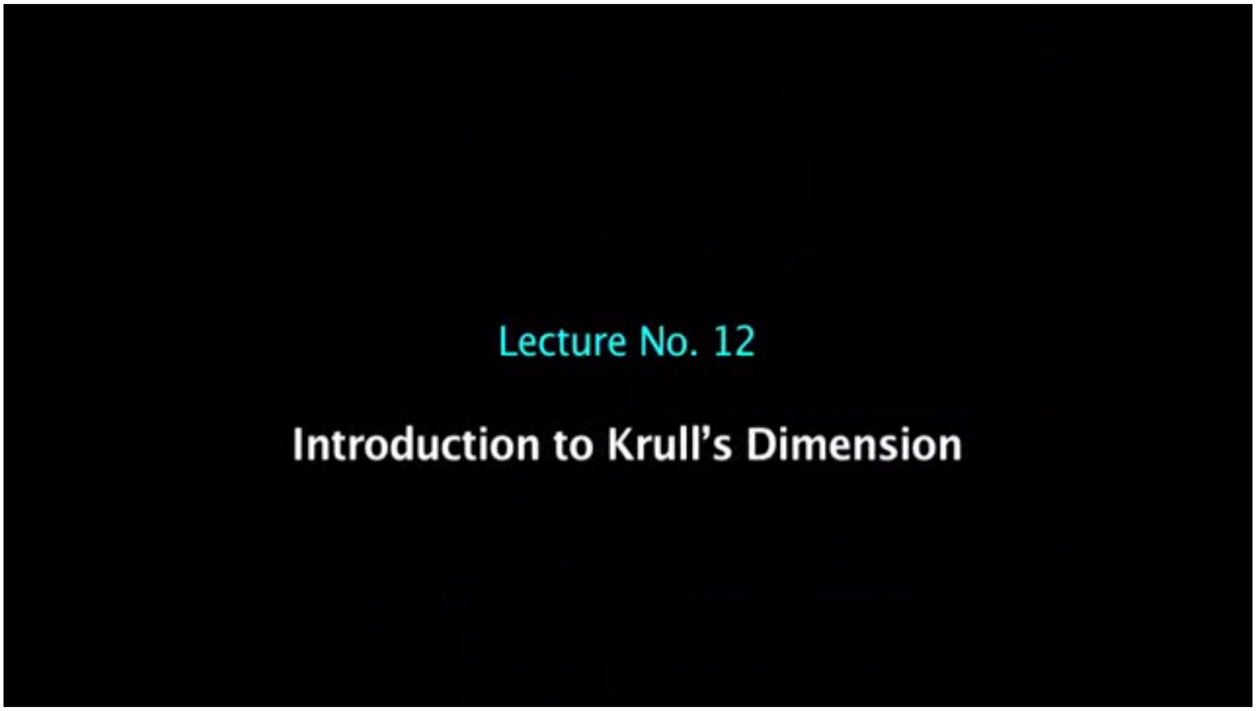
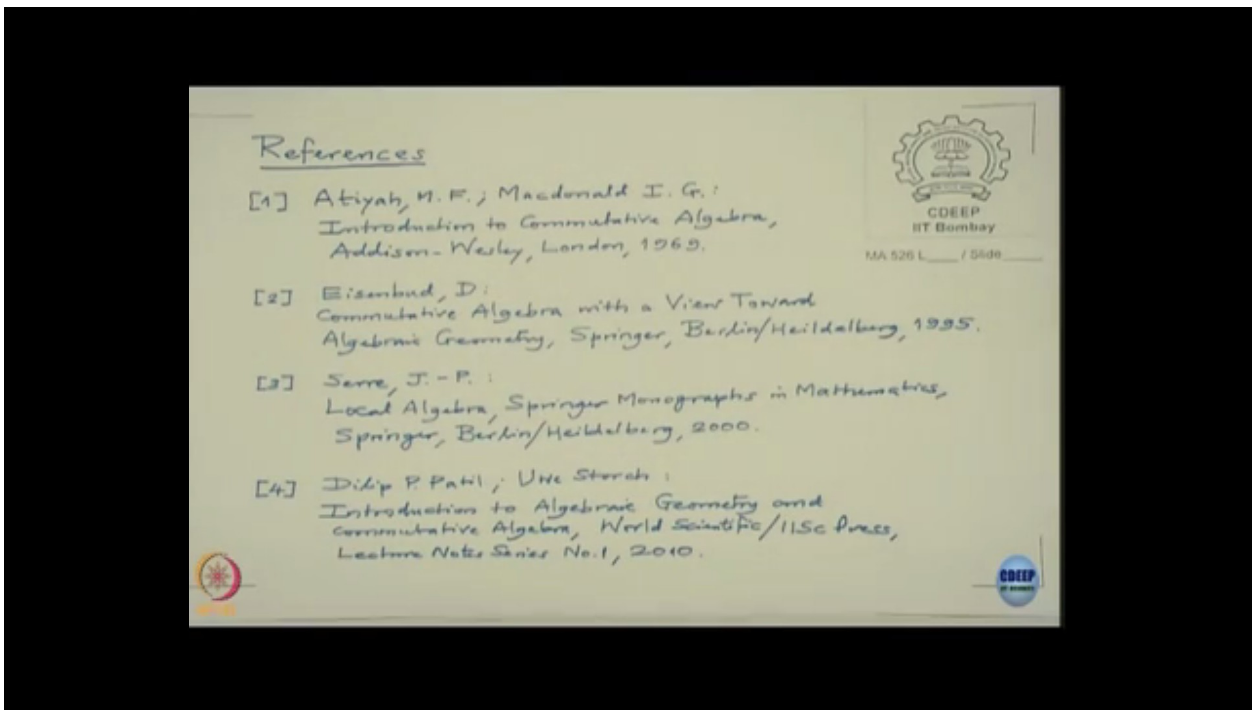
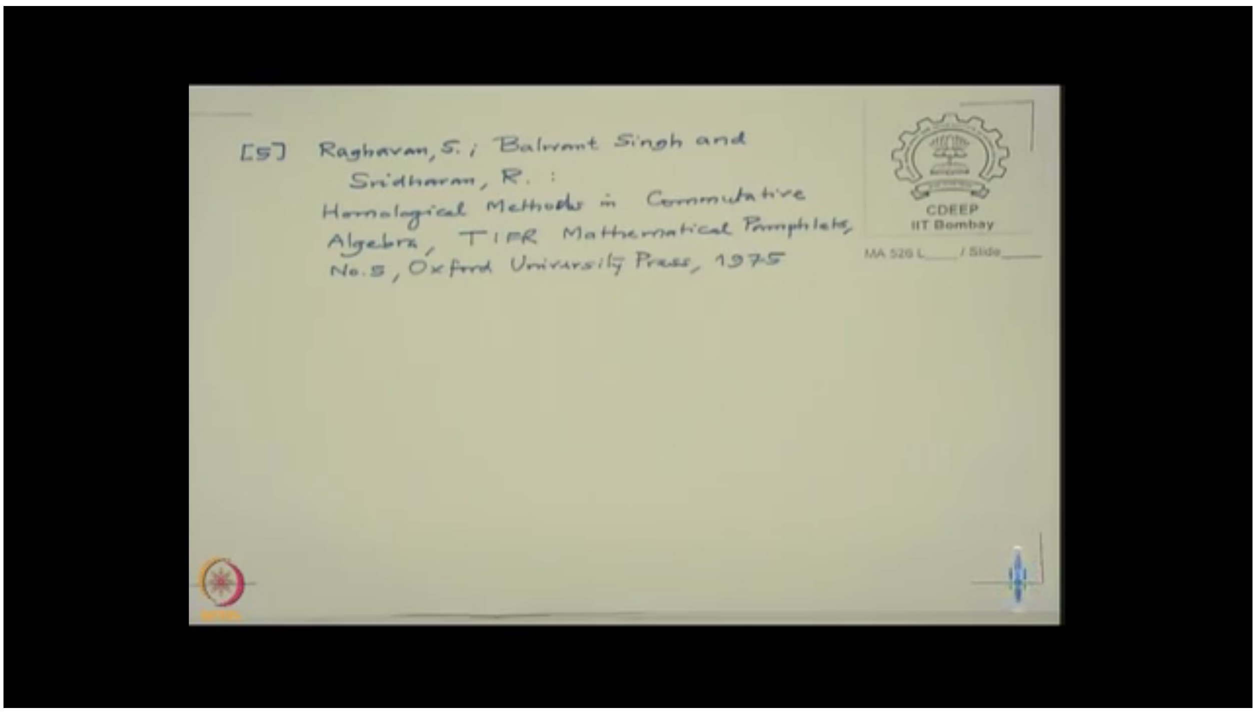




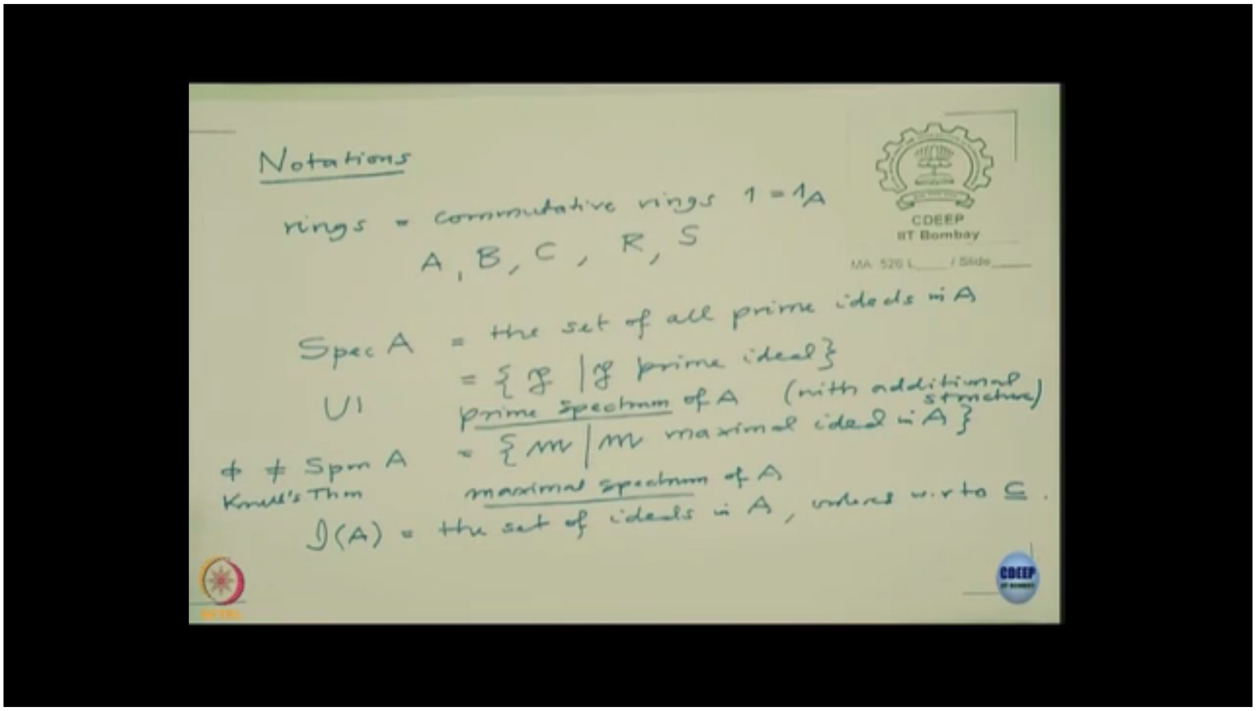
Prof. Dilip P. Patil: So first I want to give some references at least. So you would have studied probably in the commutative algebra 1, algebra 2, you call it algebra 2, right. I call it commutative algebra 0, let us say, right. So that is Atiyah & McDonald, Atiyah & McDonald is a very famous book, very, very well written and a lot of exercises. And the other book is Eisenbud Commutative Algebra, either with a view toward algebraic geometry. I myself have not read these books very much, because by the time I finished my studies, et cetera, these books were not there. So I will read it this time if possible, but when I read a few pages, I felt a little bit -- it’s not rigorous. So I don’t know.

I am not going to follow either Atiyah & McDonald or Eisenbud or any book which I have listed here word to word. I will keep changing the proofs et cetera, right. So another one is this book which is I co-authored with my senior that is Storch, and this book actually is -- first three chapters if you see, they are the preparation for proving a theorem called Reimann-Roch Theorem. So first three chapters are commutative algebra basically. It is very fast and many people told me it is very, very touched to read it. So basically, in this course, I am going to fill up more in between, and this is a good opportunity for me also, because I am preparing a second edition of this book.



Okay, and one more thing, when homological algebra, basic homological algebra, I will deal with later part of the course. At that time, I will also use this pamphlet, TIFR mathematical pamphlet for homological methods in commutative algebra. Actually, you can download these from TIFR website.

With that, let me start recalling some of the things which you probably know, but I want to be very sure, and also I want to take this opportunity to set up the notation. As you know, notation is very important, and that if you have a good notation, you can avoid many confusions. Notations, definitions and examples, these will be the most stress of this course. You will see the theorems will come out of this. It will be easier to prove theorems if your notation is clear, if your definitions are clear, and the examples you have dealt with, they are more wide rage examples.



So notations, so all rings are commutative. So rings in this course for us are commutative, and I will also assume they have unity and that usually I will write it 1. When I want to specify -- okay, the notations or rings I will use normally as A, B, C and sometimes also R, S, and usually, I will use the letter A for the ring, and 1 is 1A. when this is chance of confusion, I will specify, okay.

So now you would have studied in your first course, there are prime ideals, right, that is -- in fact, we have studied that there are maximal ideals when the ring is non-zero, that was Krull's theorem. So I am going to denote Spec A, this is the set of all prime ideals in A, and usually, I will write the German letters to denote the ideals and prime ideals, therefore German P. initially, you might difficult or you might get irritated, but I can't help, because all these books which I am going to -- another book I didn't say, it was the Jacobson's book on not the basic algebra, but the earlier books, the three volumes lectures on abstract algebra 1, 2, 3.

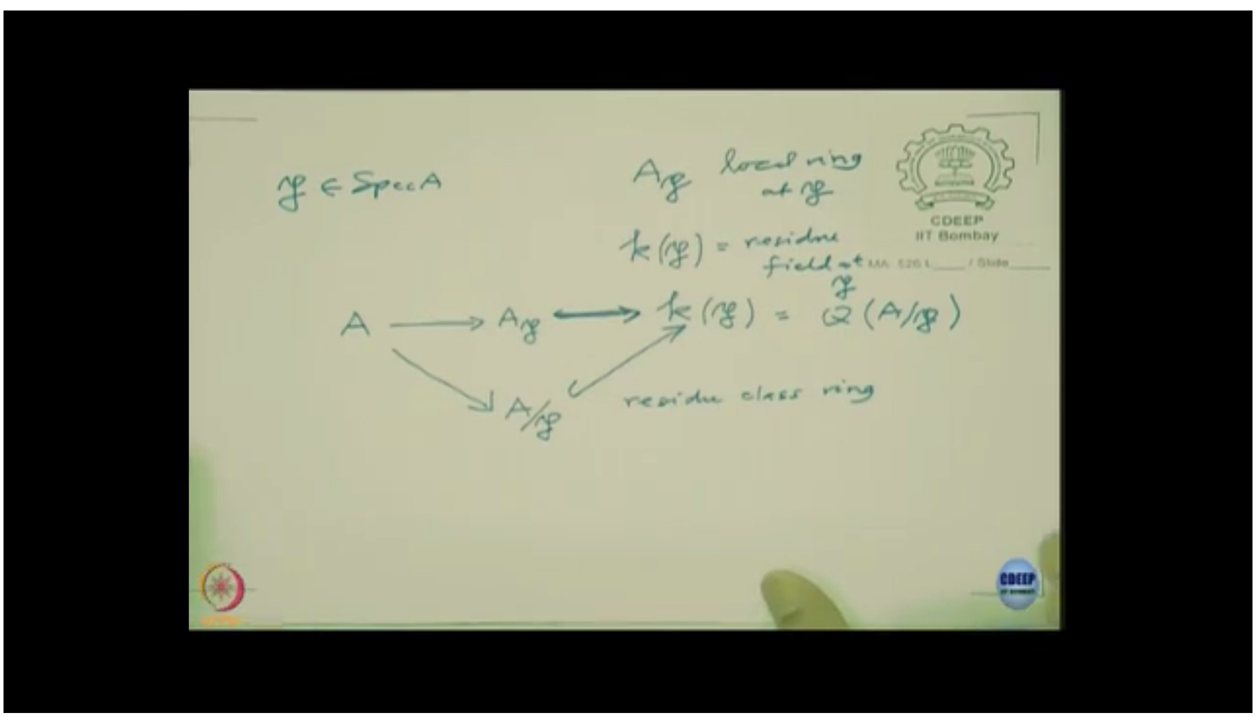
So P is the prime ideal. This is the set of all prime ideals, and this contains maximal ideals Spm, this is a set of all maximal ideals {m|m maximal, so these are the maximal with respect to the inclusion. So along all ideals in A, that set sometimes I'll denote bi I(A), it'd be the set of ideals. This set has an order, namely the inclusion. When I say order, most probably used to know partial order, but my objection to that is that partial word is not doing anything, and if you see the books written before 1950, for example, if you see the books by 07:00, it's order. Order means reflexive relation. It's a relation, reflexive, transitive and antisymmetry. So a set with such a relation is called an ordered set, and these I(A) is an ordered set, ordered with respect to inclusion. And the maximal elements in ordered set is clear with the definition, right. So these maximal ideals are precisely the maximal elements in this ordered set. And your ordered set may have maximal elements, may not have maximal element, but here we have used Krull's theorem to show that if a ring is non-zero, then it has a maximal ideal.

So this set is non-empty. This is Krull's theorem, very important theorem. Also, if you recall, it has used Zorn's lemma. Zorn's lemma is a big tool, an abstract. So usually, the philosophy is as far as possible, one should avoid using Zorn's lemma, but as you know, many times it is not possible avoid, mathematics will not go without Zorn's lemma. So whenever necessary, you should use and whenever possible we should not use Zorn's lemma.

So there is also many of the references I read to have abused Zorn's lemma in the sense that when it is not necessary also they are using. If you use Zorn's lemma, the proofs are existing chain and not algorithmic. So presently, as most phases in algorithms, we should avoid Zorn's lemma wherever possible, okay. And then you learn every maximal ideal is a prime ideal. That means we have this containment, and you would have seen example, this containment could be proper. That means there are prime ideals which are not maximal ideals, okay.

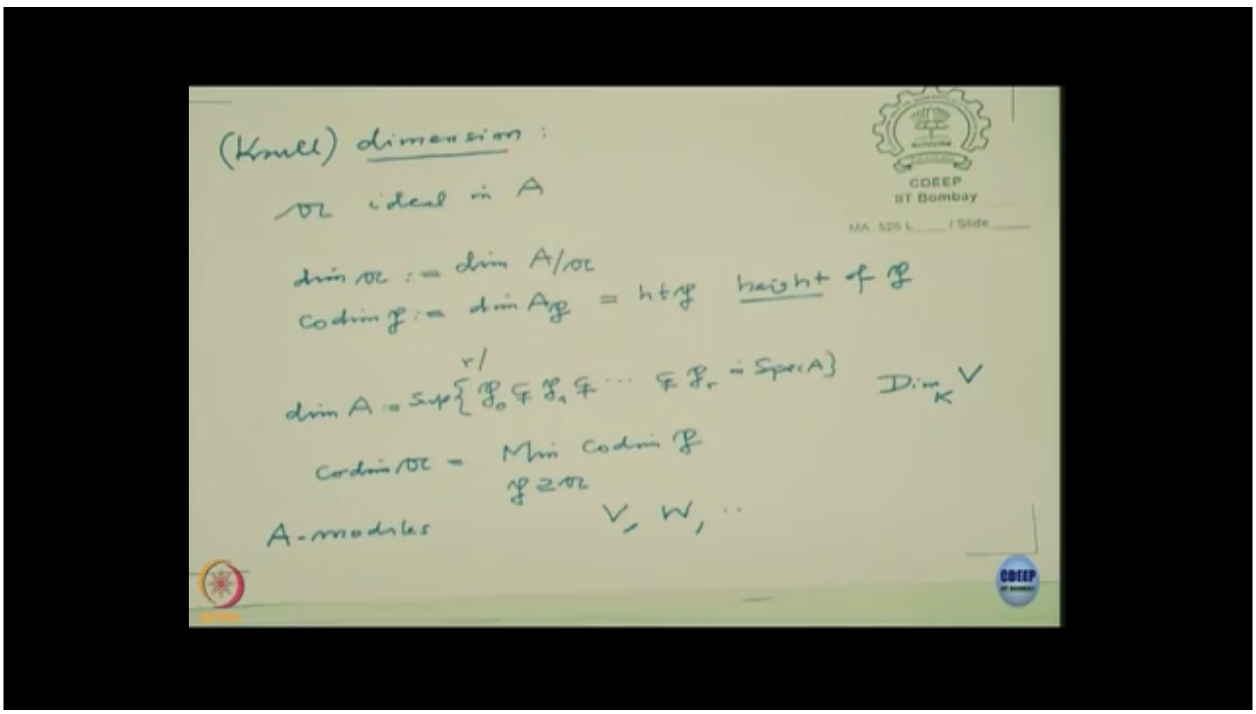
For example, if you have an interior domain, zero ideal is prime, but it is not maximal if you integral domain is not efficient, right, and integral domain precisely means 0 is a prime ideal, okay. So later on, as the course progresses, later on I am going to put more structure on this, Spec A, this is called a prime spectrum of A, and this is with additional structure I will put. Right now it is just an ordered set. Actually, this I(A) is also lattice. So some of these words, you might have not heard formally, but as we go on, we can pick up the definitions. It is useful language.

Okay, prime spectrum. So the additional structure is, for example, I am going to put a topology, we'll make it a topological study, or even more that we keep. So it becomes a geometric object to study, and then this interplay will become commutative algebra, algebraic geometry that would become very important to study. Okay, this Spm is called the maximal spectrum.



Okay, so now we have a prime ideal p, we have a local ring associated to that namely A localized at p. For every local ring, there is a field associated to that, namely the residue ring. So this is local ring at P, and usually $K(P)$, this is the residue field at P. and we have natural maps, $\frac{A}{P}$ and $K(P)$. This is an inclusion, so this is not -- this is, yes. This is residue field, no inclusion sorry.

On the other hand also, we have this $\frac{A}{P}$, this is the residue class ring. And this residue class ring and this residue field, what is the relation? This is quotient field, right, this one is a quotient field of this $\frac{A}{P}$, so there's natural inclusion method. So this is the quotient. Normally, I will write quotient field as $Q(\frac{A}{P})$. There is residue map here. More on this you will get when I start giving some proofs, okay.



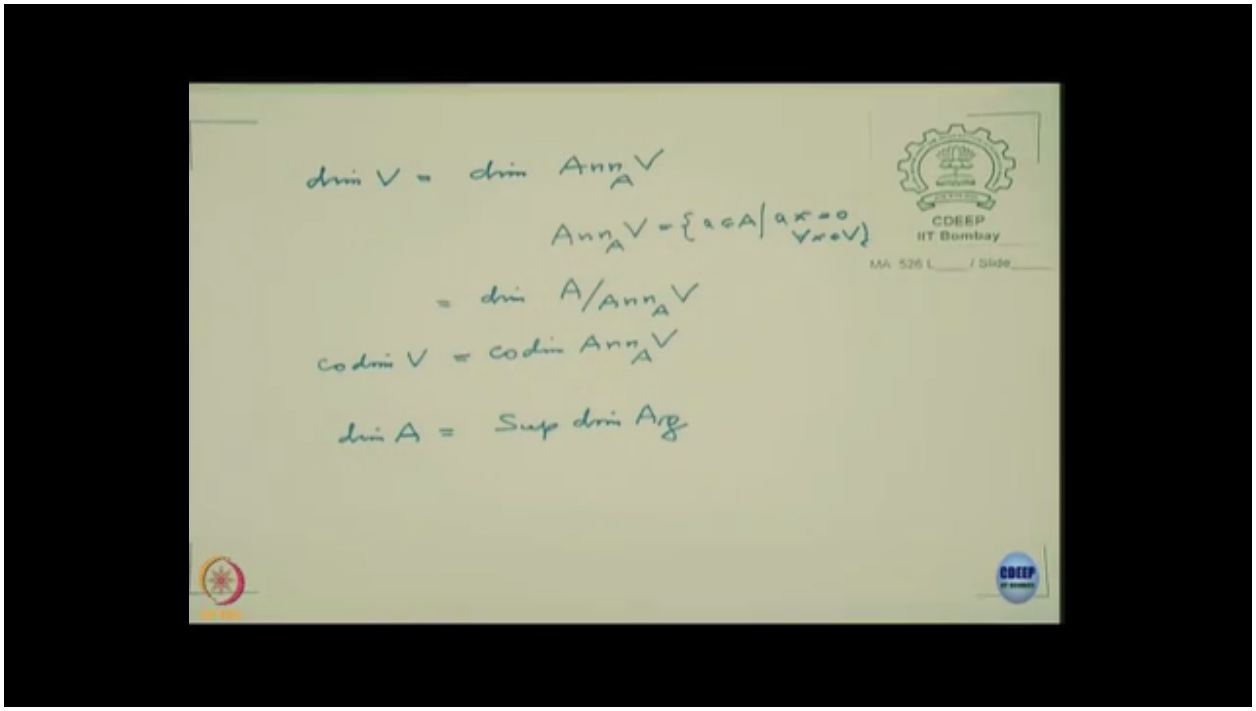
Okay, now you would have -- I just want to recall a definition of Krull dimension, and every time I will no use the work Krull. Krull was the guy -- Krull was the professor who actually proposed this algebraic definition, looking at the geometric definition. Geometric definitions are older and they were more into two also. So Krull introduced this algebraic definition.

Okay, so let me -- so you have an ideal in a, then we have this residue class ring $\frac{A}{a}$ -- oh before -- so what is the Krull dimension of the ring A that I will denote simply by dim A, and one should not get confused with the vector space dimension, because normally vector space dimension I denote by Dim and there is a suffix. For example, we have a vector space view or K where K is a field, then this dimension you know already, right. This is the cardinality of a basis. So this dimension I denote by small d. Okay, this is -- look at the chains of the prime ideals, so P_0 contained in P_1 proper chains… P_r , these are the chain Spec A. And take their length, so length is r -- so this is you get the integers r, natural numbers r, so that they have a chain of length r in the spectrum, and take this supremum.

Now there are several questions, supremum may not exist, right. So from this definition, it is not clear how do we conclude dimension for example, how do we think about dimension. So in the first few lectures, I am going to specialize the rings, and finding another ways to, another characterizations of the dimension, which will enable us to compute it, and that is the idea of this. So first topic I will take up with where the rings are specialized, rings are polynomial rings over a field or rings are finite type algebras over a field. Those that I will do first. Then I will do for local rings, and then we will do general dimension theorem for Noetherian rings, okay. So this will take some time.

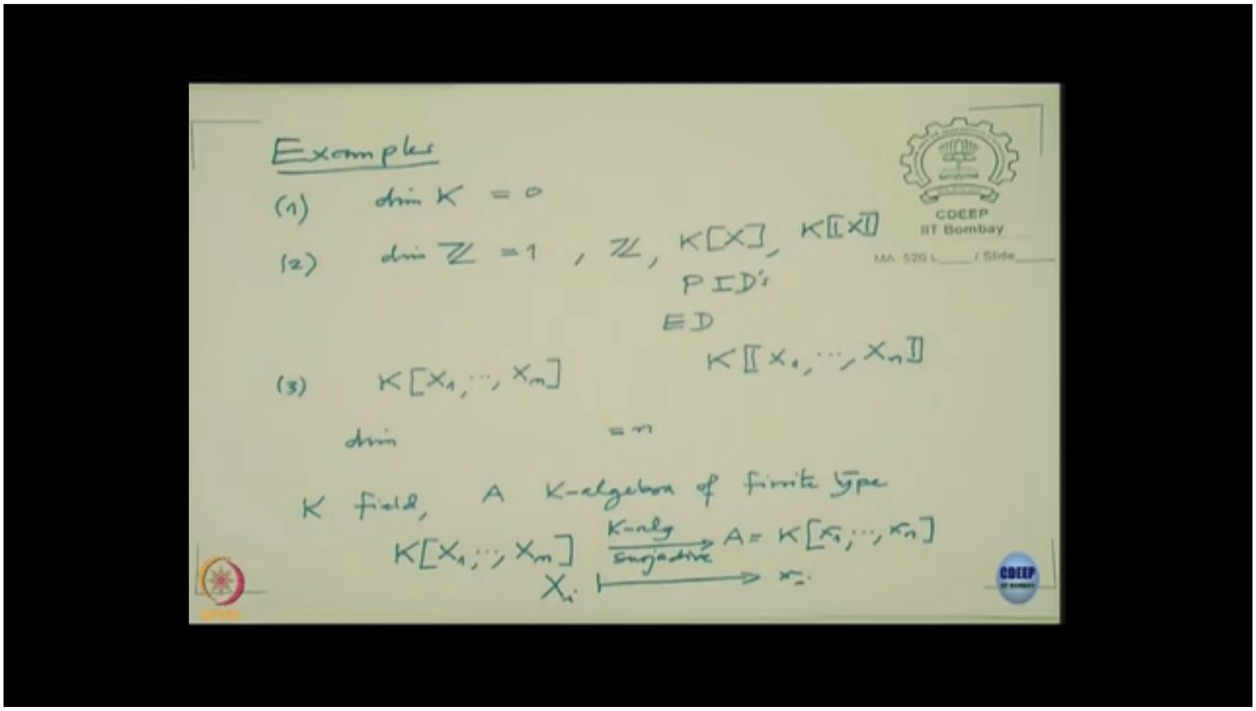
Okay, so this is the dimension, Krull dimension of the ring A. Now if you have any ideal A, the dimension of -- when you have dimension of A, that is by definition dimension of the residue class ring. So codimension, codim, codim is usually first defined for prime ideals, so that is for codimension of a prime ideal is by definition the dimension of the local ring, which is obviously the supremum of the lengths of the prime ideals which are contained in p, because when you localize the prime ideals which are not contained in p, they will become unit ideals. So that is -- this is also called a height of p, denoted by $ht\,P$, height of p. And then once you have defined it for prime ideals, then you can write it for arbitrary ideals. Codim A is by definition minimum of codim P where P contains A, minimum over P contains A. That is codimension.

Similarly for module, if I -- okay, so modules A-modules, I will denote it by letter V, W et cetera. I am not going to denote by M, N or what you are used to it probably. I am going to denote by V, W et cetera, because they are vector spaces, vector spaces over a ring, all right. The same definition, the only thing, only tool, only assumption we don't have is the basic ring may not be a field, and therefore the module may not be free modules. So modules may not even be finitely generated and so on, right, but it is good feeling that when you denote a vector space, it's similar notation, okay.



So dimension of a module is by definition dimension of the annihilator of, annihilator of module is a -- what is annihilator of a module? Annihilator of a module is by definition all those elements of the ring A, which kills everybody in V, $ax=0$ for all x and v, which is annihilator, and an annihilator is an ideal in the ring. So we already defined dimension of the ideal. So in other words, this is a dimension of the ring $\frac{A}{Ann_A}V$. Similarly, we have a codim, this codim $Ann_A V$.

Okay, and one of the most important properties of the dimension, if you know from topology geometry, et cetera, that it is a local, definition is local. What it means in algebraic is dimension of the ring is precisely the Sup dimension of $\frac{A}{P}$. This is, if you study geometry topology more carefully, this is what the local will mean. When I have the topology on that, I will explain why is it same like the earlier.



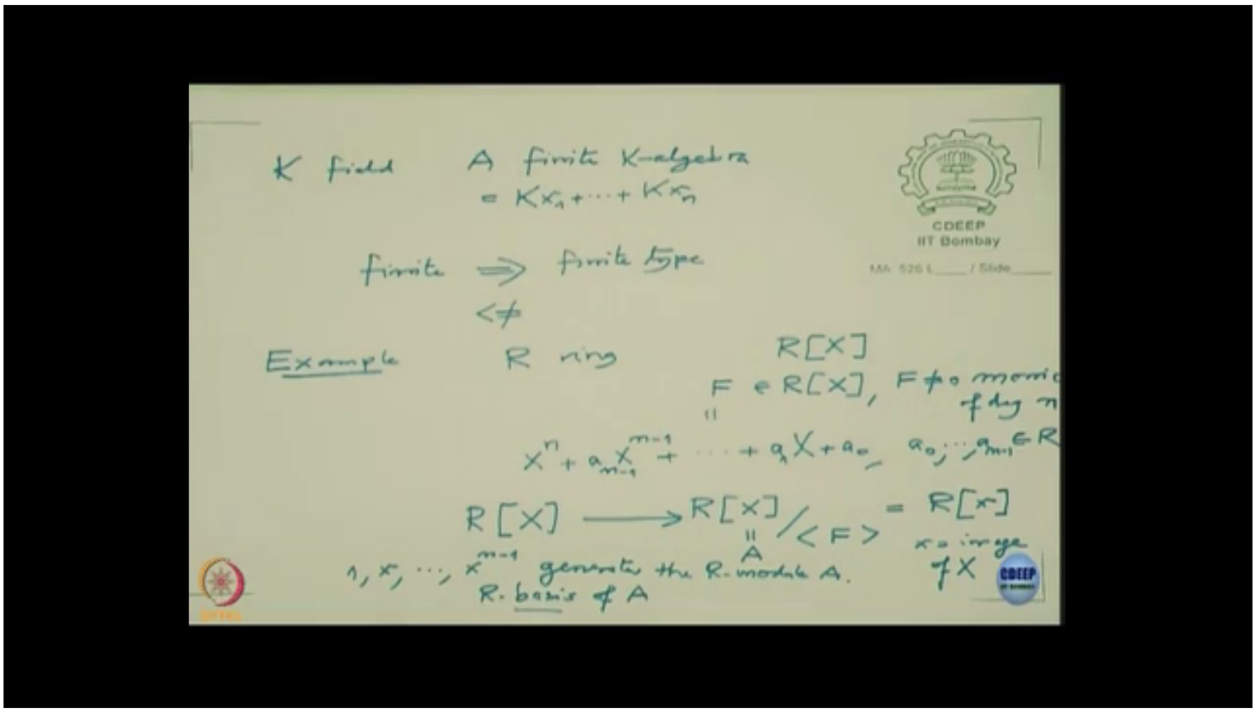
Okay, so now before I close it, we should at least know dimension of some rings. So let us see some examples. All of this may be repetition for you, but it is better to recall. So, for example -- some examples. Dimension of a field, usually I denote the K letter to be field. Dimension of the field is 0, because there is only one prime ideal in the field, namely 0. So there is no chain. Chain length is 0. So it's 0. (2) Dimension of $\mathbb{Z}$ ring of integers is 1, because you know every non-zero prime ideal is maximal also. So the chain will start at 0, and the moment you go, next one will be a prime ideal, definitely prime numbers are there, and the next one doesn't. So dimension is 1. And nothing special about Z. This is will for any PID, because you know in a PID, every non-zero prime ideal is maximal.

So the typical examples of a PID are, obviously, $\mathbb{Z}$, polynomial ring in one variable over a field K and polynomial power ring in one variable over a field. These are the typical examples of PIDs, and essentially, these are the only examples. Okay, this we will check sometime.

Now a little bit more, these are also Euclidean domains, ED, right. You would have heard the Euclidean domain about. So these are the Euclidean domains. There are more examples of PID, but Euclidean function, these are the essentially only examples.

Okay, third one, now this questions raises, what do we do with if a polynomial ring in several variables, $K[X_1,\dots,X_n]$ or power of this ring in several variables. This will correspond to what we have studied in analytic geometry, $\mathbb{R}^n$ or $\mathbb{C}^n$, right, and the dimension we have been dealing with, I don't know in college days or school days, the dimension should be n. So that is the first we will prove, the dimension of these rings is n. And more generally, we will try to see -- we try to compute dimension of -- now if we have a field K, K is a field, then -- and suppose A is K-algebra of finite type. Everybody knows this term I think, right. This means as an algebra over K, it is generated by finitely many variables, or equivalently this A is a residue class ring of a polynomial ring and finitely many variables.

So this is A is generated as an algebra, so the notation will be this, $K[x_1,\dots x_n]$. These are algebra generators of A. This $x_1,\dots,x_n$ may not be algebraic independent, but in any case, there is a subjective K-algebra homomorphism from the polynomial ring in n variables $X_1,\dots,X_n$, to this rare X_i go to x_i . This is subjective. This is evaluation method. Any polynomial you evaluated at the small x or X. So I will be very, very careful always to write my variables as capital letters, and small letters are arbitrarily letters in a ring, right.



And one more thing, so when I say finite algebra, so suppose K is field and A finite K-algebra. Finite means finitely generated module. It doesn't mean cardinality finite, finitely generated as a K-module, and even these earlier concepts, finite type over a field or finite over a field. The base ring cannot be a field. One can make a definition when we have arbitrary base ring, r, and algebra over r, when do I say that algebra is finite type over r. That means as an algebra over r, it is generated by finitely many variables or equivalently it is quotient of a polynomial ring over r in finitely many variables.

Similarly, finite over r means, as a module over r, it is finitely generated, and obviously, finite will imply finite type, but not conversely, but not this. And in this case, it's simple to write. This is $Kx_1+\dots+Kx_n$. This is what finite means. That is $x_1,\dots x_n$ are K-module generators of A. so this notation is also clear that we are taking a K linear combination of x side, all K linear combination of x side, and that will exhaust A.

So one typical example, which we will deal very often is take any -- now let us do any ring R, R is any ring, and let us take a polynomial ring in one variable, R[X]. This are obviously finite type R algebra. It's not finite, but it's finite type. See the more often you use this word, it will become more and more clearer, okay.

Now if I take a polynomial F in R[x], non-zero polynomial and monic. Monic means the top degree coefficient is 1. So monic of degree, let us say, n, so that means F looks like $X^n + a_{n-1}X^{n-1} + \ldots + a_{1X} + a_0$. This $a_0, \ldots, a_{n-1}$, there in R. They are the coefficients of if n is the degree. And if you go more into the residue class ring, if you go mod F, ideal generated by F, this is the notation for ideal generated by (F). This is -- and let us denote there is subjective map from R[X] to this natural subjective map, and let us denote image of X to be x, so this is R[x]. So x is the image of X in this residue class ring.

Now $1, x, \ldots, x^{n-1}$, this is genetic set of let us call this residue class ring as A, this generates the R-module A. This is because any polynomial you can divide by a monic polynomial and have a reminder. You don't need field for this, you know the coefficient should be -- it should be monic. And not only it generates a module, it's a basis. There is actually a basis, this is R basis, r basis of A, and because remainders are unique, right. This is a very, very important example which will be used for generations.

Okay, so in this case, I will keep -- so one say that -- I say that finite algebra of rank n, that means it is a free model and you probably have learned the theorem that if I have a free module over a commutative ring, then the rank, that is the number of cardinality of a basis, is well defined, and that is called rank of a module, and this is not true if the ring is commutative. So the modules, free modules can have basis, which have different cardinalities. So commutativity is also very important. So this we will use some time.

Now I think we should stop.