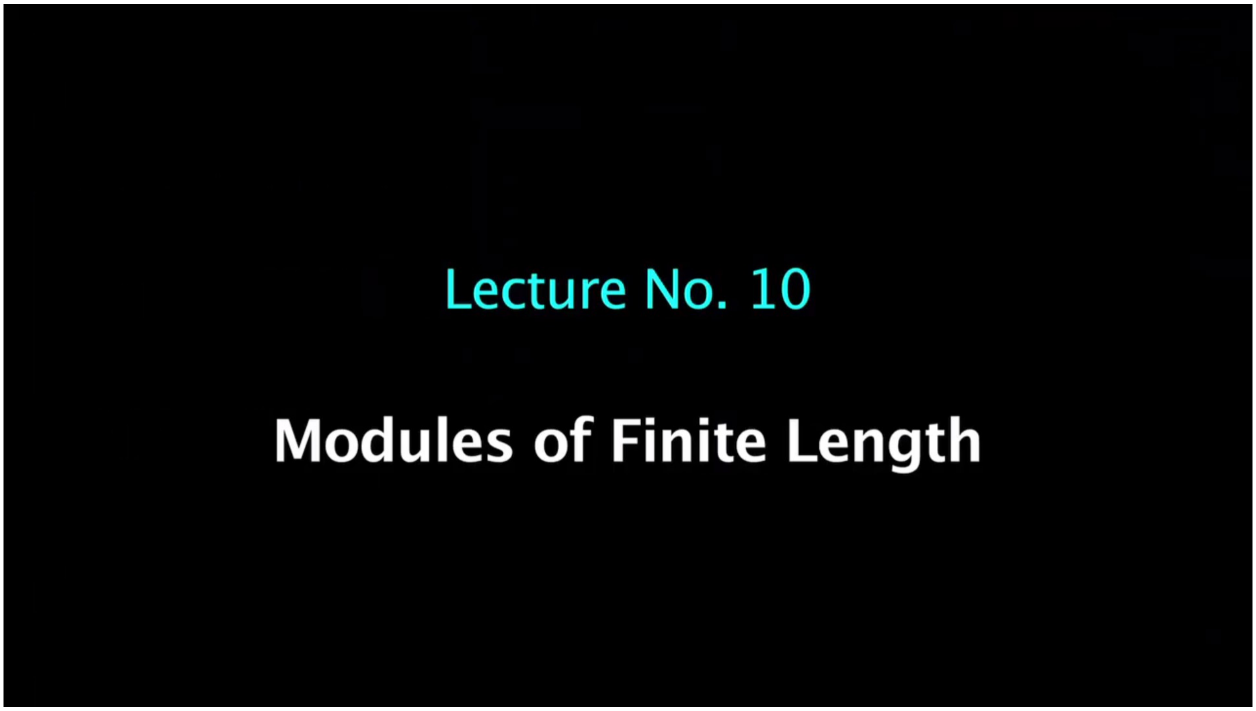
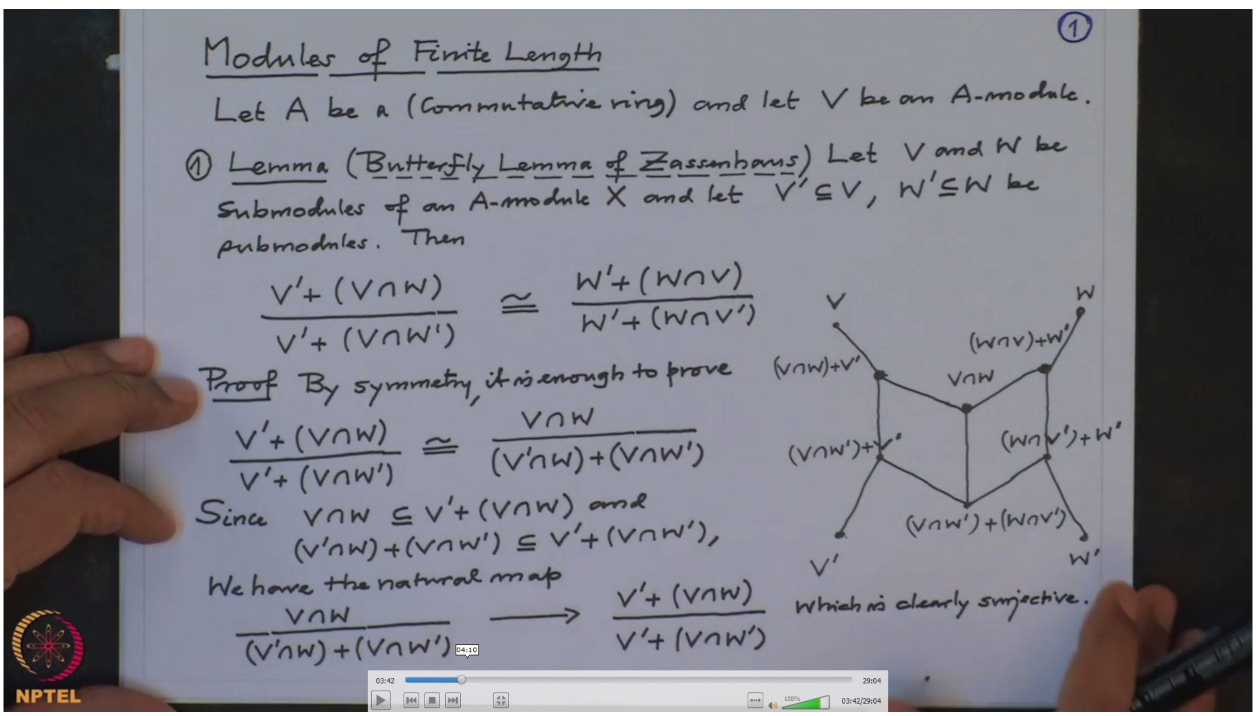


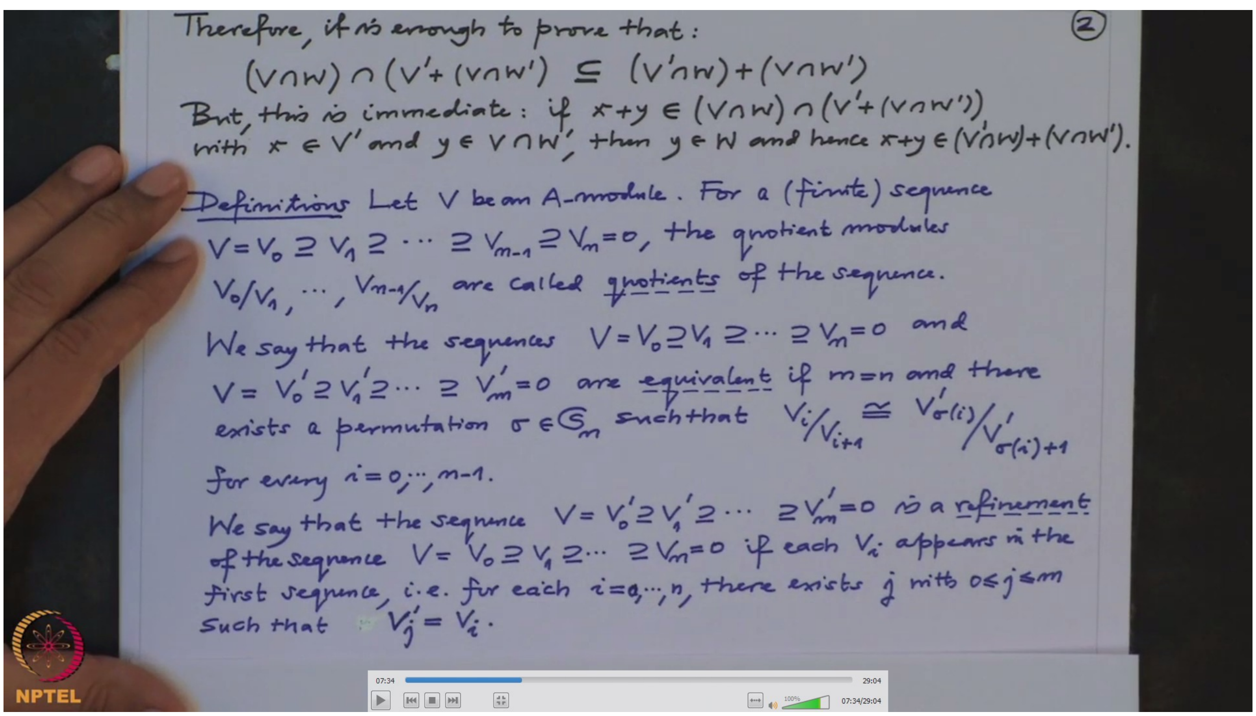
Prof. Dilip P. Patil: So in the last couple of lectures, we have discussed associated primes support and a primary decomposition of a module or an arbitrary commutative ring.



Now today, we are going to discuss modules of finite length, and as usual we will assume our ring is commutative and V be an A-module. Now to prepare, first I want to observe an interesting Lemma, which is also known as Butterfly Lemma of Zassenhaus, which said that if I have V and W are submodules of A-module X and let $V' \subseteq V$, and $W' \subseteq W$ be submodules of V and W respectively. Then if you look at this question module $\frac{V'+(V\cap W)}{V'+(V\cap W')}$, this is submodule of the above, because W' is a submodule of W. This is isomorphic to $\frac{W'+(W\cap V)}{W'+(W\cap V')}$. So we want to prove these modules are isomorphic, and while we call the Butterfly Lemma, I've drawn a picture her. So this picture looks like a butterfly that is why it is called the Butterfly Lemma.

Okay, how does on prove this? So by symmetry, see this -- by symmetry it is enough to prove that this left hand side module is isomorphic to $\frac{(V\cap W)}{(V'\cap W)+(V\cap W')}$, because if I you have proved this, and then this equal to this, so that is very easy to see.

Now $(V\cap W)$ is a submodule of $V'+(V\cap W)$, this is bigger one. Similarly, $(V'\cap W)+(V\cap W')$ is submodule of $V'+(V\cap W')$, because this one is contained here and this one is contained here. So these are the submodules, and now we have therefore the map. We have a natural map. This is a submodule of this, this is a submodule of this. So I take this going to this, this question module, and this one -- because this is a submodule of this. So we have a natural map and this map is clearly subjective, because any element here is coming from this one, because any element here, module of this, it is an element in $(V\cap W)$, so that is coming from here. So this map is clearly subjective.


Therefore, it is enough to prove that: ②

$(V\cap W)\cap(V'+(V\cap W')) \subseteq (V'\cap W)+(V\cap W')$

But, this is immediate: if $x+y \in (V\cap W)\cap(V'+(V\cap W'))$ with $x\in V'$ and $y\in V\cap W'$, then $y\in W$ and hence $x+y\in(V'\cap W)+(V\cap W')$.

Definitions Let V be an A-module. For a (finite) sequence $V=V_0\supseteq V_1\supseteq\cdots\supseteq V_{m-1}\supseteq V_m=0$, the quotient modules $V_0/V_1,\ldots,V_{m-1}/V_m$ are called quotients of the sequence.

We say that the sequences $V=V_0\supseteq V_1\supseteq\cdots\supseteq V_m=0$ and $V=V_0'\supseteq V_1'\supseteq\cdots\supseteq V_m'=0$ are equivalent if $m=n$ and there exists a permutation $\sigma\in\mathfrak{S}_m$ such that $V_i/V_{i+1}\cong V'_{\sigma(i)}/V'_{\sigma(i)+1}$ for every $i=0,\ldots,m-1$.

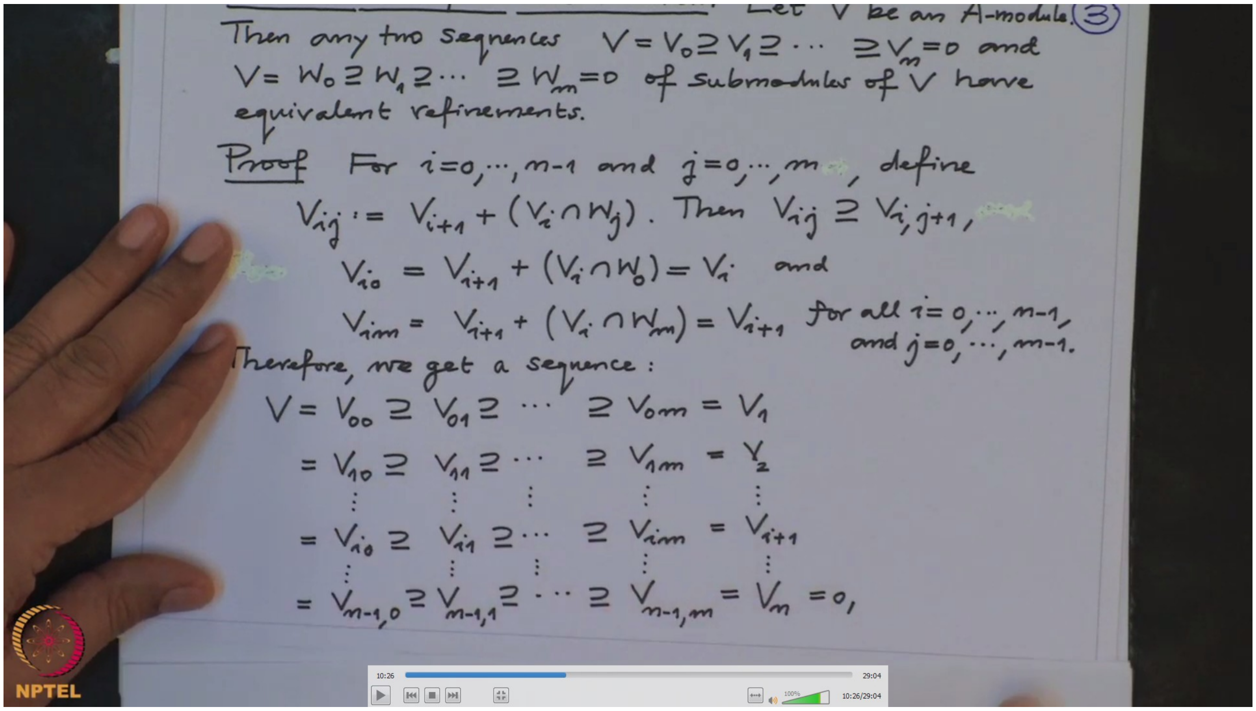
We say that the sequence $V=V_0'\supseteq V_1'\supseteq\cdots\supseteq V_m'=0$ is a refinement of the sequence $V=V_0\supseteq V_1\supseteq\cdots\supseteq V_m=0$ if each V_i appears in the first sequence, i.e. for each $i=0,\ldots,n$, there exists j with $0\le j\le m$ such that $V_j'=V_i$.

Now therefore, we want to prove that this map is injective, but injective means the kernel should be contained in -- so remember here, this is the map. I want to prove this map is subjective. That means I want to prove that the kernel should be 0. That means, what is the element in the kernel, the one which goes to 0 here. That means it is in the intersection of (V∩W) intersection with this module, and if I have proved that it's contained in this that will prove the kernel is 0, but this is immediate. I have to prove this equality. Look I have to prove this equality.

So if you take element in the left hand side, that is of the form $x+y$ where $x+y$ is in this as well as in this where that x is in V' and y is in $(V \cap W')$. Then y is in W, because W' is a subset of W, so y is in W, and therefore, $x+y$ is in $(V' \cap W)+(V \cap W')$, because that's obvious. So that proves the kernel is 0. Therefore, the map is in isomorphism, and therefore it too is Zassenhaus Butterfly Lemma.

Now how am I going to use it? So first let us define -- let us make a definition of module V, and suppose I have a finite sequence of submodules decreasing, where it starts with V and ends with 0, and then the quotient modules, successive quotient modules, $\frac{V_0}{V_1}$ and so on $\frac{V_{m-1}}{V_m}$, these are called quotients of the sequence. So we say that the sequence, the decreasing sequence of submodules and another one, decreasing sequence of submodules, they are equivalent if the number of submodules are same, $m=n$, and there exists a permutation on n letters, such that the quotients are isomorphic, the $\frac{V_i}{V_{i+1}}$ and $\frac{V'_{\sigma(i)}}{V'_{\sigma(i)+1}}$. If these quotients are isomorphic for all, $i=0,\ldots,n-1$, then we call these two decreasing sequences of submodules to be equivalent.

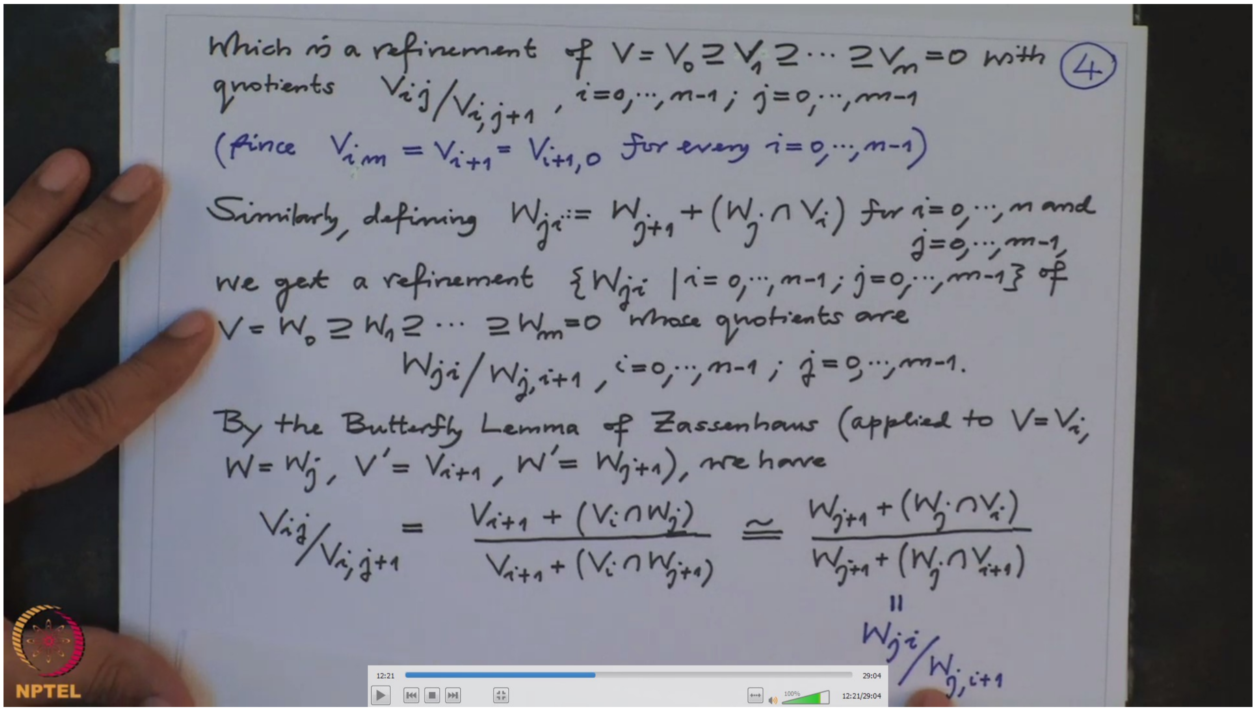
Also, we say that sequence of decreasing submodules which starts at V and ends with), there is a refinement of a sequence V_0 to V_m to V_n, the decreasing sequence of submodules, the lengths are different, but we say that this is a refinement of this, if each Vi appears in this sequence, V'_0, V'_1 and so on, if it appears there somewhere, then we say that this is a refinement of this. In other words, we should be able to -- this V' sequence is obtained by inserting some more terms in the given sequence, then we call it a refinement. So that is what it is, that all these V_is appear in the sequence, then we call it a refinement.



Okay, then. Now very important theorem of Schreier. Schreier's Refinement Theorem says that if I have a module and if I have two decreasing sequences, which starts with 0 and ends at V, this is another one of some modules, they have the equivalent refinements, so the terms may not be the same here, but again, both of them will be refinement of some bigger sequence. So we want to prove that this -- we want to refine them both, so that they become equivalent, all right. So for i equal to -- I am going to insert more terms in both the sequences.

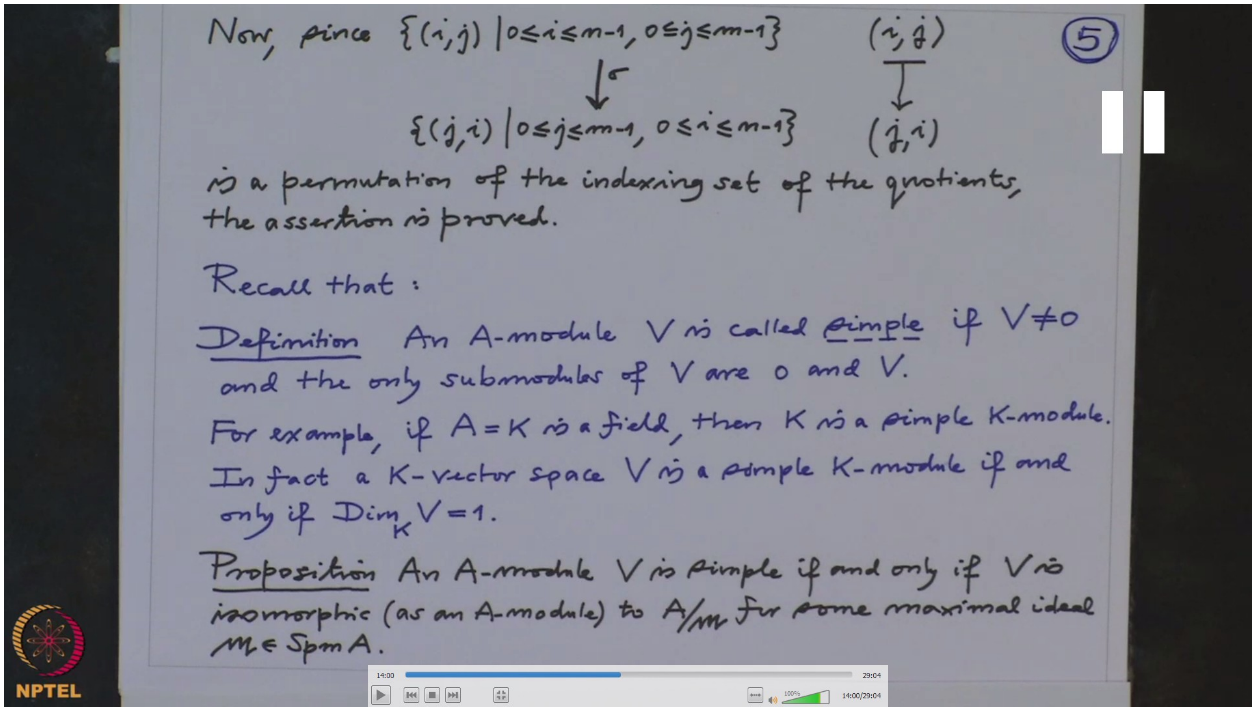
So let us define for $i=0,\ldots,n-1$ and $j=0,\ldots,m-1$, m is the number of submodules in the second sequence. So $V_{ij}=V_{i+1}+(V_i \cap W_j)$ and it is clear that $V_{ij} \supseteq V_{i,j+1}$, because W_js are decreasing. And what is V_{i_0}, V_{i_0} is by definition $j=0$, so it is $V_{i+1}+(V_i \cap W_0)$, but W_0 is whole V so this is V_i only and this is V_{i+1}, and that is decreasing, therefore, it is V_i. So therefore -- and what about $V_{i_m}, V_{i_m}=V_{i+1}$, by definition, $(V_i \cap W_m)$, but W_m is 0, so this is nothing, so this is -- V_{i_m} is V_{i+1}. This is true for all, i from $0,\ldots,n-1$ and j from $0,\ldots,m-1$.

So therefore, we have V here, which is V_{00}, which contains V_{01} and we go on decreasing to V_{0m}, but $V_{0m}$is V_1, because i=0, so this is V_m, V_1. And now you start with V_{10}, V_{10} is V_1, so these are actually same term, therefore, I write equality here. This is again the same, V_{11} goes on to be V_{1m}, but V_{1m} is V_2, and so on. So we have such a decreasing sequence and where this equality is equal to this, this is equal to this and so on. And this last one is V_n, which is 0.



So therefore, this sequence which I have defined is a refinement of the sequence. It's a refinement of this given sequence of decreasing submodule V_0 to V_n, all right. And what are the quotients? Quotients are of the refinement, $\frac{V_{ij}}{V_{i,j+1}}$, but these are quotients. So that is because we have noted $V_{im}=V_{i+1}$, which is $V_{i=1,0}$, so this is the refinement. So similarly, I can now use the other sequence to define $V_{ji}=W_{j+1}+\left(W_j \cap V_i\right)$, and these are for $i=0,\ldots,n$ and $j=0,\ldots,m-1$. So therefore, by the same way, we get the refinement Wji. Now i is running from $0,\ldots,n-1$ and j is running from $0,\ldots,m-1$, and this is refinement of this, and the successive quotients are $\frac{W_{ji}}{W_{j,i+1}}$.

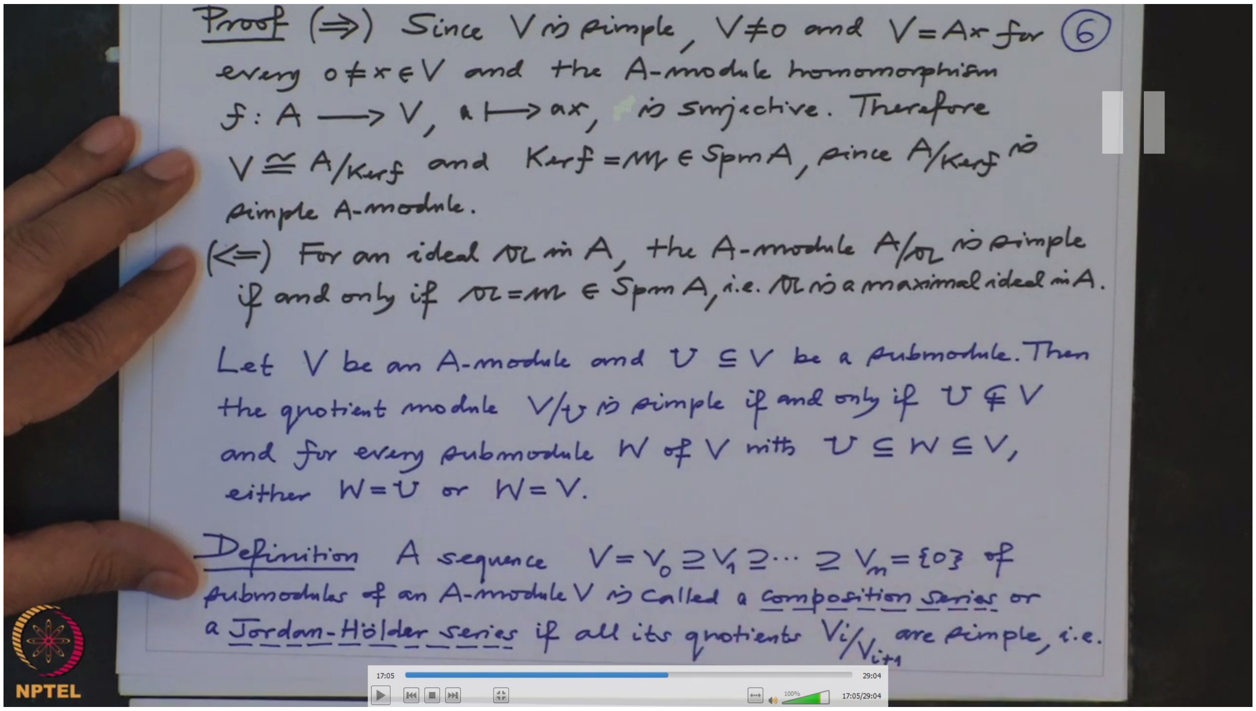
But now we want to prove that these quotients are same. So Butterfly Lemma will tell us, when I apply to $V=V_i, W=W_j, V^{'}=V_{i+1}$ and $W^{'}=W_{j+1}$, we will have this quotients. Just put on the definitions and apply Butterfly Lemma, then you will get this converted into this submodel and where this is the quotient of the refinement serial. So we have proved that these refinements have the same quotients.



But now also, we have to say that the permutations. So because you get the {(i,j) i is running from $1,\ldots,n-1$ and j is running from $0,\ldots,m-1$, and the other way, just interchanging the coordinates. So the numbers are same. These two indexing sets have the same cardinality, and the σ is a permutation when you switch the coordinates, it's a permutation. So therefore, this is a permutation, so therefore, we have put the assertion that these two sequences have the common refinement, okay.

Also, I want to recall a definition of a simple module. Simple module means -- first of all, the module is non-zero and the only submodules are 0 and V. For example, if you are over a field, if the base ring is a field, then K is simple k-module. More generally if you have a vector space, then vector space is simple if it is non-zero and the dimension should be 1. So when the dimension is 1, this is non-zero, so I don't have to say it is non-zero. So vector space is simple, if and only if the dimension is 1.

All right, now how do you test some modules are simple or not? We say a module is -- we want to test a module is simple, I want to generalize this from the vector spaces. So a module V is simple if and only V is isomorphic to as an A-module, $\frac{A}{m}$ where m is a maximum ideal in A. Proof is very simple.

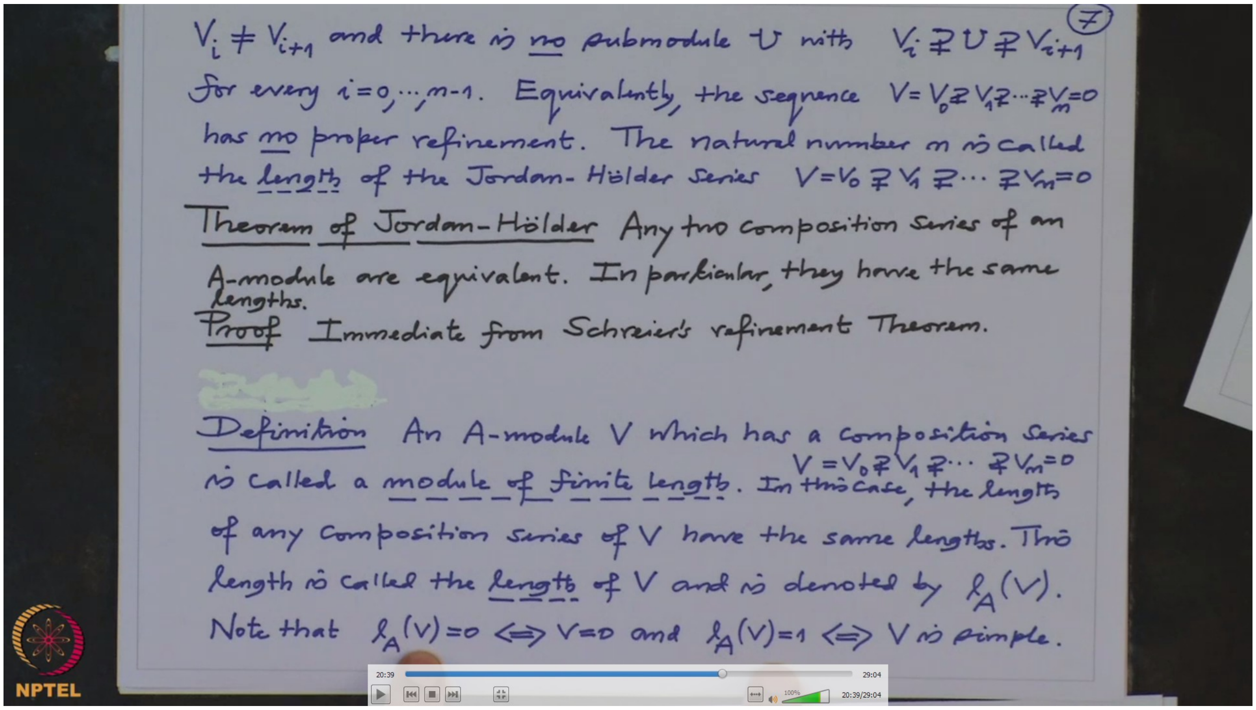


So suppose -- first assume V is simple, then we know V is non-zero and if I take any non-zero element of V and take the submodule generated by x, that should be the whole V, because it's a non-zero and the only submodules are either 0 or the whole. So therefore, V generated by x and we have a module homomorphism, $f: A \to V$ just mapping $a \to ax$. This is clearly subjective, because V is generated by x, therefore, V will be isomorphic to A-module of the kernel of f and the kernel of f has to be maximal, because A-mod kernel of f is a simple module. It's isomorphic to V, therefore, it is a simple module, therefore, it can't have any proper ideals. So therefore, $\frac{A}{kernel\, of\, f}$ has to be field, therefore, m is a maximal ideal of A.

So one way we have proved. The other way, if you have an ideal in the ring A, then the module A/a is simple if and only if a is a maximal ideal, i.e. -- this is also very easy, because this module is simple means it is non-zero first, so therefore, A is a proper ideal and there is no ideal in this residue class ring, because the proper ideal of this ring will give you proper submodule of $\frac{A}{a}$. So that proves the proposition.

All right, if you have a module V and a submodule, then the quotient module is simple, if and only if, first of all U is a proper submodule and every submodule W in between either V = W or W = V. So this is like a maximal ideal. So submodule -- the quotient module is simple if and only if -- there is no proper submodule in between a module V and submodule U. Now this allows us to make a definition, but decreasing sequence of submodule is called a composition series or Jordan-Holder series, if all its quotients are simple submodules.

First of all, that means these are proper inclusions and you can't insert anybody in between. That means you cannot make the sequence refine. So that's one has written.



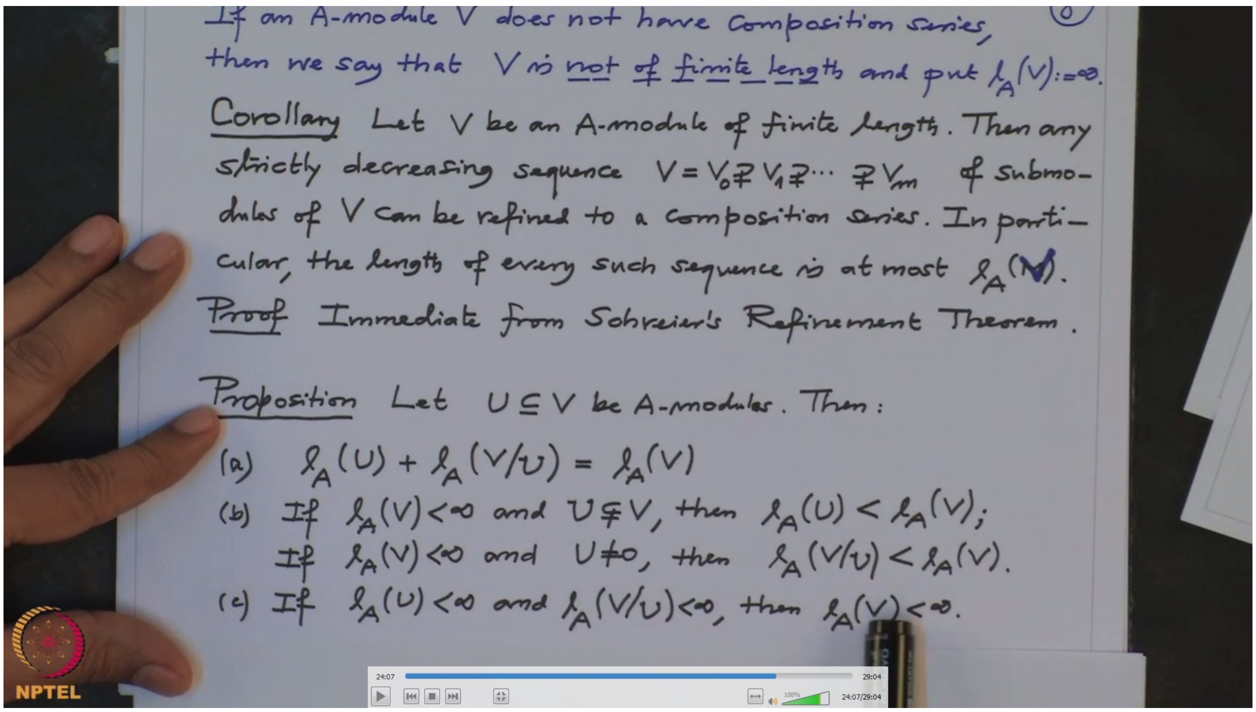
So the composition series or Jordan-Holder series means in between any two terms where we cannot insert any proper submodule for every i. Equivalently, this given decreasing sequence of submodule has no proper refinement, and in this case, the natural number n, that is called the length of this Jordan-Holder series.

So the next theorem is Jordan-Holder Theorem, which says that any two compositions series of a module are equivalent, that in particular they have the same number of terms, so that means in particular, they have the same lengths. This is very important theorem. It was proved for the groups first, the abelian groups first. That is because Jordan was studying Galva theory, and therefore, it was very important to consider abelian groups and Jordan-Holder series like that. More generally, also consider, one considers such a series for arbitrary groups, but you have to assume that in the decreasing sequence of subgroups, each one is normal in the next one and so on, because then the quotients will make sense, okay.

The Jordan-Holder theorem is clear from the Schreier's Refinement Theorem, because once you have a composition series, you cannot refine anymore, and therefore, it has to be refinement of itself only. So therefore, if I take two composition series, Schreier's refinement theorem says that they have a common refinement, but each one is a refinement of itself. So therefore, these two, any two compositions series are equivalent, and in particular they have the same number of terms and the same quotients also. So that is Jordan-Holder theorem.

So remember that the Schreier's refinement theorem is more general and it was proved later than the Jordan-Holder theorem. But now the proofs, I have used the theorem, which was proved later, because it was more general, anyway. So now, we can make a definition, an A-module V, which has a composition series that is called Module of Finite Length, and in this case, the length of that composition series of V have the same length. This is very different, because we have just proved in a Jordan-Holder theorem that any two have the same length. So this length is well-defined and that length, we call that is a length of V and it is denoted by lA(V), so length of V as an A-module.

Okay, note that when will he length be 0, that means the Jordan-Holder series, it has a composition series and that has only one term, namely 0. So V is 0, that is equivalent to saying V is 0. When will the length be equal to 1, that means V has to be simple, because the composition series starts with V and ends with 0, and there can't be any more terms, because all the terms are not there because the length is 1, and therefore, the quotient is simple, that means V_0 is simple so that is V is simple.

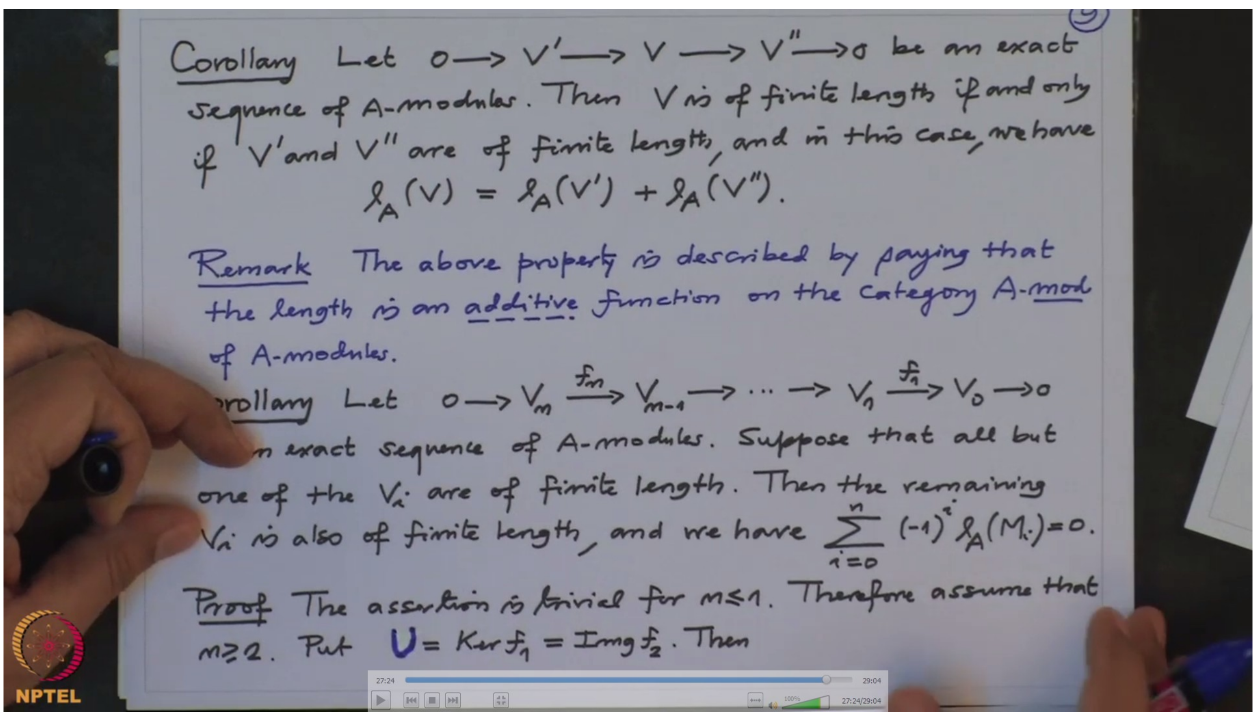


So if a module does not have composition series, then we will say that module V is not of finite length, and in that case, we will put length of V to be infinity. All right, now we have to make a criterion. How do we decide whether a module has a composition series or not? And now we are discussing that. So the corollaries, which I deduced from Jordan-Holder theorem that will lead to some answers. All right, so suppose I have a module of finite length, then any strictly decreasing sequence of submodules can be refined to composition series, but the assumption is V should have finite length, okay. In particular, the length of every such sequence is at most, the length of the module V. This should be (V).

So this is also immediate from the refinement theorem, because given the sequence, I will keep refining it and already I know there is a composition series for V that exists, because V is a finite length, and these I have by inserting more and more terms, I refine, and make it the composition series. But then these two composition series have the same length, and therefore, these m will be at most length of V, because length of V by definition length of a composition series.

Now this one, the next proposition will allow us to compare the length of a module and its submodule and the quotient module. So if U is a submodule of V, then length of U as A-module and length of $\frac{V}{U}$ as A-module, that is same thing as length of V. All right, so this formula is even true for when the length is not finite, okay. So that is -- the next statement is if the length of V is finite and you U is a proper submodule of V, then length of U is strictly smaller than length of V, because we can increase the composition series at least by 1. This is triviality.

So length of V is finite and U is non-zero, then length of the quotient module is strictly less than length of V. If length of U is finite and length of $\frac{V}{U}$ is also finite, then length of V is also finite. This is also easy because length is finite, so U has a composition series and this is finite, so it has a composition series. So lift the elements in the composition series and you will get a composition series. So I will not prove this proposition. It is left as an exercise, all right.

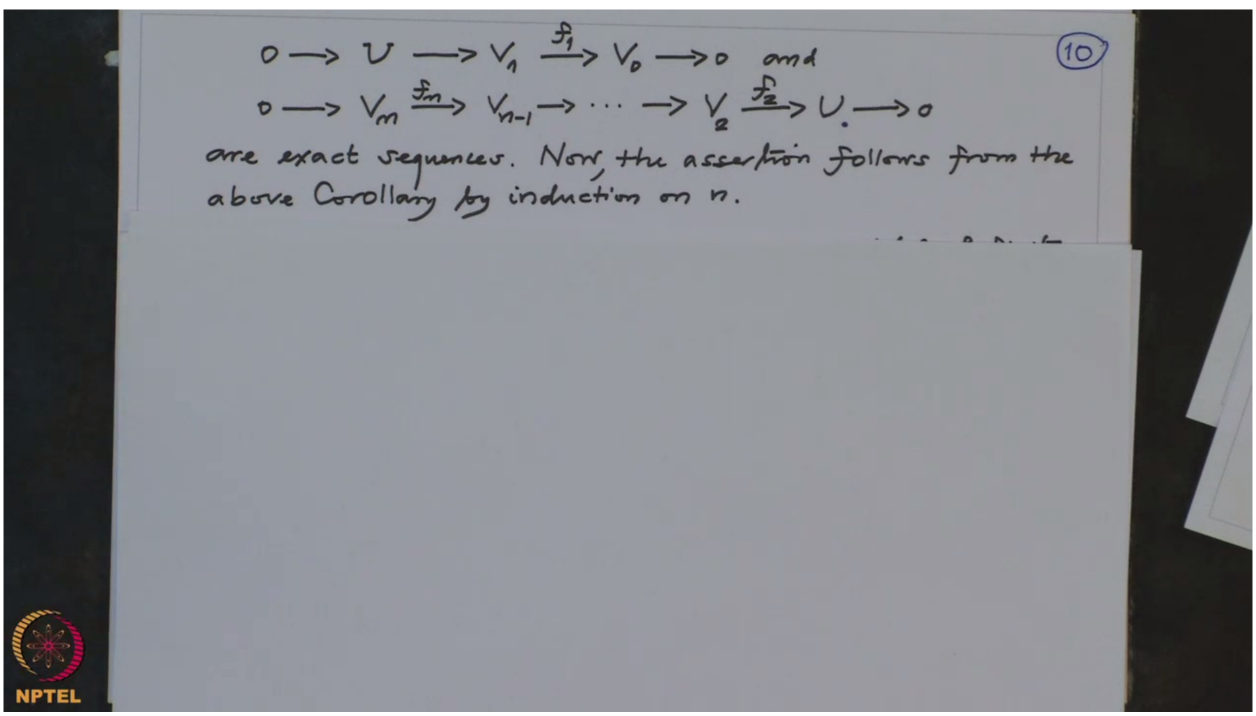


So similarly, we can deduce immediately from this. If I have short exact sequence of A-modules, $V' \to V \to V''$, this is a short exact sequences, means here the kernel of this map is equal to the image of this map; this map is injective and this map is subjective. That is the meaning of the fact that this sequence is an exact sequence of A-modules. Now, the assertion is if V is of finite length, V is finite length if and only if -- this middle one is finite length if and only if the outer ones also are of finite length, and in that case, the length is additive. That means length of the middle one is length of V' plus length of V''.

All right, this is because you can think this is a submodule of this, and V'' is quotient module of this. So the earlier preposition will tell you this length is additive. So this is also -- see this property can be described by saying that length is an additive function on the category of A-modules, okay.

More general version of this corollary is, if you have a long exact sequence like this, which has many terms, so that means you have the maps and at each stage, kernel equal to the image, kernel of the later map is the image of the earlier map. There you call such a sequence should be exact sequence of A-modules.

Suppose if this sequence of modules, all but one of them is -- or suppose that all but one of the Vi are of finite length. That means only Vi may not be finite and all others are finite length, then the assertion is the remaining Vi is also finite length, and in that case, we have alternating sums of the length is 0. So this assertion is trivial for any $n \leq 1$, because there is only either 0 term or only 1 term. In that case, the associated -- this is isomorphism, so this is equal to this, therefore -- so assertion is clear for $n \leq (n-1)$. So I am going to prove the assertion by induction on n. Therefore, we assume first n is at least 2 and I am going to put this kernel -- so we know kernel of f_1 is equal to image of f_2, and I am going to call it U.



Then we can break this long exact sequence into two exact sequences, namely if U is a kernel of this f_1, so therefore, we have short exact sequence, and now, this U is also the image of f_2, so I will forget V_1 and put it at U, and therefore, this sequence is clearly exact. So these are two exact sequences. Now this length has -- I can apply the induction, and therefore, the assertion follows by the above corollary, because for this alternative sum is 0 and for this, I will replace the length of U by coming from here, alternating sum again, so therefore, the assertion is clear by induction on n.

And we will continue after the next break, so that we will connect this concept of finite length of with the support and associated prime ideals and also relation with the Noetherian and Artinian modules. So thank you.