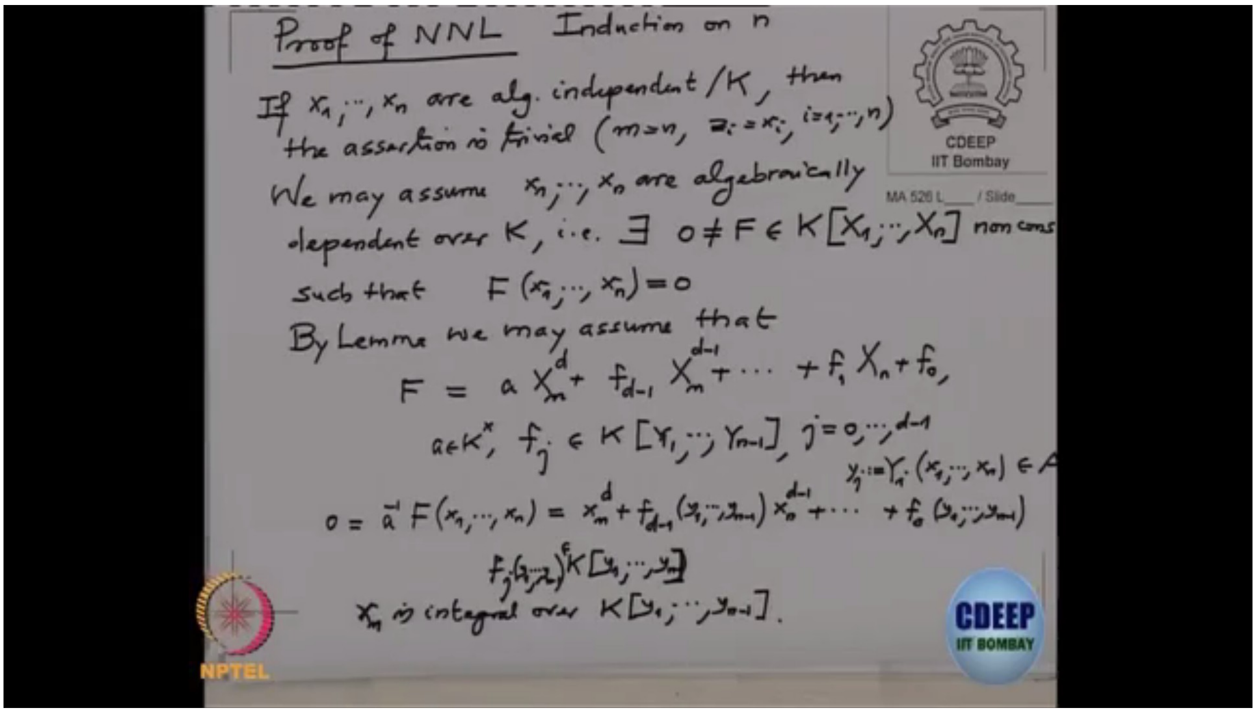
So that means by lemma we may assume that F looks like some non-zero constant, $X_n^d + f_{d-1} X_n^{d-1} + \ldots + f_1 X_n + f_0$, where these f_j are polynomials in $K[Y_1, \ldots, Y_{n-1}], j = 0, \ldots, d-1$ and Y_j s are the changes or images of those X_i s.

So when I plug it in now small xi these become 0. So I multiply by a^{-1} of F, this is a non-zero constant, $(x_1, \ldots, x_n)$. This is 0, because $F(x_1, \ldots, x_n)$ is 0. On the other hand when I plug it in what do I get? This a^{-1} I cancelled it, so it becomes monic in x_n. So x_n^d + now this Y_i, they are polynomials in X_i, so I can substitute in this x_i. This, I'll call it Y_j. They are the elements in A. So we will get here, $f_{d-1}(y_1, \ldots, y_{n-1}) x_n^{d-1} + \ldots + f_0(y_1, \ldots, y_{n-1})$, clear. So these coefficients are in the sub algebra generated by $K[y_1, \ldots, y_n]$, f_j we evaluate at $(y_1, \ldots, y_{n-1})$, they are actually elements here. So this satisfies monic xn, satisfies monic equation over that. So this simply means x_n is integral over $K[y_1, \ldots, y_{n-1}]$.

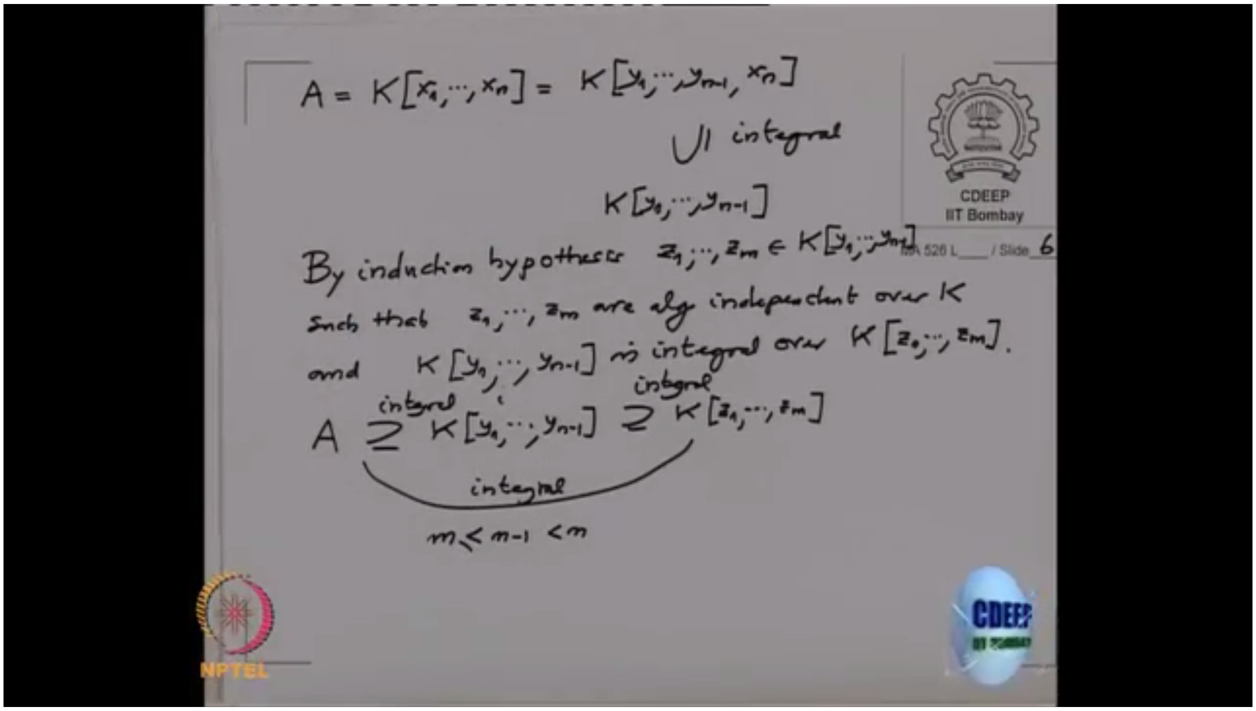


So we have A here, and A is generated by $K[x_1, \ldots, x_n]$, which was $K[y_1, \ldots, y_{n-1}, x_n]$, and x_n is integral, so this contains $K[y_1, \ldots, y_{n-1}]$ and this x_n is integral over this, so therefore, this extension is integral, because there is only one extra element and that is integral so this is integral extension. Now you see, I am going to apply induction to this, induction hypothesis to this new algebra which is finite type and generated by n-1 elements as an algebra.

So by induction hypothesis there exists elements $z_1,\ldots,z_m$ in this $K[y_1,\ldots,y_{n-1}]$ such that -- see what was our requirement. I just want to show you NNL -- such that they are algebraically independent, $z_1,\ldots,z_m$ are algebraically independent over K, and $K[y_1,\ldots,y_{n-1}]$ this is integral over $K[z_1,\ldots,z_m]$. This is an inductive hypothesis. So we have checked that A is integral over this and this is integral over this, so by transactivity of the integral extension, $A\supseteq K[y_1,\ldots,y_{n-1}]$ and this is integral over $K[z_1,\ldots,z_m]$. This is integral and this is also integral, so this one is integral. So that is what the second requirement, and just one thing that by induction this $m\leqslant n-1$, which is <n, strictly <n. So it's true the moreover part also.



You see moreover part is, I'll just bring it here. Moreover part is if $x_1,\ldots,x_n$ are algebraically dependent, then that m is strictly <n.



So that comes here.