cINDIAN INSTITUTE OF TECHNOLOGY BOMBAY

IIT BOMBAY

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

CDEEP IIT BOMBAY

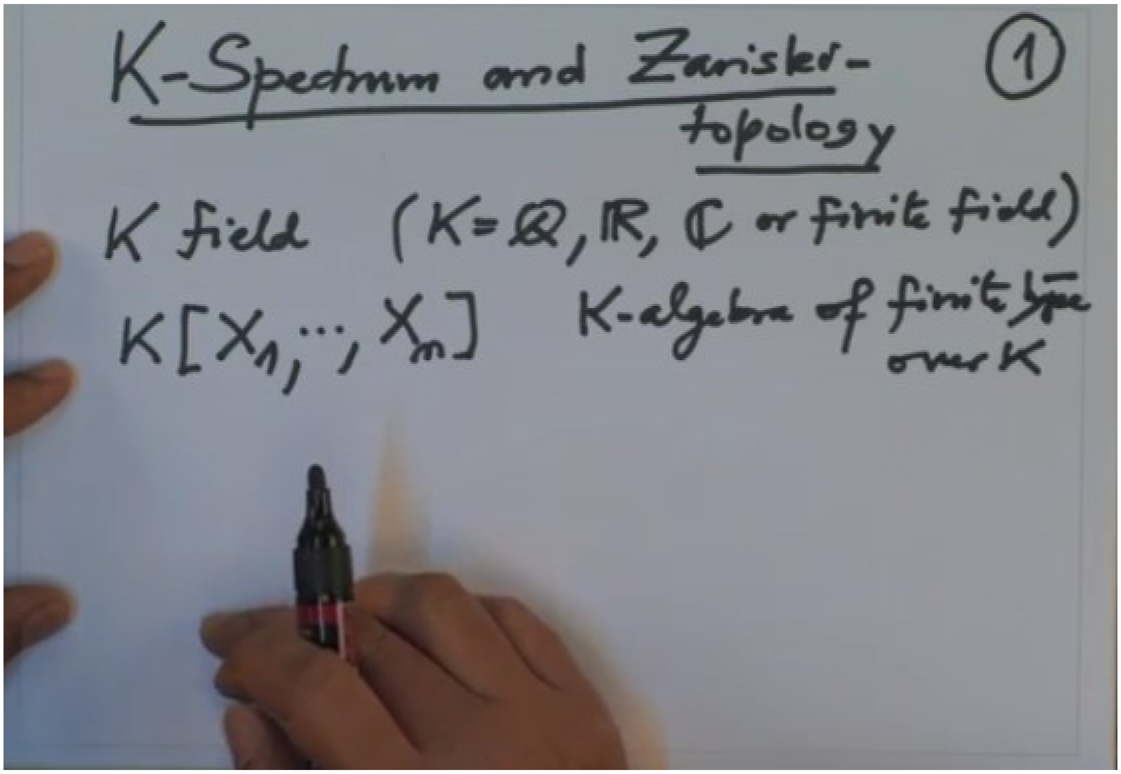
COMMUTATIVE ALGEBRA:

PROF. DILIP P. PATEL
DEPARTMENT OF MATHEMATICS,
IISc Bangalore

Lecture No. – 01
Zariski Topology and K-Spectrum

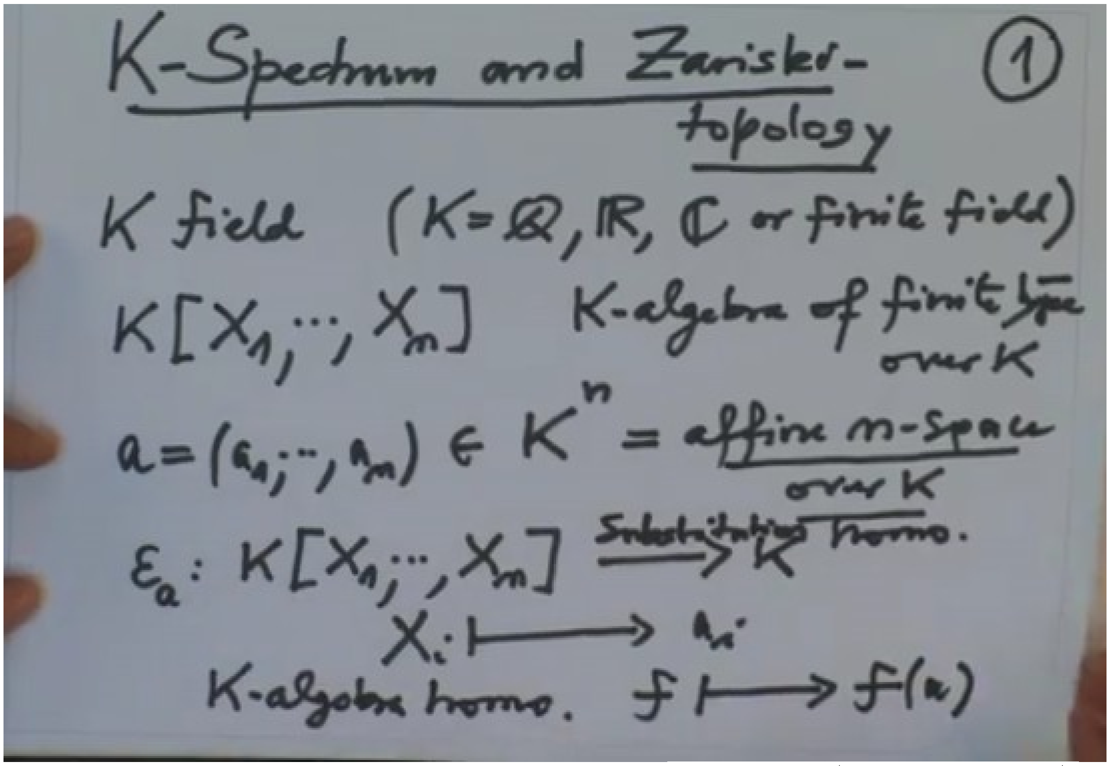
So in this lecture I will explain results on Zariski topology and its connections with commutative algebra. So I will do it first for the very special case and then we will do it for more general case after that, so this set up is like this, so this lecture I want to explain what is the K-Spectrum, and Zariski topology, alright.

So where K is a field, for example one could take K = $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ or finite field, alright, and then we have seen the polynomial algebra in N variables, $K[X_1,\dots,X_n]$ this is K algebra, this is a finite type over K,
(Refer Slide Time: 02:11)



okay later on we will drop this some chain and we will study for general commutative rings but to give a better understanding we have to do this case first, so we have seen that the universal property of the polynomial algebra say that if I have any point $a=(a_1,\ldots,a_n)$ this is a point in K^n, this K^n is also sometimes called affine and space over K, the coordinates are in K, these are the tuples which coordinates are in K, therefore it is called affine N space over K.

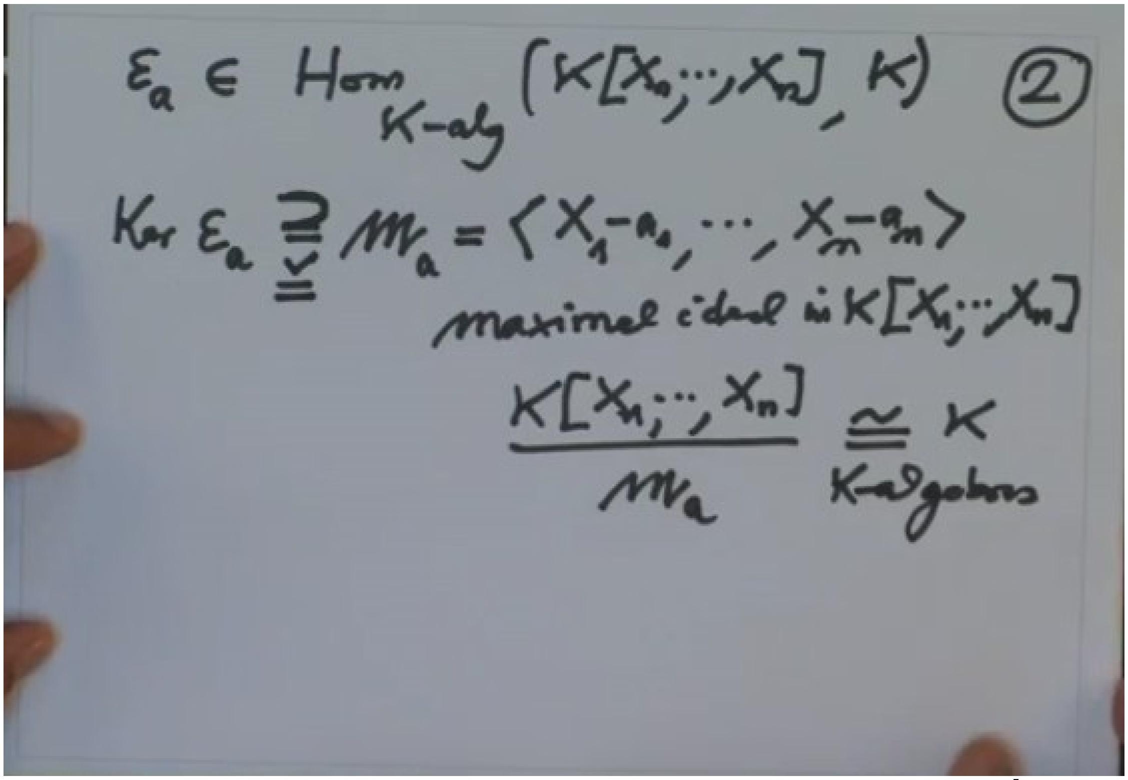
And to each point we have a algebra homomorphism, epsilon A this is a homomorphism of K algebra from $K[X_1,\ldots,X_n]$ to K, namely if I have to give, this is the substitution of homomorphism by a, so XI's are mapped on to a's, so this is the K algebra homomorphism, this is also called substitution homomorphism, substitution homomorphism, this is a K algebra homomorphism, so any polynomial f in n variables, f will go to f evaluated at a, alright,
(Refer Slide Time: 03:59)



so this is, therefore this ϵ_a therefore an element in $Hom_{K-algebra}(K[X_1, \dots, X_n], K)$. What is kernel of this? Kernel of this, it's very easy to check Kernel of this, ϵ_a homomorphism is precisely, these I will denote by m_a, this is the ideal generated by $\langle X_1 = a_{1,} \dots, X_n - a_n \rangle$, this is very easy to check that it is obvious that all this polynomials $X_1 - a_1$ etcetera, they are clearly contained in this kernel because X_1 goes to a_1 so this polynomial go to 0.

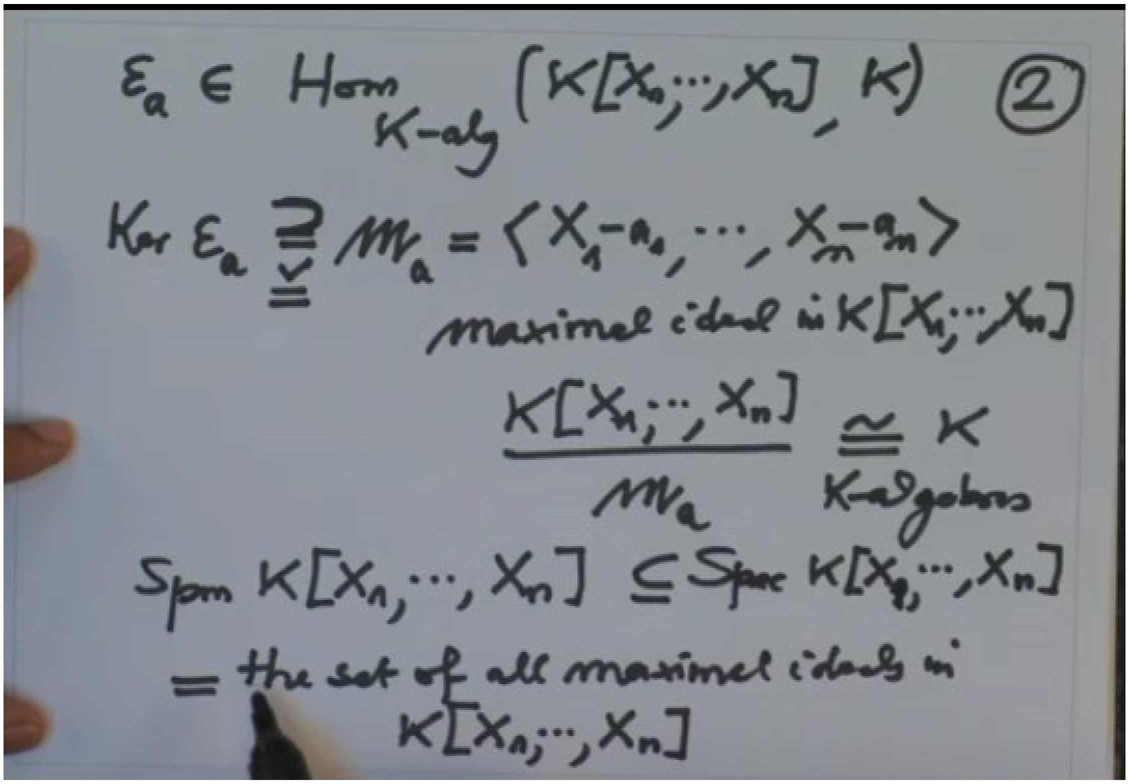
Also on the other hand this ideal here is a maximal ideal in $K[X_1, \dots, X_n]$, that can be easily checked by going modular there, so in fact if I take $\frac{K[X_1, \dots, X_n]}{m_a}$ is isomorphic to K, as K algebras, therefore it's a field, therefore this is maximal ideal, so Kernel cannot be, because this ϵ_a maps 1 to 1 this cannot be the whole ideal therefore it will be equality here, that is how one proves this is a maximal ideal.

Now therefore what we have done, we have done the following,
(Refer Slide Time: 05:57)



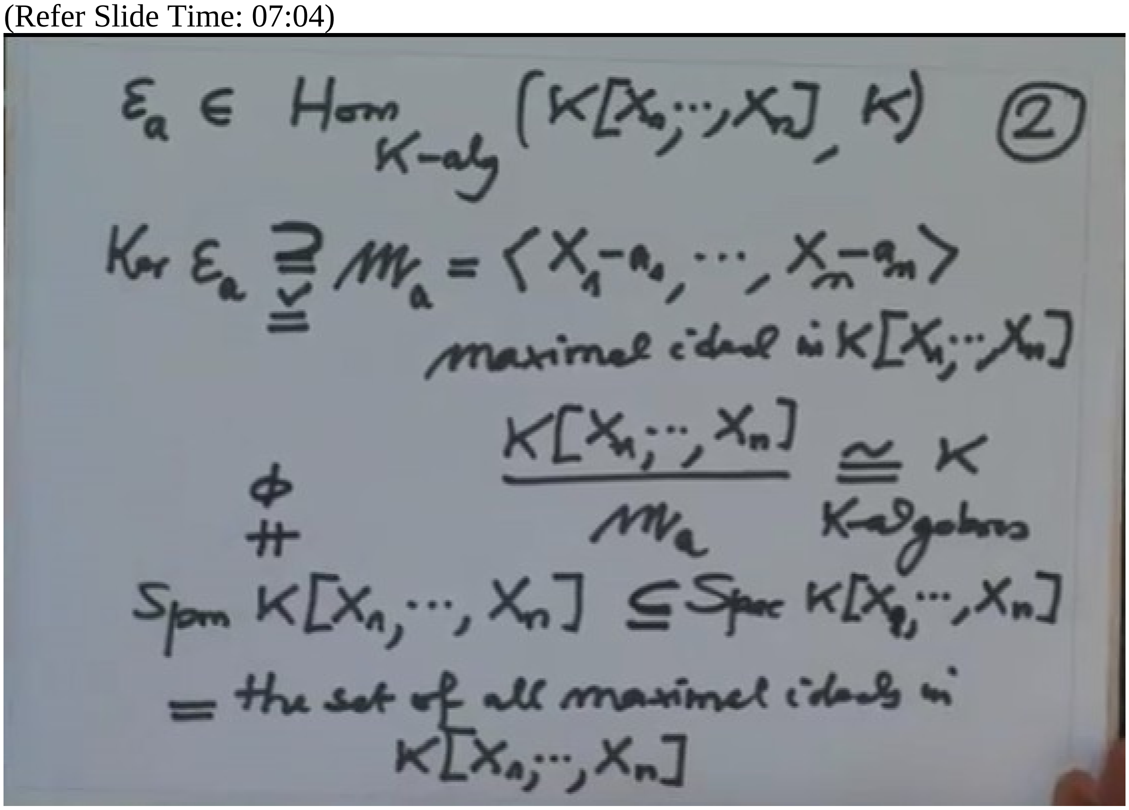
we have defined, so before that also I want to remind you that if I write $spm\,K[X_1,\ldots,X_n]$, this is precisely set of all maximal ideals, this is the set of all maximal ideals in the polynomial ring, this is a subset of $spec\,K[X_1,\ldots,X_n]$, spec is set of all prime ideal, this is set of all prime ideal, this is set of all maximal ideals and we know that every maximal ideal is prime ideal,

(Refer Slide Time: 06:56)



therefore this contentment also Krull Schmidt say that this is a non-empty set, anyway this is we can also prove it much easier than the Krull's theorem,

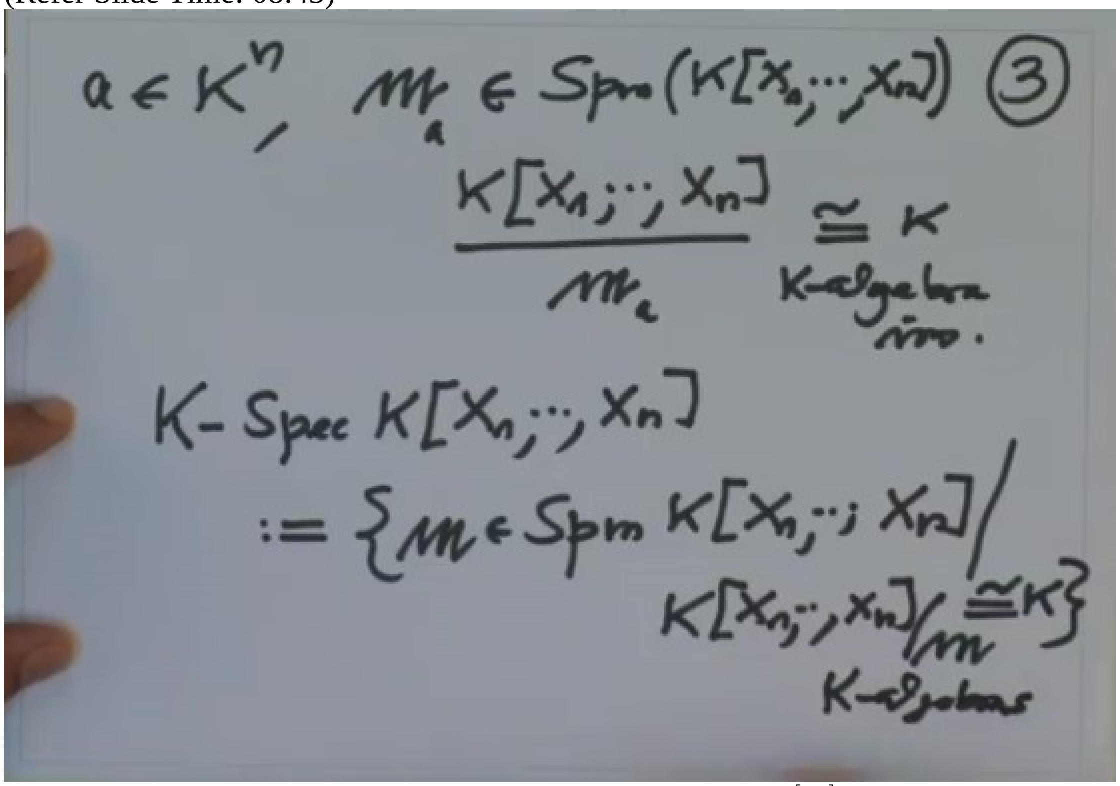
(Refer Slide Time: 07:04)



alright, so and this m_a, this is corresponding to $a \in K^n$, this m_a is not only maximal ideal, it is a special maximal ideal, this is a maximal ideal with the property that when I go mod it I don't get arbitrary field, but I get back the residue field; $\frac{K[X_1,\ldots,X_n]}{m_a} = K$ this is as K algebra, K algebra homomorphism.

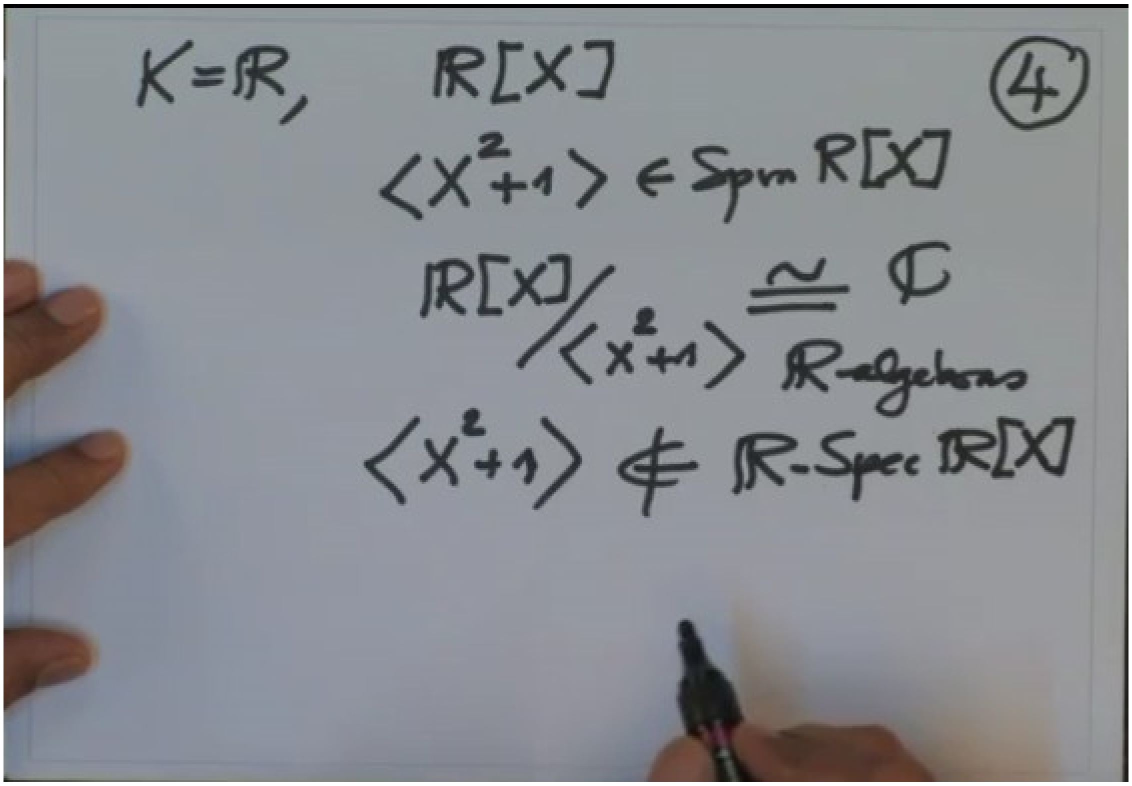
So these are the special maximal ideal, so that's why I'm going to denote by K spectrum, $K-spec\, K[X_1,\ldots,X_n]$ is by definition, all those maximal ideals M in spec in the maximal ideal SPM of K X1 to XN such that the residue field $\frac{K[X_1,\ldots,X_n]}{M}$ is isomorphic to K as K algebras, so note that every maximal ideal doesn't belong, here is just one example I'll write it.

(Refer Slide Time: 08:43)



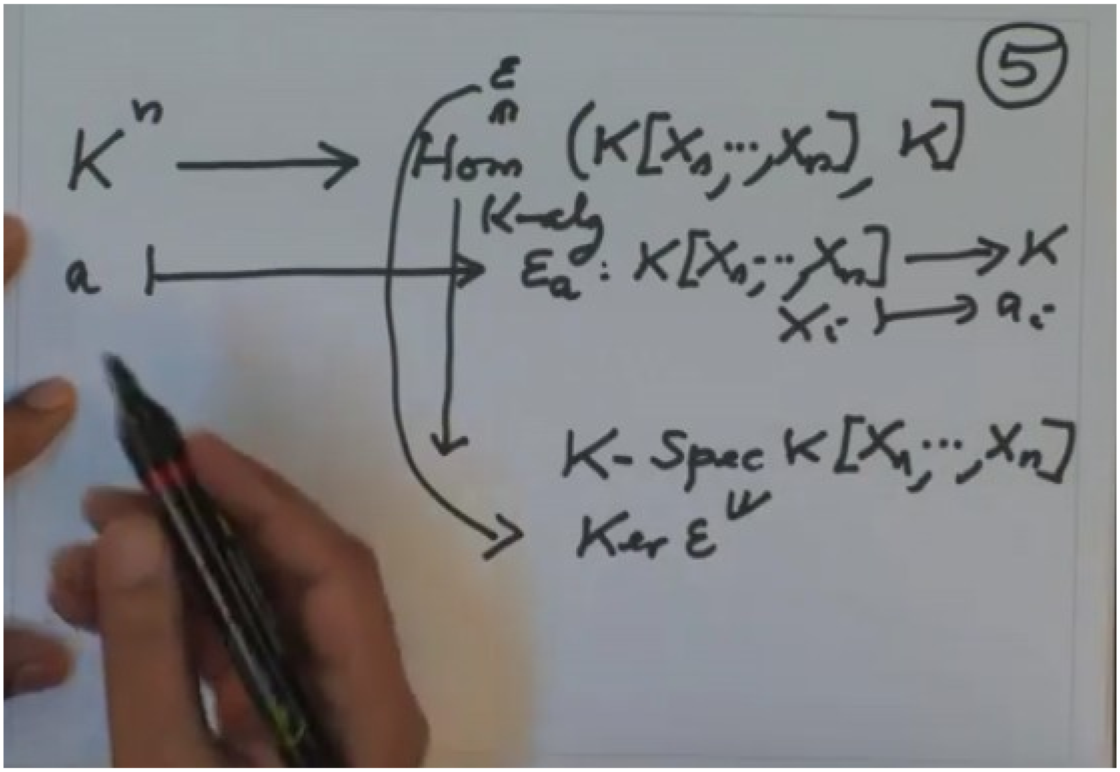
So suppose you take K = $\mathbb{R}$ and only one variable K is so $\mathbb{R}[X]$, in this, this maximal ideal $\langle X^2+1 \rangle$, this is a maximal ideal, this is in $spm\, \mathbb{R}[X]$, but it is not when you go mod this $\langle X^2+1 \rangle$, I don't get real numbers but this is isomorphic to complex numbers, as $\mathbb{R}$ algebras, so this shows that this maximal ideal $\langle X^2+1 \rangle$ is not in the $\mathbb{R}$ spectrum. The reason being $\mathbb{R}$ is not algebraically closed, as soon you will see it,

(Refer Slide Time: 09:46)



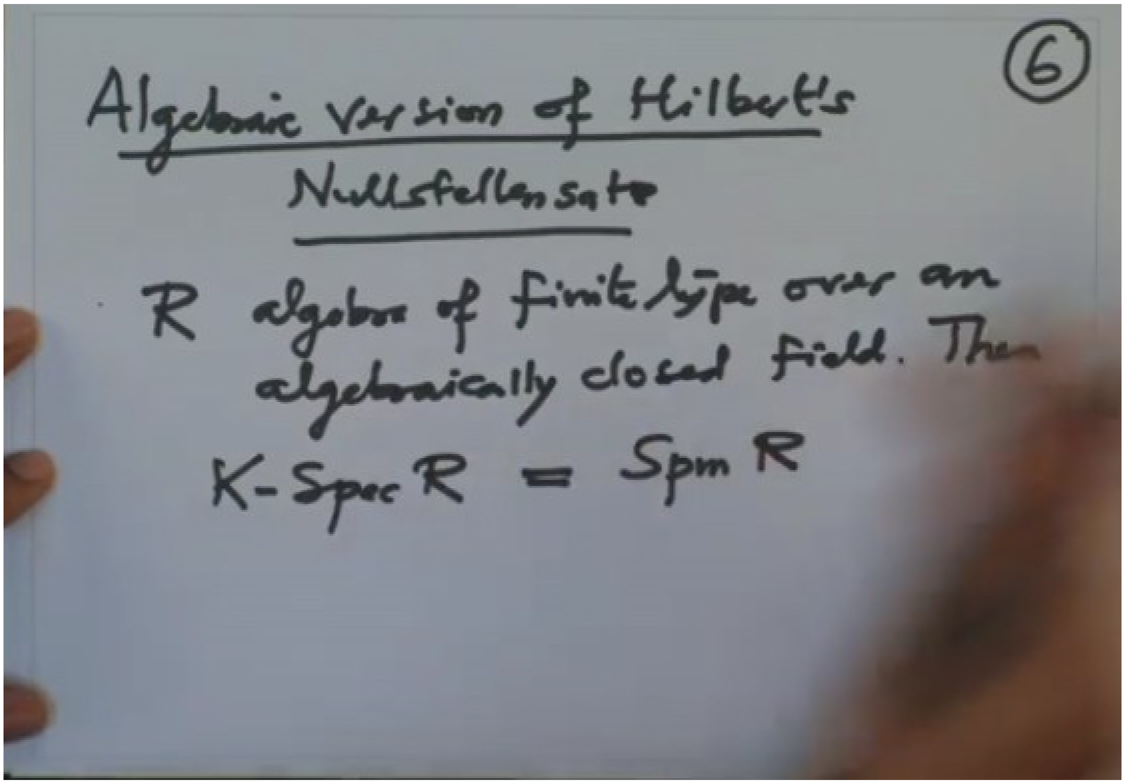
so we have therefore done the following, we have K^n here, we have defined $Hom_{K\,algebra}(K[X_1,\ldots,X_n],K)$ and this K spectra, K spec of $K[X_1,\ldots,X_n]$, so we have a map here namely any a that goes to the K algebra homomorphism ϵ_a . $epsilon_a$ is K algebra homomorphism which is a substitution homomorphism X to a is, this is X_i is going to a_i 's, and we have a map, so from $Hom_{K\,algebra}(K[X_1,\ldots,X_n],K)$ to $K-spec\,K[X_1,\ldots,X_n]$ any homomorphism epsilon that we are sending it to the kernel, so any epsilon in $Hom_{K\,algebra}(K[X_1,\ldots,X_n],K)$ is mapped on to kernel of epsilon, and note that kernel of epsilon has a property that first of all it is maximal ideal because it's a kernel, this map ϵ is subjective its clear because it's a K algebra homomorphism and this is therefore an element in the residue, when you go mod to the kernel you get the image, so this modular the kernel of epsilon is K, therefore it is by definition it is an element in the K spectrum.

(Refer Slide Time: 11:43)



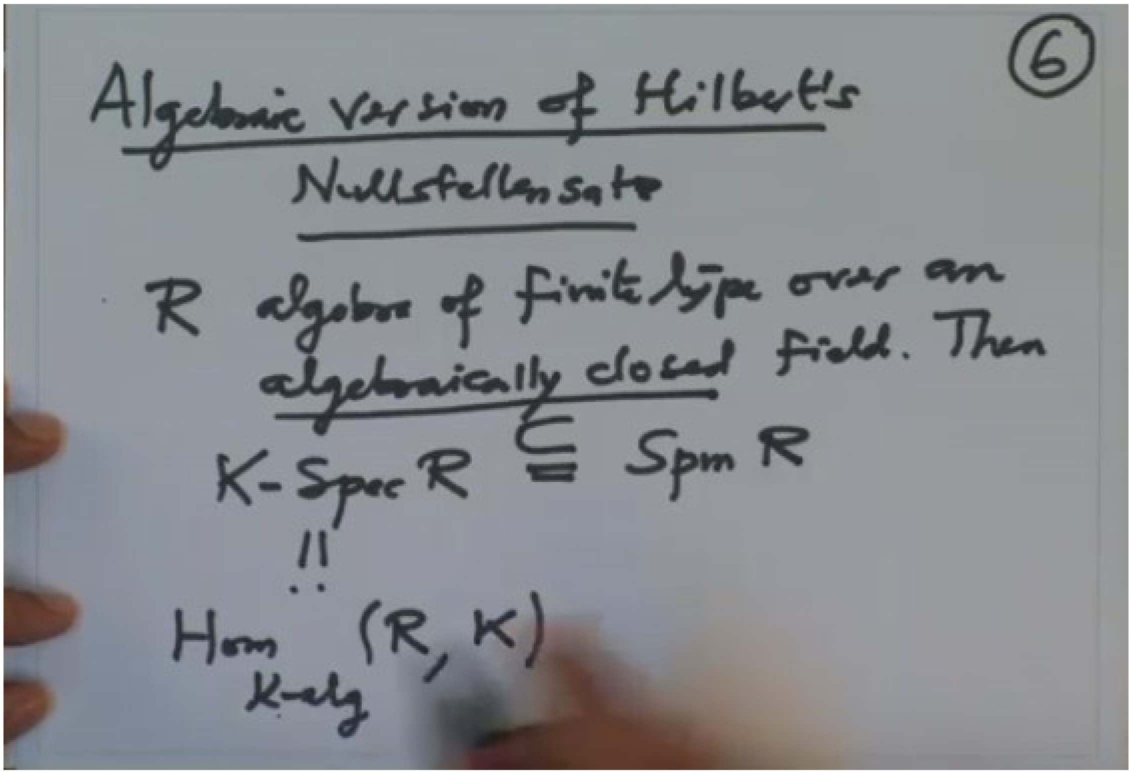
And now I want to say that these are the identifications, so these are bijective maps. So this is very important because of the following observation we already have made in the lectures, so that I just want to recall the algebraic version of Hilbert Nullstellensatz which says that if you have an algebra R of finite type over an algebraically closed field K, then in the K spectrum of R is precisely spm R.

(Refer Slide Time: 13:30)



So where, let me just recall that when I say what is the K spectrum for a finite type algebra that means this is by definition $Hom_{Kalgebra}(R, \mathrm{K})$, this is by definition over K spectrum is, and in case of polynomial algebra this we have identified with the points in the fine space and for arbitrary finite type K algebra this is just to think of this is a homomorphism, algebra homomorphism from R to K, and so this was how, it was not shaded in this format because I didn't have a definition of K spectrum that time, but what did we prove? The content of this thing is every, so this is clearly contained here that is no big deal because by definition these are all maximal ideals whose residue fields are actually K, so we are proved that Hilbert Nullstellensatz say that if you have the base to be algebraically closed then every maximal ideal is of the form M_a , so that was what it was proved in that format,

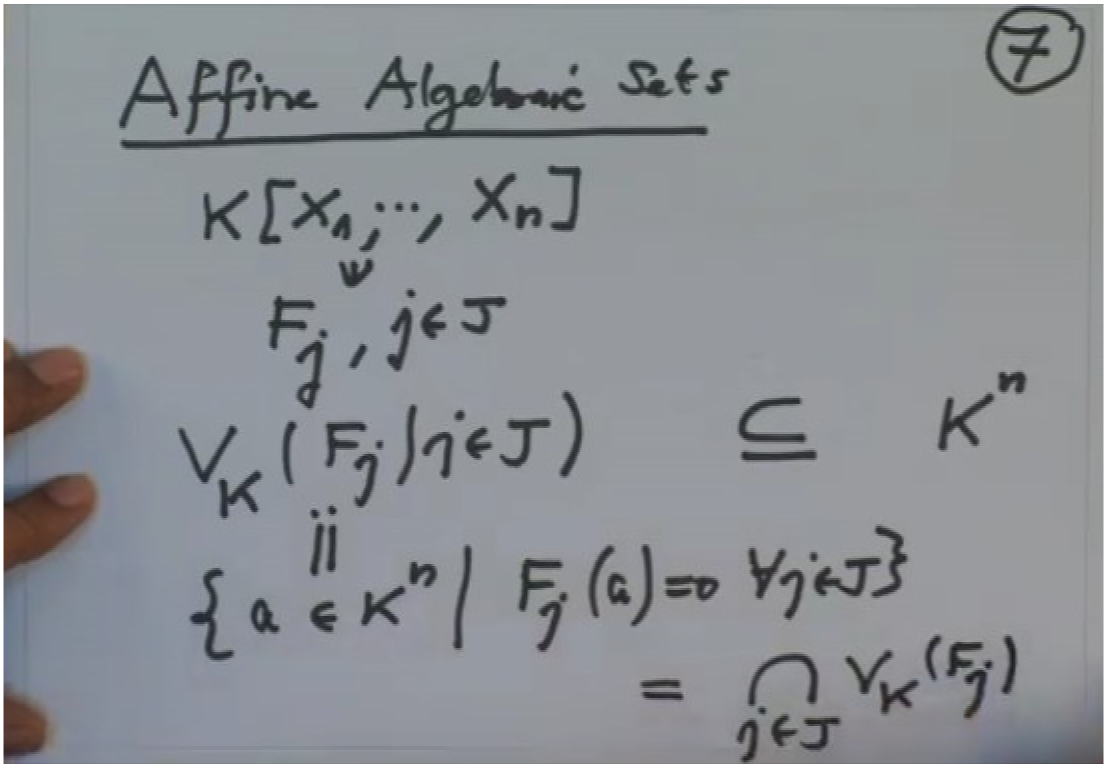
(Refer Slide Time: 14:47)



so this is precisely the Hilbert Nullstellensatz, so the K spectrum is the whole maximal spectrum that is how it is said, alright.

So I want to define now what is the Zariski topology on K^n , but before I do that I want to also consider the polynomial functions, but maybe I'll do this later so I want to define now what is so called affine algebraic sets, and this affine algebraic sets will give us a topology and that topology will be called Zariski topology, alright, so what is the affine algebraic set? So I have a polynomial rings $K[X_1,\ldots,X_n]$, and I have some polynomials, set of polynomials $F_j, j\in J$, they maybe finite, may not be finitely j in J is are elements of this, so I have lot of polynomial, and now I want to define some subset of K power N, see ultimately I want to define a topology on K^n so $V_K(F_j : j\in J)=\{a\in K^n \textit{ such that } F_j(a)=0 \textit{ for all } j\in J\}$, note that this is also same thing as intersection, they're common 0's, so intersection running over $j\in J$, $V_K(F_j)$, so this is the set of,

(Refer Slide Time: 17:15)

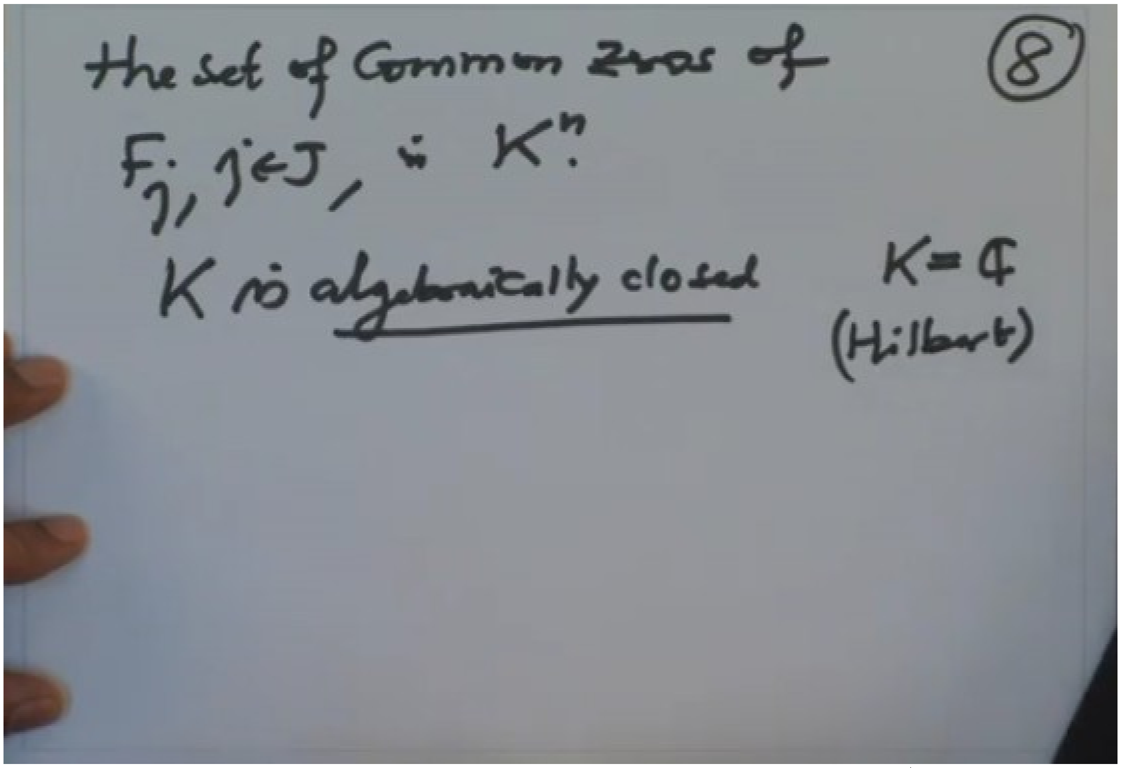


I'll write in the next page, the set of common zeros of F_j 's in K^n .

Now it might happen that this maybe empty set that is one of the biggest difficulty in studying algebraic geometry, so therefore one usually assumes we will not do that but one usually assumes, one needs to assume that K is algebraically closed, but we will not do this so K for example, K = $\mathbb{C}$ it was done in the classical, classically K = $\mathbb{C}$, this was done by Hilbert, so anyway we are not going to assume K is algebraically closed, but we will allow this could become empty set, alright.

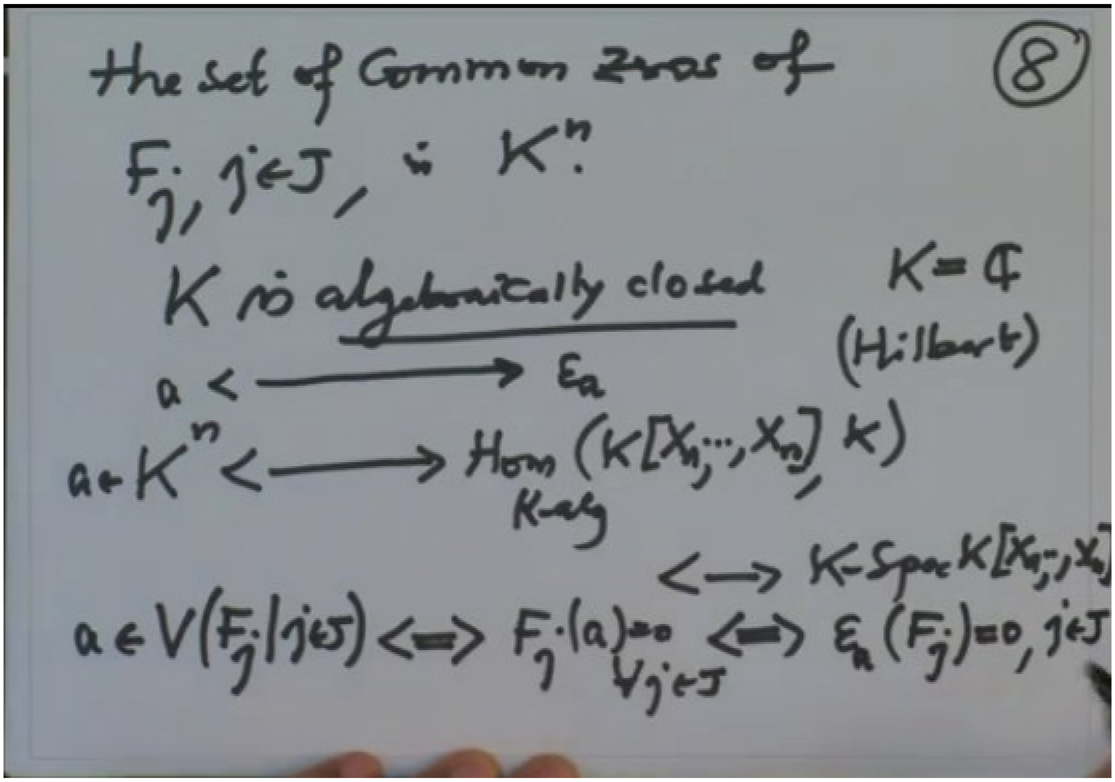
So now first of all note that, so I just want to translate this in terms of that the algebra homomorphism,

(Refer Slide Time: 18:39)



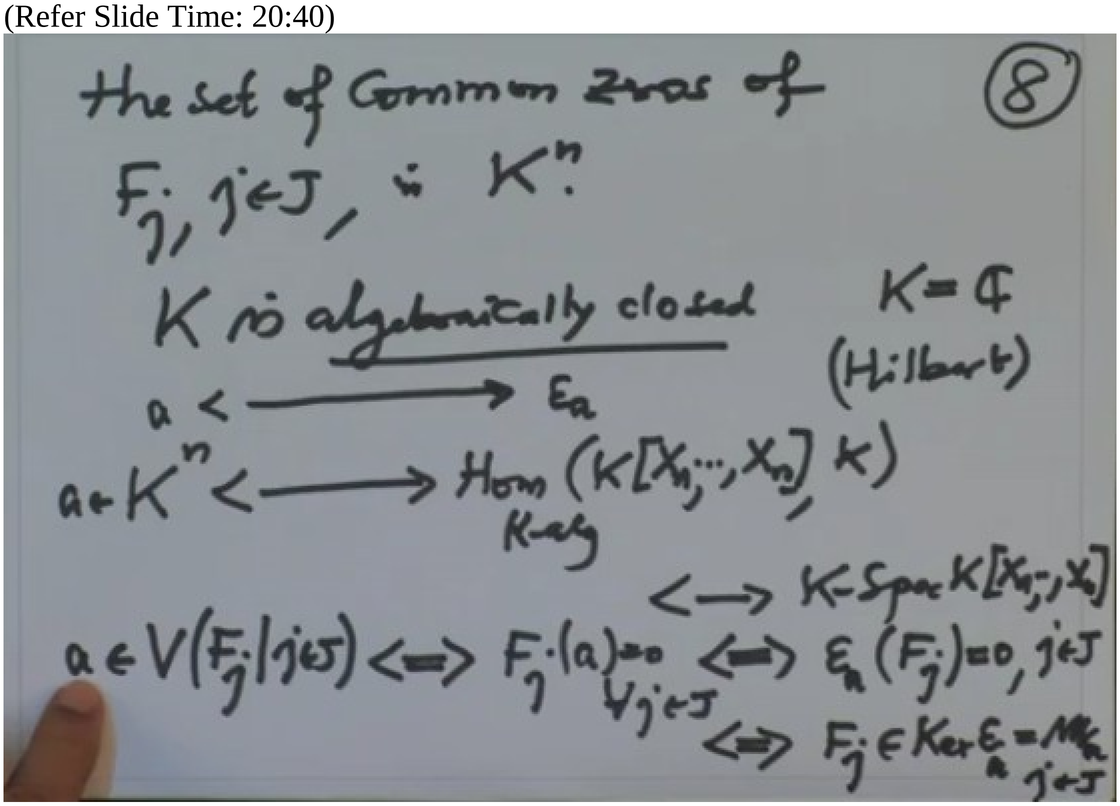
so note that under this K^n we have identified this with $Hom_{K\,algebra}(K[X_1,\dots,X_n],K)$ and then this with, and then $Hom_{K\,algebra}(K[X_1,\dots,X_n],K)$ with $K-spec\,K[X_1,\dots,X_n]$, so I just want to translate when will given $a\in K^n$ will belong to, when will this a belong to $V(F_j)$ then it'll be common 0 of this, so that will get translated into all this polynomials will vanish it so F_j at a is 0 for all $j\in J$, but this will mean that all this polynomials will be in the kernel of, so this a is identified with this substitution homomorphism, so this a is identified with this, so therefore this condition will become now $epsilon_a(F_j)\,j\in J$ all this polynomials are, they will go to 0 under this algebra homomorphism, this is true for all $j\in J$.

(Refer Slide Time: 20:20)



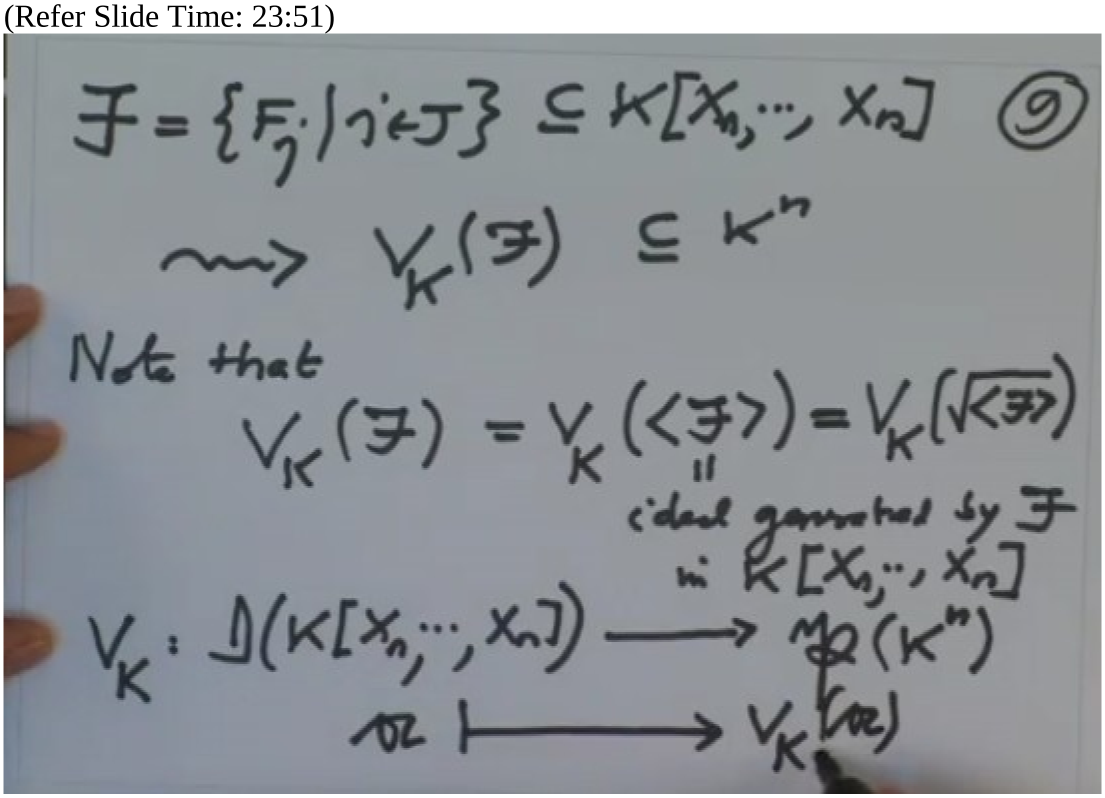
But that mean this F_j 's are in the kernel so that is if and only if all the polynomial F_j 's belong to kernel of ϵ_a and that we know this is M_a , and this is true for every $j \in J$, so therefore to say that a point belong to this V,

(Refer Slide Time: 20:40)



I want to keep track of V_K , there also I wrote, so this one is get translated a point belong, so in this case that means ϵ_a at F_j is 0 or in the K-spec K is, that is F_j belong to the maximal ideal M_a .

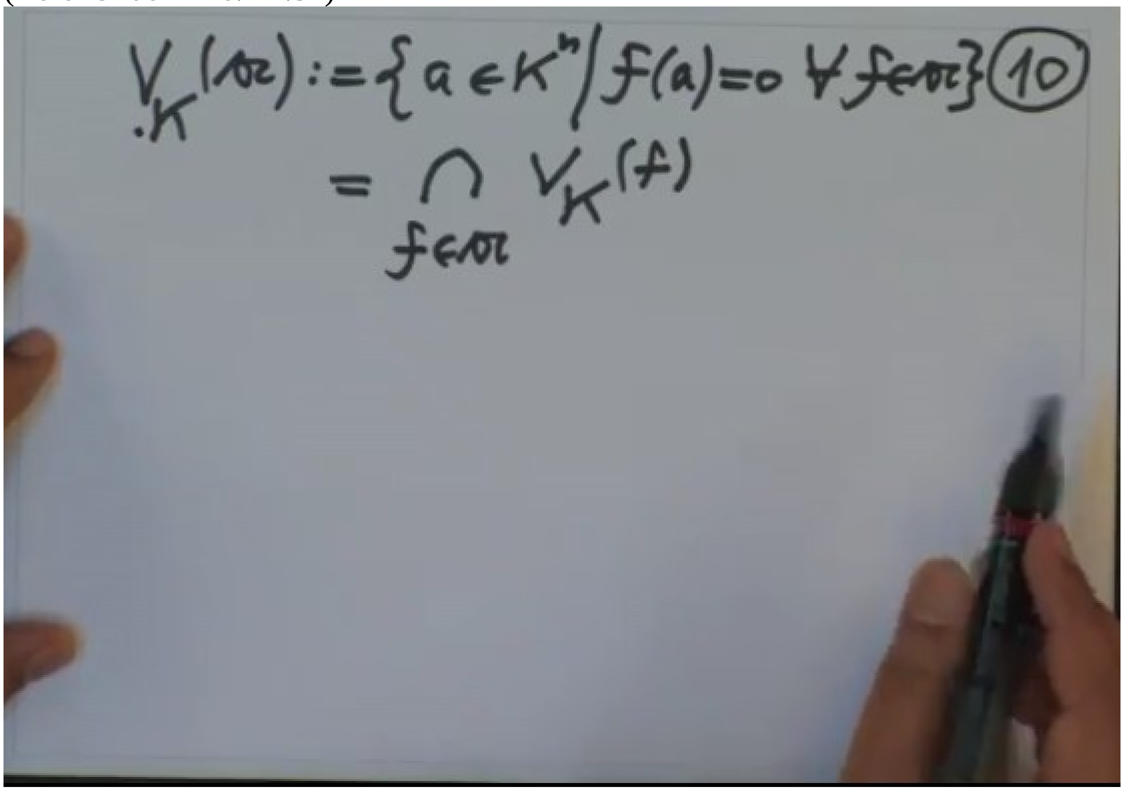
So now before I go on I want to make some couple of remarks, so that means to a bunch of polynomials to a subset, let me call it F, to a subset F, this subset F_j of some subset of the polynomials in several variables over K, we have defined a subset $V_K(F)$, this is a subset of K^n , so first of all note that these are very easy to check, note that $V_K(F)$ is same thing as $V_K(\langle F\rangle)$, so this is ideal generated by F, this is ideal generated by the set F in $K[X_1,\ldots,X_n]$, because if all, this is the set of all polynomials, all points which, all the polynomials in F vanishes but these are all polynomial linear combinations of F, so if all polynomial vanishes then the polynomial linear combination will also vanish, so this is really easy and second equality is actually V_K of the ideal, not only the ideal there radical of that ideal that is also equal because you know the polynomial belong to the radical is the power belongs to the ideal, so if the power vanishes because images are in, we are taking the images inthe field K, so if the power vanishes then the original polynomial also vanishes, so therefore this equalities, so we could have defined, so this V_K is really you can think of V_K in the map, think V_K the map from the set of ideals, $I(K[X1,\ldots,X_n])$ to $P(K^n)$, to the power set of K^n , so any ideal A were mapping on to the $V_K(A)$.
(Refer Slide Time: 23:51)



$V_K(A)$ is by definition all those points in K^n such that all polynomials in A vanishes it, so this is, so I will write on the next page, so $V_K(A)$ is by definition all those points A in K^n

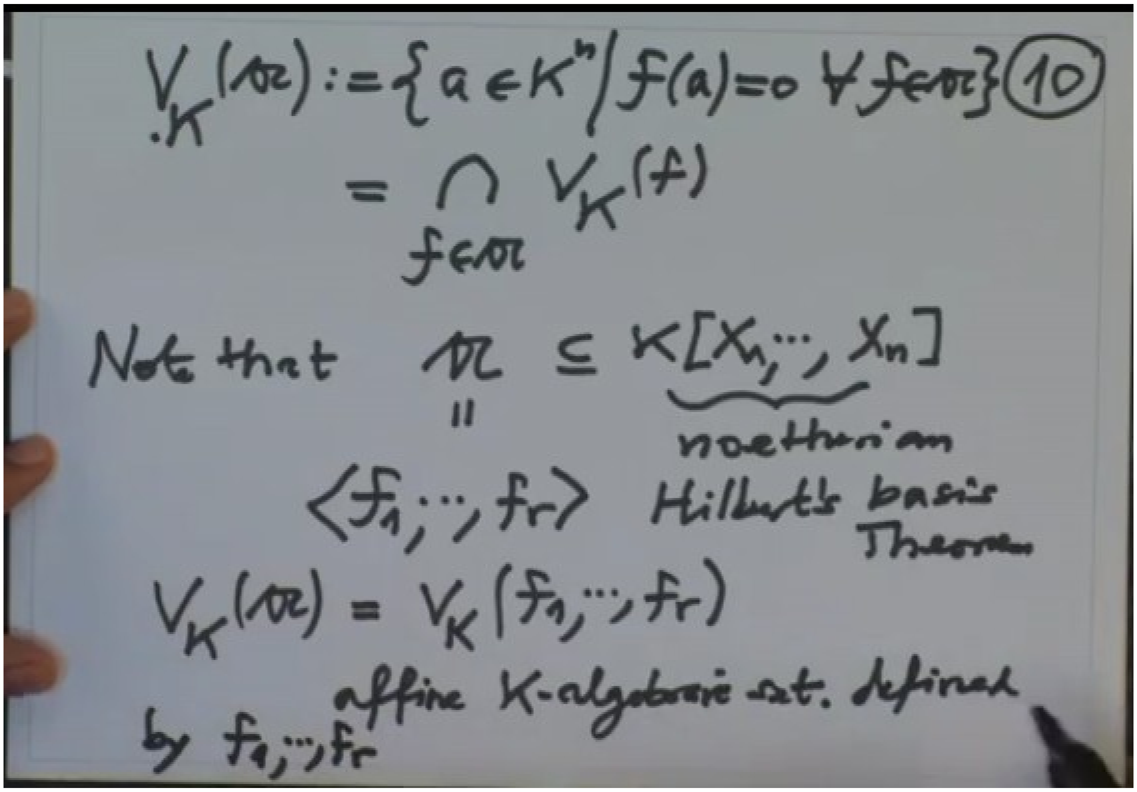
F(A) = 0 for all F in the ideal A, so this is same thing as intersection F in A such that $V_K(F)$ is so it is this, this is clear.

(Refer Slide Time: 24:37)



So we have defined a map from the ideals in this ring to the power set of this, and now we would like to check when is it bijective or what is the possible inverse for this or such thing, so and study the property of this sets and I will list some properties which are mostly easy to check, so for example so let me write in the form of proposition, so before I write that note that A is ideal in the polynomial ring and we know the polynomial ring over a field in finitely many variables it's the noetherian ring, this ring is noetherian, therefore every ideal is finitely generated, so therefore this ideal is really finitely generated, it is generated by f_1 to f_r, therefore this is, this was precisely proved by Hilbert and it is known as Hilbert's basis theorem, therefore V_K of, when you want to study the set $V_K(A)$ it is enough to study VK of this generators, this is finitely many, so this is a common zero set of finitely many polynomials, so this is also called affine K algebraic set define by f_1 to f_r, this is a subset of K^n, alright.

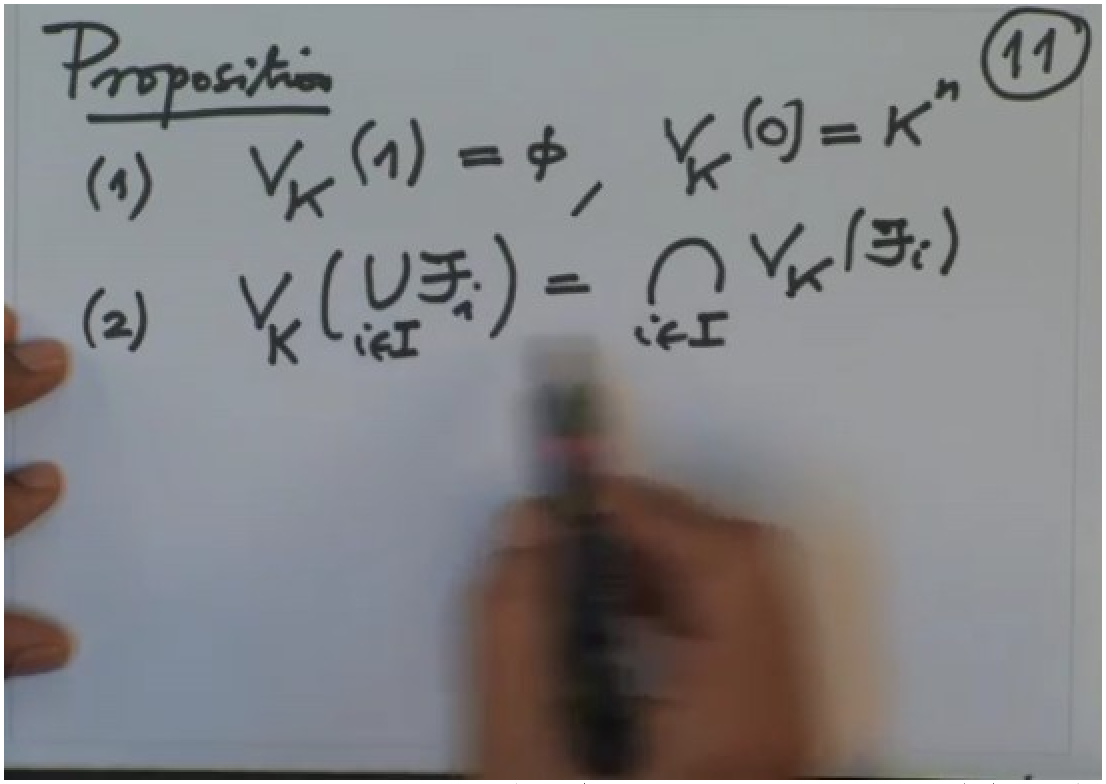
(Refer Slide Time: 26:44)



Now before, actually I want to state the proposition, so proposition which proves, we'll leave it to check because once it is stated precisely it is very easy to check, so all over notation as it is, okay, so the first one is if I take V_K of this single polynomial 1 that means the 0 set of the polynomial constant, polynomial 1 this is obviously empty set. And if I take V_K of this 0 polynomial, this is obviously the whole K power N, because 0 polynomial vanish it every point and one polynomial one vanish only at 1, okay.

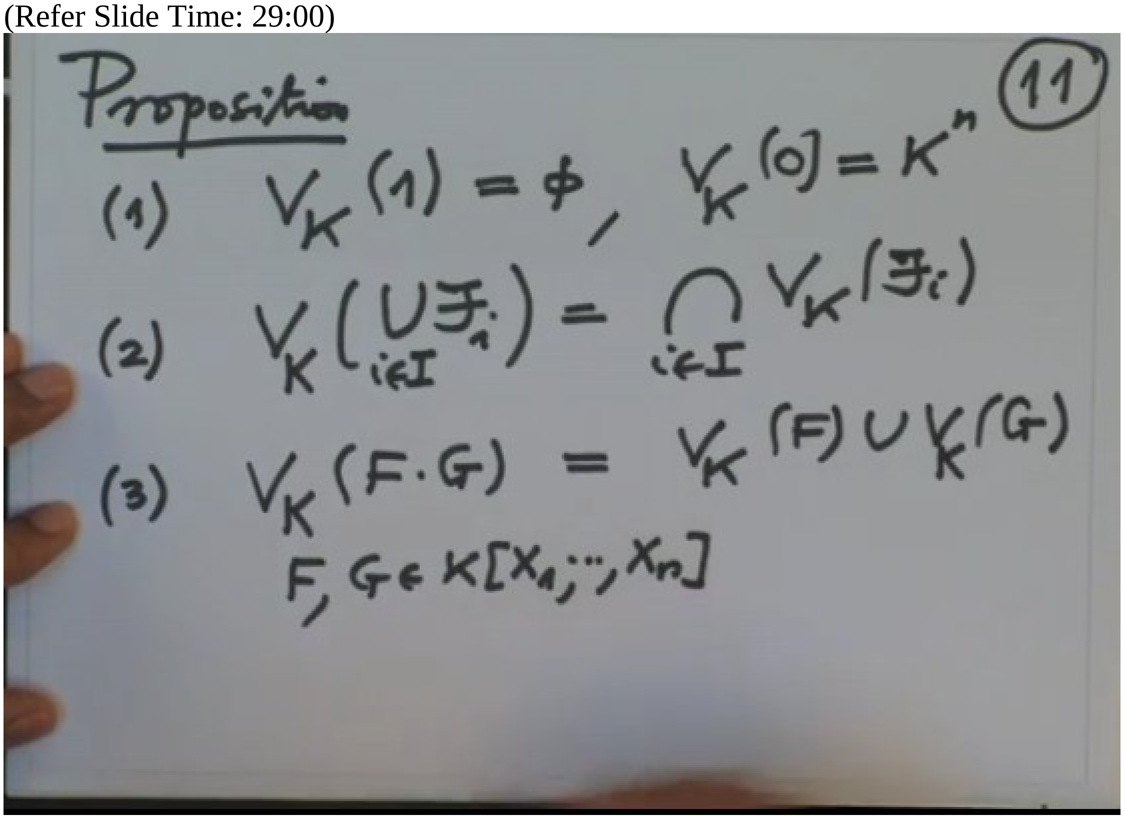
So this is 2, if I take the union V_K of union of the subsets f_i 's arbitrary union that is same thing as intersection $i \in I$ $V_K(f_i)$, because this is what? This is all those points in K^n where all the polynomials in all f_i 's are 0, that means they are all common so therefore if the union will become intersection.

(Refer Slide Time: 28:16)



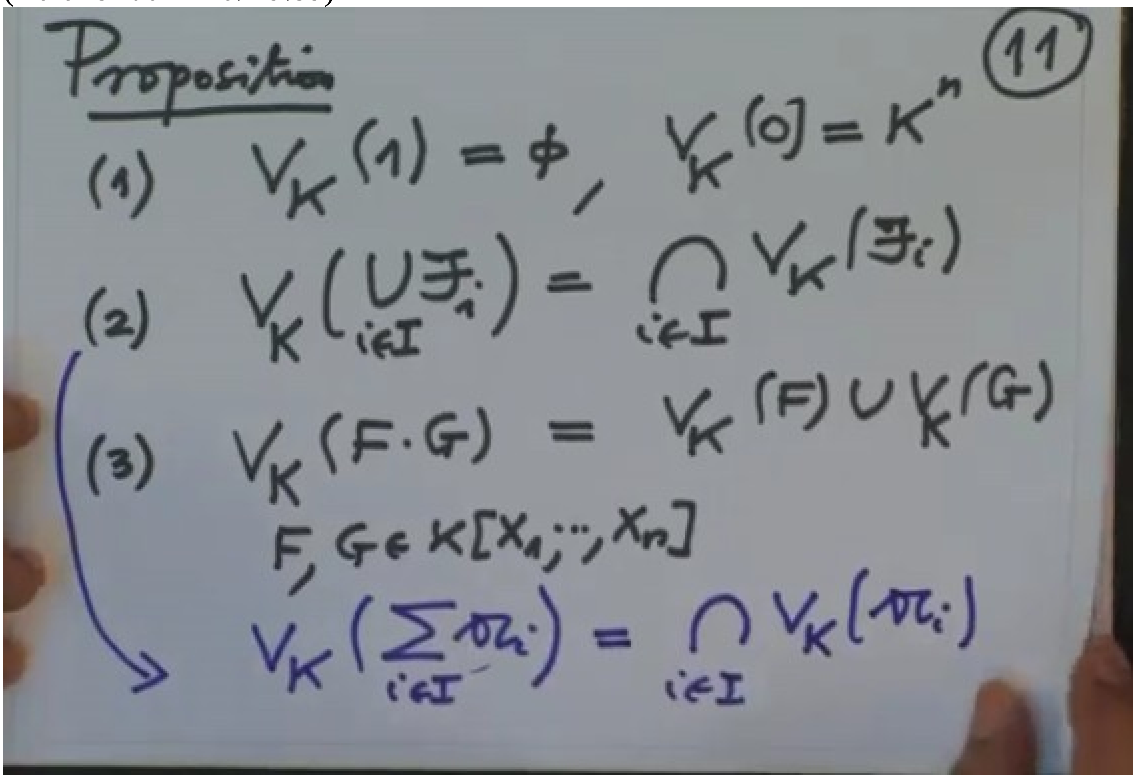
Third one, if I take 2 polynomials $V_K(F.G)$ this is same thing as $V_K(F) \cup V_K(G)$, because if a product vanishes at some point, then as a F vanishes or G vanishes, so this will become union.

(Refer Slide Time: 29:00)



So you can more generally state this for ideals, so what do be the corresponding things for the ideals, I'll write in blue colour, so this one for the ideals, so V_K of the sum ideal where A_i 's are the arbitrary ideals, family of arbitrary ideals index by I, this is same thing as intersection $i \in I V_K(A_i)$ because this means all, all polynomial vanish and therefore this,

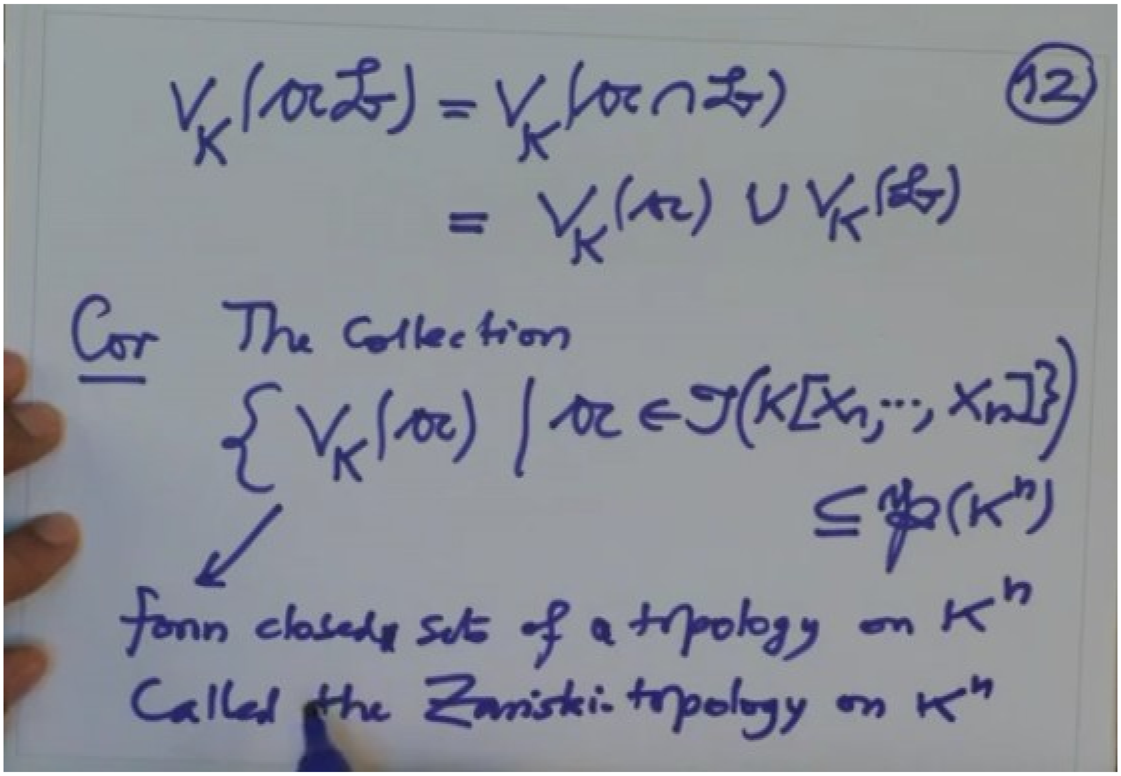
(Refer Slide Time: 29:39)



and the third one will become, third one for ideals will become V_K of the product ideal, if I have 2 ideals A and B, then V_K of the product ideal is same as V_K of the intersection ideal is also same thing as union this 2 ideals $V_K(A) \cup V_K(B)$.

Now it is clear that this, so therefore so corollary is the collection V_K of ideal A where A is that is in the ideal of the polynomial ring, this collection, this is a subset of the power set of K^n , this collection form closed sets, closed sets of a topology on K^n and that topology is called the Zariski topology on K^n ,

(Refer Slide Time: 31:14)



so just one minute I'll check how, what do we have to check? You have to check that, if you want to check that some collection form a closed subsets of a topology we have to check that empty set is there, whole space is there and it's closed under arbitrary intersection and also it is closed under finite union, so now you see here this precisely this what I have checked, this is one is empty set is there, two is this second part is K power N is there, so empty set is there, whole space is there and then the intersection is there, and then the union is there, therefore it forms a topology, this topology is called as Zariski Topology, and next time we will generalize this to arbitrary ring, and there we will have difficulties, but we'll have to overcome that difficulties by enlarging the set K power N, so that we will do it in the next half.

Prof. Sridhar Iyer

NPTEL Principal Investigator
&
Head CDEEP, IIT Bombay

Tushar R. Deshpande
Sr. Project Technical Assistant

Amin B. Shaikh
Sr. Project Technical Assistant

Vijay A. Kedare
Project Technical Assistant

Ravi. D Paswan
Project Attendant

Teaching Assistants

Souradip Das Gupta **Hemen Gogoi**

Bharati Sakpal
Project Manager

Bharati Sarang
Project Research Associate

Riya Surange
Project Research Assistant

Nisha Thakur
Sr Project Technical Assitant

Project Assistant
Vinayak Raut