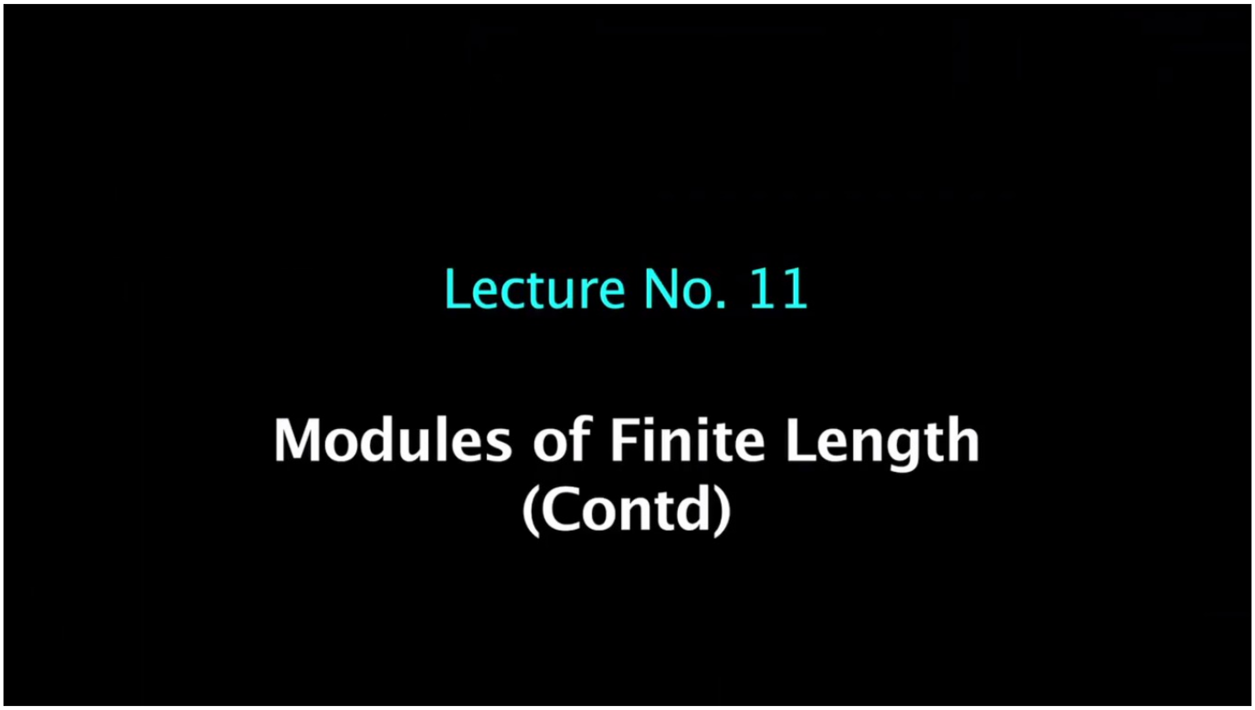
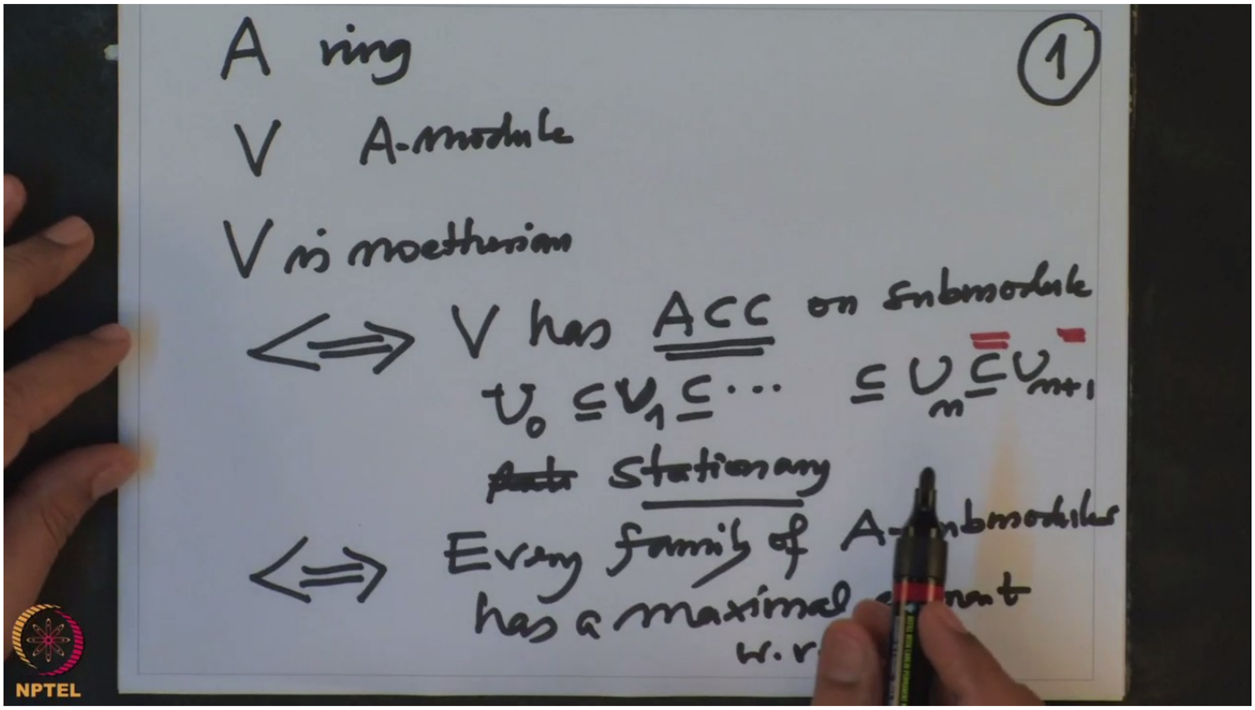
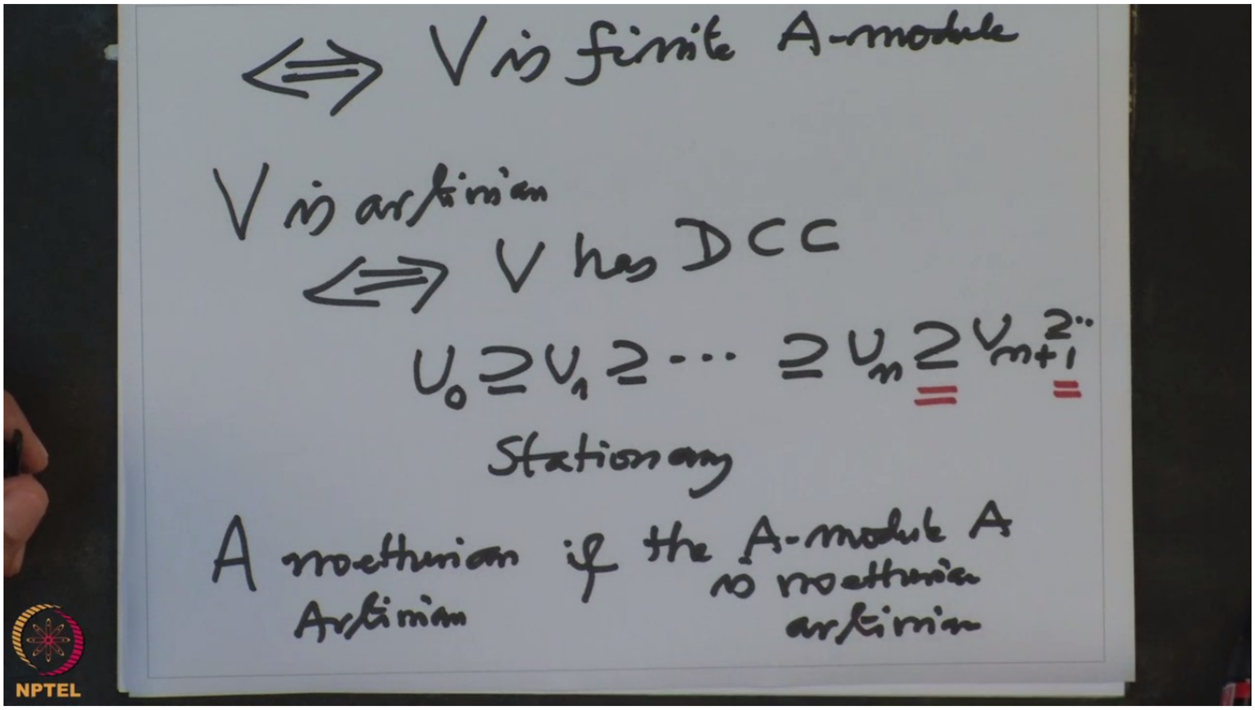
Knowledge is supreme.



So recall that now I want to connect models of the concepts, the models of Noetherian and Artinian modules. So let us recall that so A is always our base ring. Commutative always. So I will not keep writing commutative so it's by understood A is our base ring is commutative and V is A model. When do you say V is Noetherian that is if and only if V has ACC on some models that means what, that means if I have an ascending chain of some modules so U_0 containing U_1 , U_1 containing etc. this is ascending chain of some modules then it should become stationary after some stage. So every this is stationary. Stationary means that after some stage it will become equal here, equal and all are equal after that. Then you call that ascending chain to be stationary and if you have ACC means every ascending chain of some modules is stationary which is clearly equivalent to saying that every family of some modules, of A-sub modules has a maximum element. Maximum with respect to the inclusion.



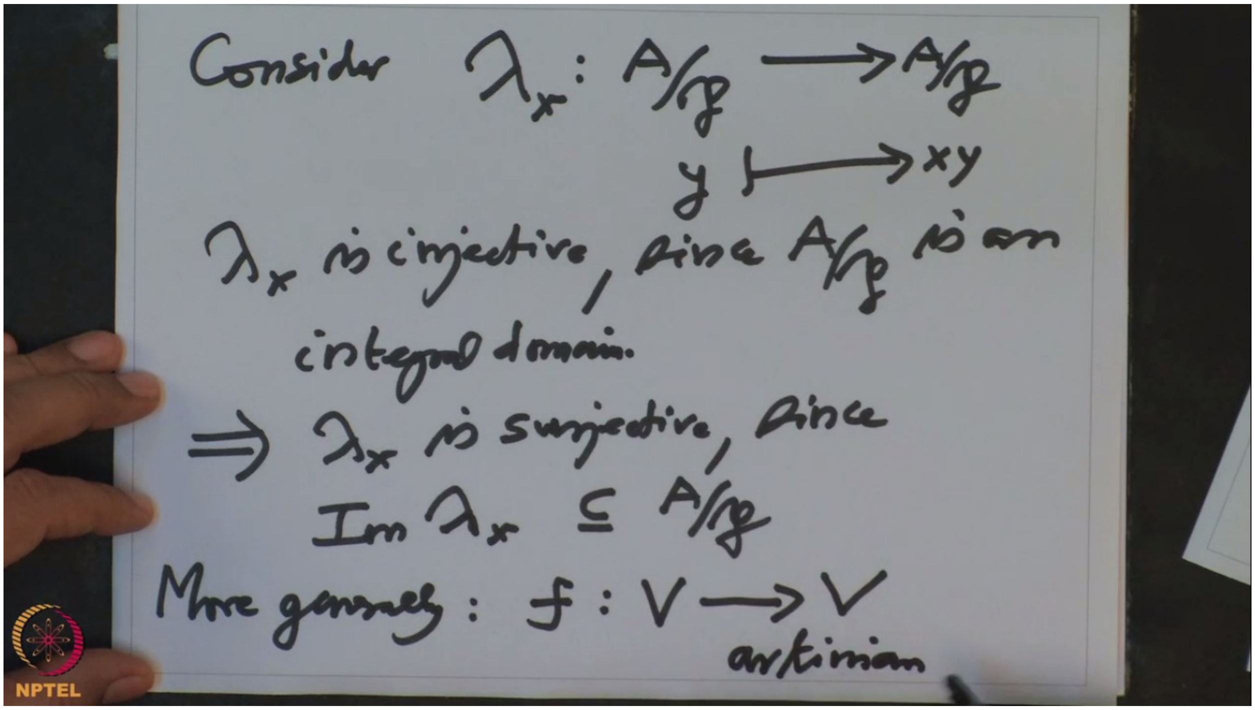
And also this is clearly equivalent to the following that V is finitely generated. Equivalently V is finitely generated. So finite A module. Our convention is when I say module is finite that means V is finitely generated module. So this clearly are equivalent. I am not going to prove that and dual concept is Artinian. So V is Artinian if and only if V has DCC. That means if you have descending chain U_0 containing U_1, containing containing U_n containing U_{n+1} these should become stationary that means after some stage it becomes equality everywhere. Then you call it an Artinian and I will say that ring is Noetherian ring is Noetherian or Artinian if the A module A is Noetherian or Artinian then you call ring to be Noetherian or Artinian. So that means this equivalent is saying that an ideals have ACC then you call A ring to be Noetherian but that is equivalent to saying that all ideals are finitely generated and Artinian means decreasing sequence of ideals should become stationary.



So now let us give some examples first. So for example any field, some examples, fields, they are no Noetherian as well as Artinian because the only sub modules are zero and the whole field so therefore it's clearly that the fields are Noetherian and Artinian. More generally if you have a vector space base ring is A vector space V then K vector space then V is Noetherian K module if and only if dimension of V is finite. If and only if V is Artinian K-module. So for vector space Noetherian and Artinian concept is same and it is equivalent to the dimension being finite.

Alright now I want to show that the next important observation I want to show that how do you test ring is the connection with the prime ideal so suppose I have a ring A and suppose A is Artinian then every prime ideal P in A that is P belonging to spec of A is maximum. That means in our notation spec of A is equal to spm of A. this is set of maximum ideal. This is a set of prime ideals and in general these inclusion is there. This is always but we are proving equality. So that mean we want to prove that every prime ideal is maximum. So proof so let P be a prime ideal and to prove P is maximal we need to prove we will prove that A by P is a field that is only we have to prove some ideal is maximal or not. So to prove it is a field we have to prove that every non-zero element there has a inverse. So start with the non-zero element in $\frac{A}{P}$ call it x. I want to prove that x has an inverse.

Alright. So how do you prove that? Alright so just let us consider the map. So consider lambda X this is a map from $\frac{A}{P}$ to $\frac{A}{P}$. The map is any x going to extend xy with the left multiplication by x this map. Alright. Now this map is clearly injective. λ_x is injective because this is an integral domain. So if xy goes to zero then only y has to be zero because x is already non-zero element we have taken in this so therefore this map is injective since A by P is an integral domain. Now I claim that therefore λ_x is surjective because what is the image since image of λ_x is a sub module of $\frac{A}{P}$ and A is Artinian therefore this residue model is also Artinian. And therefore if I look at so this is 2 in general actually. If you look at in general more generally if I have homomorphism of endomorphism of module V and V is Artinian then it is surjective.



So I will leave that for you to check so once it is λ_x is surjective that means what? That means one is in the image of λ_x but that means one equal to λ_x of some y which is xy so that means these y is the inverse of x. So therefore y is equal to x inverse and then we have finished proving that every prime ideal is maximal.

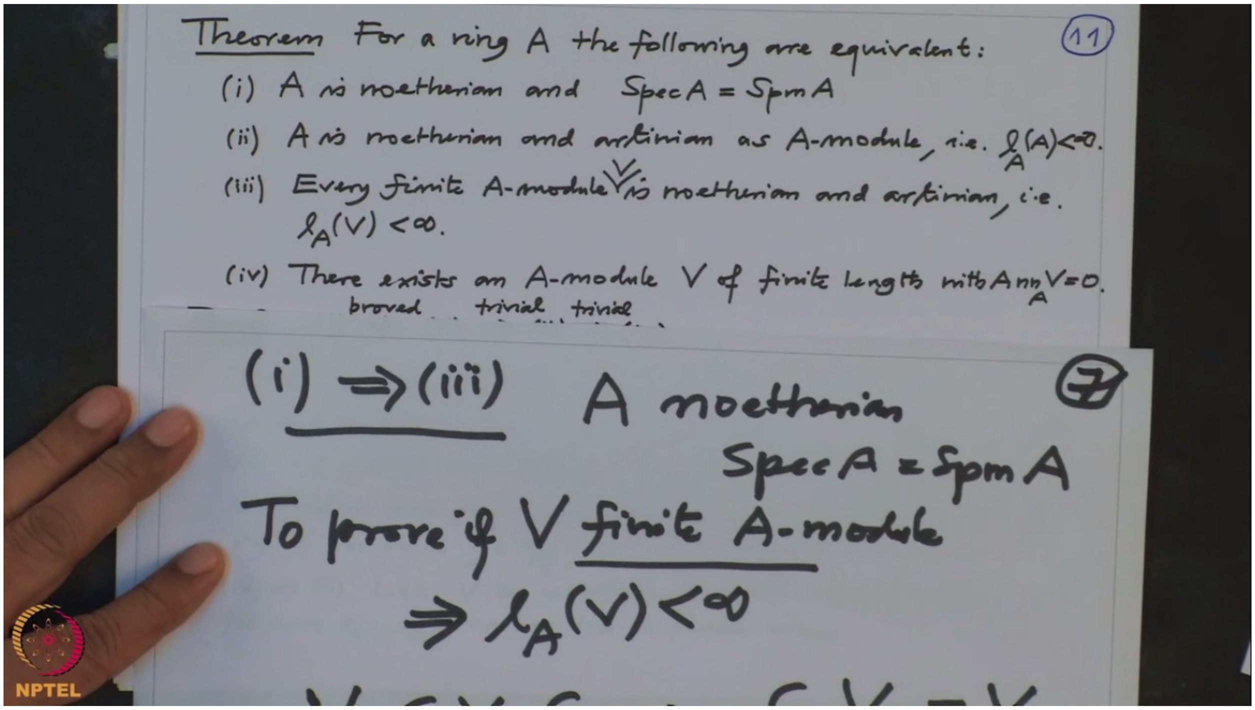
Now this I want to prove the following theorem. The theorem is if you have a ring A the following statements are equivalent. Number A is Noetherian. And spec of A is equal to spm of A that means every prime ideal is maximal. The second statement is A is Noetherian and Artinian as A modules but this will mean the length of A as module is finite because it satisfies ACC and DCC also.

The third one is every finite A module V is Noetherian and Artinian that is length of V is finite. So the third statement say that every finite A module has a finite length. And that is equivalent to saying that the base ring is Noetherian. And every prime ideal is maximal. Okay. fourth statement is there exists an A module V of finite length with annihilator is zero. You see here this is very important. This fourth statement is about existence of module of finite length with annihilator zero. And the conclusion we are drawing about the ring A being Noetherian or prime ideals being maximal and so on.

So how am I going to prove this? So first note that A implies – one implies three I have proved. I will explain you when. And three implies two is really trivial because every finite A module is

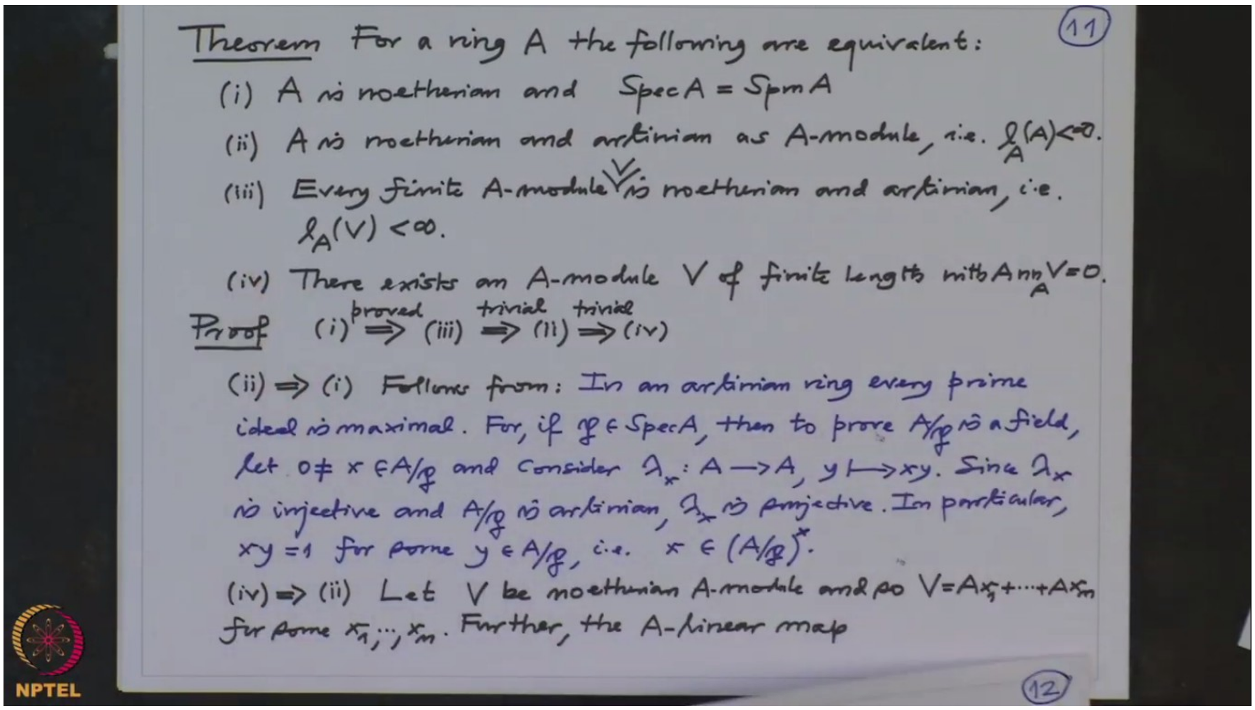
Noetherian and Artinian but so if we assume three and two is you want to prove ring is Noetherian and Artinian but ring is a finite module over itself therefore I can apply this three to V equal to A and that will prove two. Two implies four is also trivial because I will take V equal to A, V is a module A itself. Then two says it is Noetherian and Artinian so length is finite. So therefore I have a finite length module namely V equal to A and letter of A module is zero. Alright.

Now how do I prove one implies three? I want to recall. So this is the proof of one implies three. So what are we assuming we are assuming A is Noetherian and every prime ideal is maximal. This is what we are assuming. Alright. Now and what do we want to prove? I want to prove that every finite module V so to prove V infinite A module if V the finite A module which is – which has a finite length then V has the finite length. This is what we want to prove. Finite module either finite length and we are assuming A is Noetherian and so on. Remember that we have proved when we discussed primary decomposition we have proved that if V is finite module we can find a sequence like this V_n which is zero containing V_1 containing etc. containing V_{n-1}, this is V_0 which is V such that the successive quotients $\frac{V_i}{V_{i+1}}$ these are isomorphic two $\frac{A}{P_i}$ where P_i s are prime ideals. So think of this as the decreasing sequence. And successive quotients are $\frac{A}{P_i}$. But now prime ideals are maximal. Therefore, these are actually A by these are fields. $\frac{A}{m_i}$ s so these are simple. So therefore successive quotients are simple. That means this series is actually composition series. And therefore the length is n. So therefore the sequence is composition series and hence length of V at A module is the number of terms in that composition series which is M. So that proves one implies three. So we have proved one implies three, three implies two and two implies four. Now we want to prove two implies one. What does two implies one say?



Two says we are assuming that A is Noetherian and Artinian as A module so that is length of A as A module is finite and from here I want to prove that A is Noetherian and every prime ideal is maximal. So take a – here A Noetherian is also given. So Noetherian given only thing I have to prove that every prime ideal is maximal. And we have given the ring is Artinian. So we have to prove that in an Artinian ring every prime ideal is maximal. That is what we have just now proved. So the proof is written here. Alright so that proves two implies one and now to complete the equivalence I only have to prove four implies two because if I prove four implies two already we have two implies four so two and four are equivalent and one to three to two and two to one already you have proved. So therefore it remains to prove four implies two. But four says what there exist an A module V of finite length and annihilator is zero. And we want to prove two. Two is A is Noetherian and Artinian as A module. So that means we want to prove that the length of A is finite that means A has a composition series.

So let given in four there exits a module V of finite length and annihilator zero. So take that module V so V is because it is finite length it is Noetherian and therefore V is finitely generated therefore V generated by a finitely elements x_1 to x_n and now we hare using these x_1 to x_n I will define a linear map for $A \to V^n$ we have a linear map. Namely map that $A \to A x_1 + \ldots + Ax_n = V^n$ is the model $V \times V \times \ldots V$ n times. So look at a going to $ax_1 + \ldots + ax_n$ this map is clearly injective. Why? Because when a goes to zero that means all these components are zero that means a annihilates x_1, a annihilates x_2, a annihilates x_n but then a will annihilate whole V because V is generated by x_1 to x_n and therefore because annihilate is zero the only element which annihilates all these guys are in the annihilator but the annihilate is zero therefore the only element which goes to zero is zero. So that prove this map is injective. Therefore via this map this is a injective model homomorphism therefore A is a sub-module of V^n and we have proved we know that V is of finite length and therefore V power n is also finite length because we do it by induction see for example V^2 this is $V \times V$ and we have an exact sequence which connects V and V here to zero. This is the projection map. (v,w) goes to v and this one is an inclusion map so the kernel should be equal to image so therefore this map is any v or any v goes to (0,w). This is this map. So the kernel of this is precisely W equal to 0 which is the image of this. So therefore this exact this sequence is exact. So this has a finite length. Therefore V also has a finite length. And because V has a finite – so V has a finite length therefore, this has the finite length, this has the finite length therefore the middle one has a finite length since this is finite that implies length of a V^2 is finite and keep doing this so that we will go length of A as V^n in finite and but A is sub module of this V^n so sub module of a finite length module is finite length that we have noted. So this will imply length of A as A module is also finite and that proves this implication four implies two of the theorem.



So this theorem is very important. This is used many times in the proofs. So now one important corollary I will write it down. The corollary is if you have an integral domain A then the following statements are equivalent. A is Noetherian and every non-zero prime ideal is maximal and second for every ideal A not zero A has a finite length. This is also clear because what so let me write down the proof so we want to prove this say for example proof one implies two we have given a non-zero ideal and looking at the ring $\frac{A}{a}$ I want to prove that this ring has a finite length but so that I have to prove that by earlier theorem I will prove one that is A is Noetherian so I have to prove this ring is Noetherian and every prime ideal is maximal. So if you take any prime ideal here that is of the form $\frac{P}{a}$ and P has to be non-zero. Prime Ideal A because A is non-zero and A is containing P therefore this is non-zero. Therefore P non-zero but we have given in one P is maximal then. Therefore we have proved that in this ring this is

Noetherian and every prime ideal is maximal here. $\frac{P}{a}$ s maximal. Therefore that implies two that it is finite length.

Two implies one. That we want to prove that A is Noetherian and so this is same because directly apply this has finite length so we have a finite length module which is Noetherian also and Artinian also so therefore the ring – so the ring is Noetherian and every prime ideal is maximal. So that proves the corollary and with this I will stop this lecture because we have completed finite length modules and we will continue in the next lecture the further discussion about the lectures. Thank you.