Basic Linear Algebra
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Lecture – 06
Systems of Linear Equations III

So let us say that or we are going to do on induction, number of rows, 1*m. There is only 1 row in the matrix. Can you say it is in the row echelon form? No basically because if it is non-0, right, some 0's may be coming. First time there will be some non-0 if it is a non-0 matrix. That is fine. There is only 1 row, so nothing to worry, right. R1 is, R is 1, it is 1*n, R has to be between 1 and n.

So it is equal to 1. So the proof that a row matrix, a row vector is in row echelon form if it is non-0. This is okay? By default, it is non-0. So it has to be in row echelon form. So induction starts, m=1, the proof is, let us assume when the number of rows is something, it is okay. When number of rows become 1 more, how to do it, right? So that is a question. So let us look at that.

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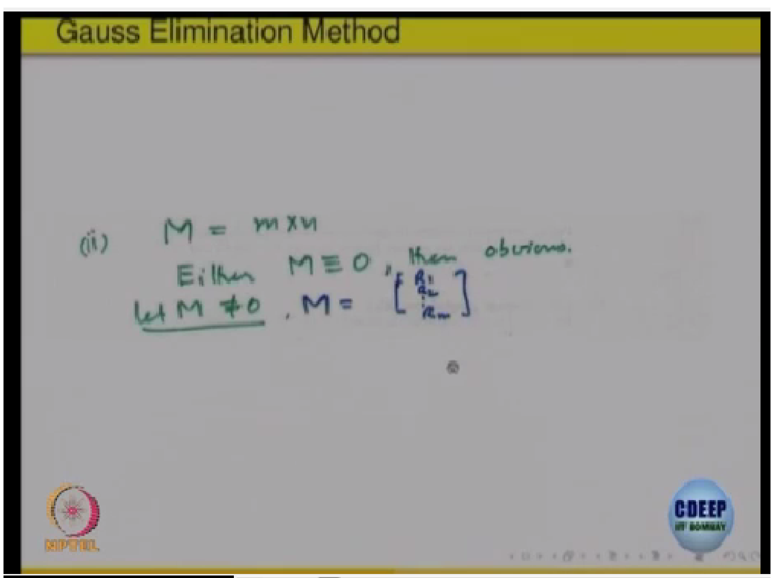

So let us say these are the rows of R1 R2 Rm. m is non-0 matrix, so it has rows, R1 R2 Rm. So what one does is?

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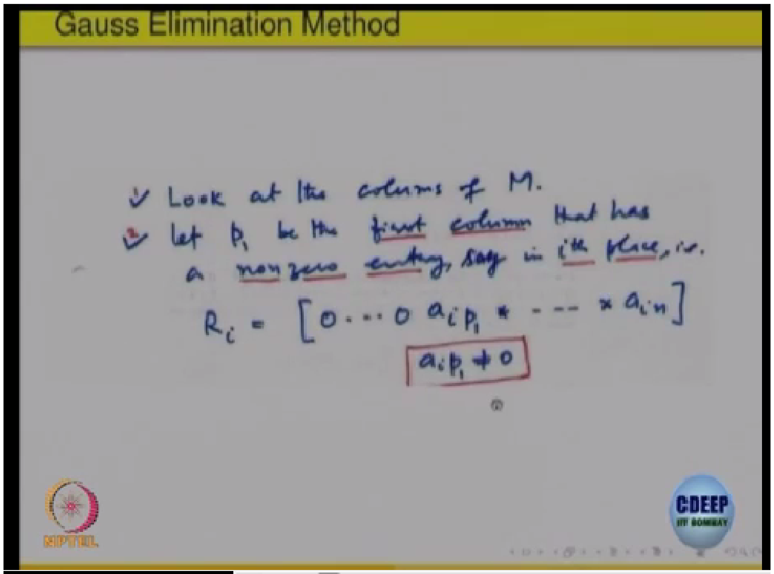

Look at the columns of m. We are given a matrix, right. Look at the first column. Possibly everything is 0 in that column, fine. So what will the machine do? It will scan the first column. No non-0 entry, go to the second column. Non-0 entry, so look at a column where the first time it spots a non-0 entry in some place. We do not know where. So let p1 be the first column that has a non-0 entry.

First column 0, second column 0. The first column where non-0 entry somewhere comes is the column number p1. So non-0 entry has come at some place, that we should make it as the first one, that should make as the row. So that row should be made as the first row. Because everything else was 0 on the left side of it. Is it clear what we are doing? Look at the column 1. Column 1 is all 0.

Scan column 2. All 0. At the p1 column, the first time a column has a non-0 entry is p1. Now it will come at some place. Say it is at the ith place. So pip1 is non-0. Everything else on the left side is 0, right. Everything at the top is also 0. So what we do is, we take this row and make it as the first row. So what will happen? Everything else is 0. The first non-0 entry is coming at the place p1, right.

So first stage is achieved. The first non-0 row as a non-0 entry at a place p1. Everything on the left is 0. At the bottom, we do not know what it is, right. But if there is a non-0 entry coming somewhere at the bottom, we can make that also 0 by row operations. So everything below that can be made as 0, right. So what we have gotten? p1 non-0, everything below that is 0, everything on the left is 0.

Now again now starts scanning the remaining part of the matrix. The remaining part of the matrix, what is the order of the remaining part? Already 1 row is gone. So 1 row less. So by induction, I should have row echelon form for that. So proof is over, right. So it is done by scanning the columns. Scan a column, non-0 entry, make it as the first row, right. Now go to the next columns.

Spot the non-0 entry. Make that as the second row. And everything below that keep on making it 0. So that will make, right, that is definite way of telling a machine that proceed this way, write systematically and in the end, everything is over. Everything you scanned, all the columns are scanned, your matrix will be in row echelon form.

So that is what is written here. So it says look at the columns of m. Let p1 be the first column that has a non-0 entry. Say ith place. First column non-0 entry. So this is what the ith row will look like. 0 0 0 first non-0 entry coming here. Now that should be made as a first one, right. So change it.

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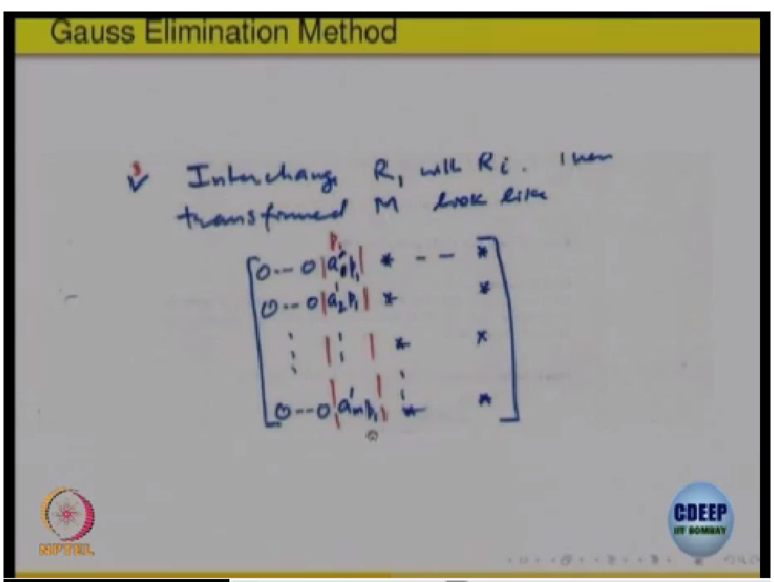

So change it to the first one. Something is there at the bottom, we do not know. But on the left, everything is 0. These may be non-0 entries here at the bottom. But what we can do is? By row operations, I can make these entries as 0, right by a suitable multiplications. This is non-0, divide by, right, multiply by the 1/of that number, that will become 1, negative of that, add it here, I can make this as 0, make this as 0, this as 0, right.

So once a non-0 entry has come at a place, everything at the bottom can be made as, in that column, can be made as 0 by elementary row operations. So I will have 0's here, I will have a non-0 entry here and this will be all 0's. So what is left? Left is this part of my matrix and that has order less than m, right. Number of rows has reduced, okay. So that will mean what? A number of, this has reduced, that means by induction, I can have that as in the row echelon form.

I can go on applying this. I look at this column now and say non-0 entry will possibly will come here and change it in this part. In this part, so I will be looking only this matrix now. So look at this column here. Non-0 entry come again, I will shift it up and so on, okay. That is the induction part, okay. So that is how.

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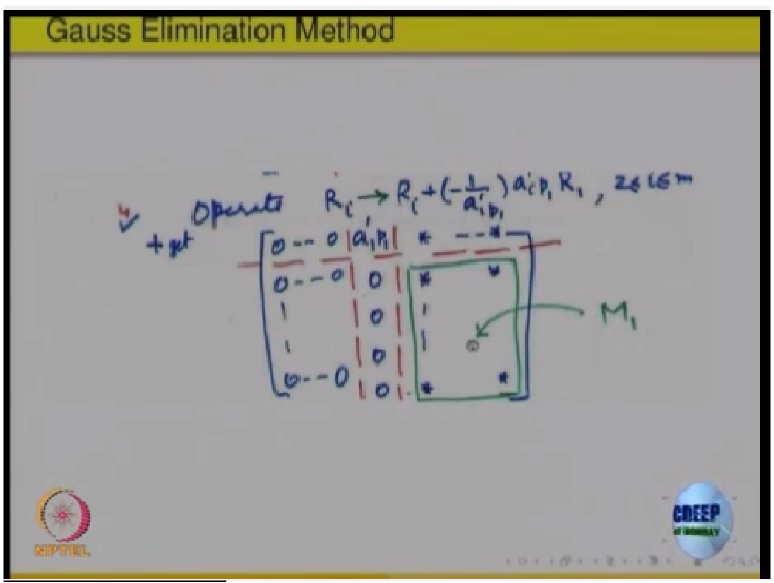

So the second step was, this is non-0, everything else is made 0, right, which I said. Now look at this part and do the same thing. Now repeat the process on this part. So in your computer language, what you will say, do loop, right. This is a loop thing. On this matrix, do nothing.

Because everything is reduced, then that will be the remaining part and so on.

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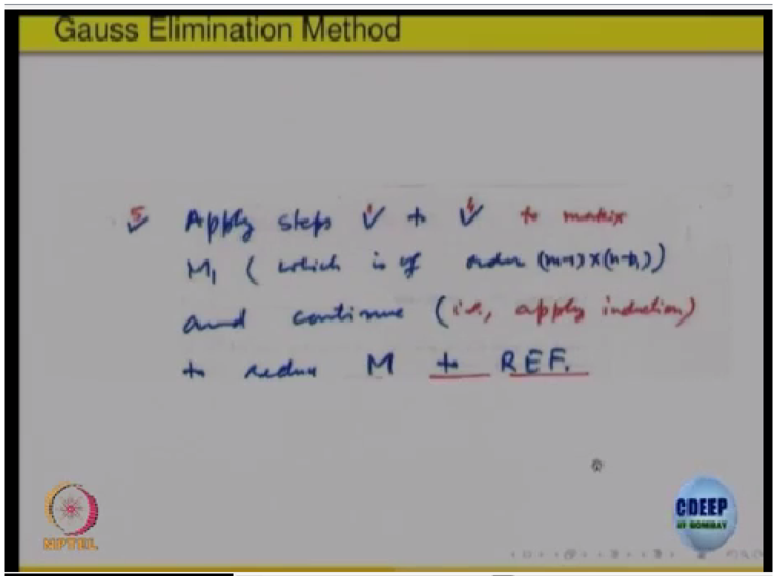

So eventually, apply this 1 and 4 in m1 and everything by induction, you should have. So this is a definite way of doing things which can be fed to a machine also, right, okay. So how do we use this row echelon form to solve system of linear equations?

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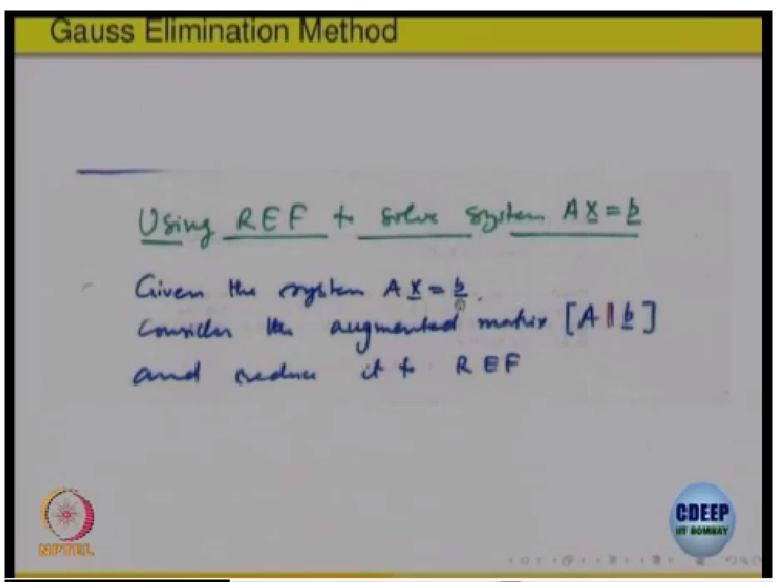

So we have a given system of linear equations AX=b, right. Then we will make the augmented matrix because b is going to important. So this is a new matrix which has got 1 more column added to it and we will reduce it to row echelon form, right.

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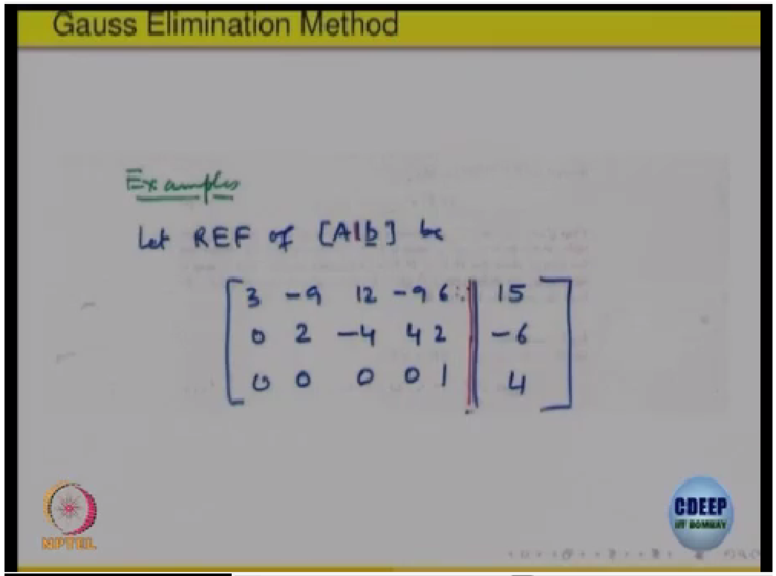

So I think let us give the video a break and try to solve this problem now. Then we will see the solution. So look at the system where the augmented matrix is given, sorry, not, actually it is in the row echelon form. So this example is already in that form, right. Is that okay? You are doing so that this matrix comes in the, A is the matrix. Let us discuss this example now. So this matrix Ab, how A is already in the echelon form.

Why? There is no non-0 row, okay. First non-0 row non-0 entry coming at 3, first column. Second at second column and last as column 1, right. So how do you read the system now? This has come from a system of linear equations, right. So what is the new system now? This was variable x1. This was variable x2. This is variable x3, x4 and x5. So what is the last, when I multiply this with this, what do I get?

Ax=b. So will give you x5=4. This is okay? The matrix multiplication. So reading is x5=, keep in mind, x1 x2 x3 x4 x5. The columns are coming from the variables, right. So what is this equation? This is x5=4. What is this equation? 2x2, this is coming from x2, 2x2-4x3+4x4+2x5=-6 and this gives you 3x1-9x2+12x3-9x4+6x5=15. So your new system is that 1, right. x5=4. So already value is known now.

So that value I can put in this equation, right. So what I will get, I will put that value 2*x5, this will be a number now in the new equation. So I will have 2*x2-4x3+4x4+8=-6. So it has got only 3 variables, is the equation in 3 variables, right. And the relation between 3 variables equal to something, constant, right. So if I put values for 2 of them, I will get the third one, right. So normally what one does is, one keeps the value where pivot is coming, you determine x2 in terms of x3 and x4.

So that is backward substitution and giving arbitrary values. x5 is already determined. You give arbitrary values to this variable and this variable, that is x3 and x4. So for each value of x3 and x4, you will get a value of x2. So x2 is known, x3 is known, x4 is known, x5 is known. Put that value back in the above equation, that x1 in terms of those values, right. So how many variables you are getting arbitrary values?

We gave arbitrary value to x3, sorry. We gave arbitrary values to x3 and we gave arbitrary value to x4. x5 was already determined. So your general solution, right, will be determined by giving arbitrary values to x2 and x4. And these 2 variables can get infinite number of values, right. Each value will give you a value for x1 x2 x3 x4 and x5. So you will have infinite number of solutions where the 2 variables getting arbitrary values, right.

You can think of, there is a degree of independence kind of thing, is 2 which are getting arbitrary. This question is in this equation, this equation is 2x2-4x3+4x4+2x5=-6. In that equation, x5=4 we can put. So I will get a relation as 2x2-4x3+4x4+8=-6. So 8 I can shift on the other side. I will get a relation between x2 x3 and x4. x3 and x4 are now given arbitrary values. So x2 is determined in terms of those arbitrary values.

Or you can say x2 can be determined in terms of x2 and x4 by shifting on the right hand side, right. So for each value of x2 and x4, you will get a value of x2. x5 is already fixed. Now for these values, you can put back in this equation, the top one, right. You will get a value of x1 which will be in terms of x4 and x3 and x4. It will involve those values of x3 and x4, right. Yes, but x1 can be, x1=, in terms of x2 and x4.

So x1 can also be determined in terms of x2 x3 and x4. x2 is determined in terms of x3 and x4. So this is what is called backward substitution. Go from the bottom to the up, substitute the backward. Whatever is arbitrary, put that arbitrary value, determine the next one from the upper equation, right. So let us, so x4 is 5.

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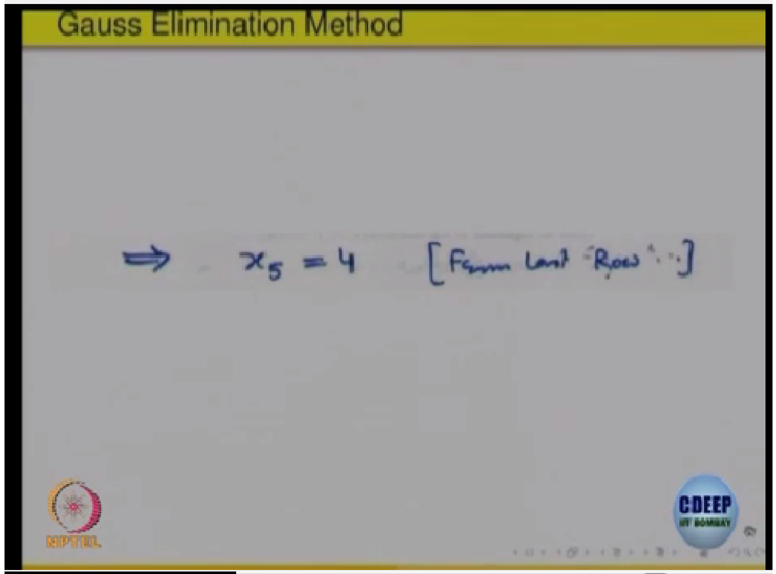

I am just repeating that solution again, right, last equation.

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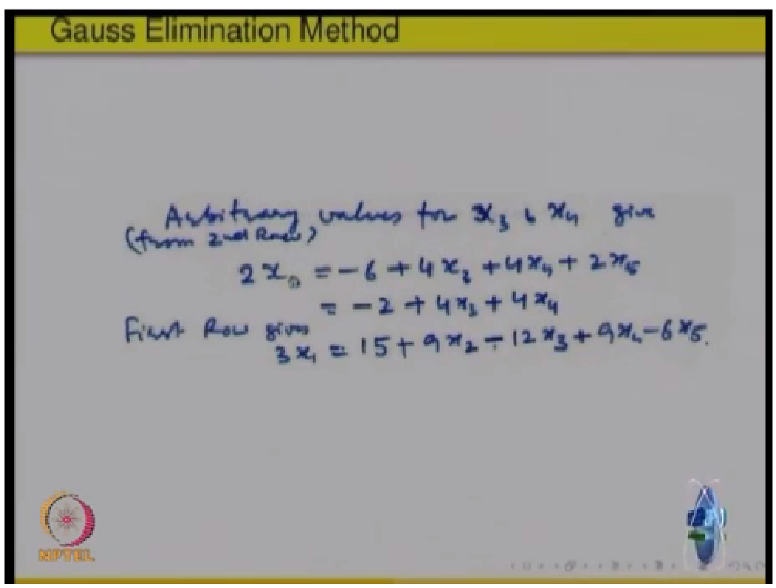

So now second equation, x2x, shifting everything on the other side. So that gives you a relation between x3 and x4, right. So from this if I put x3 and x4 arbitrary values, I will get x2, right. But I can keep this in terms of x3, x4 and put it in the first equation backward, that will give me 3x1 in terms of this where x5 is already known, right. x3 x4 in terms of, x2 in terms of these 2. So if I put this value of x2 here, I will get x1 in terms of x3 and x4, right.

So x1 in terms of x2 x3 x4. x2 in terms of x3 x4. x5 is already known. So your solution will involve only 2 variable quantities, x2 and x4 which get arbitrary values, right. So that is what the solution.

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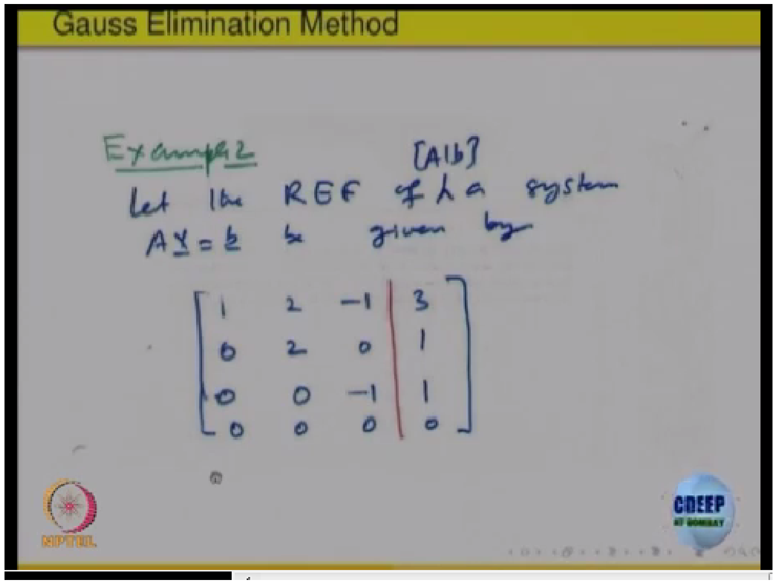

So let us do one more illustration so that you understand. So this is a system given, okay. Once again I already purposefully put in the row echelon form so that we can analyze what is the backward substitution. Otherwise, you write A and b and reduce it to row echelon form, you will get something like this, right. Here it is already done. That A~ b~. So last row does not give you anything, is 0, forget about it, no information.

What does this equation give you? This variable, which is this variable? x1 x2 x3. That gives you -x3=1, right. -1*x3=1. So x3=-1, okay. Look at this equation. It says 0*x1, so gone, 2x2, this is gone, 2x2=1. So x2 is known. So x1 is known, x2 is known. From this equation, what is the first equation now? x1+2x2-x3=3. But all those 2 are already known. Put those values here backward, you will get x1, right.

So in this system, there is only 1 solution possible, right. In this system, there is only 1 solution possible. And keep in mind, what is 4 here? r is 3 which is equal to n, right, is the minimum of, it is equal to n, the number of variables. See r=n, that means pivots are going to, each pivot is going to come in each column. If r=number of columns, then where going to be the pivots. First pivot is in column 1, right.

1 has p1, 2 has p2. If the p1 p2 pr are equal to n columns, each column has to get a pivot. That means there will be a unique solution. If each column has a pivot, it is going to be unique solution, right, that is what is happening in this. So these observations we are putting, we will put them together.

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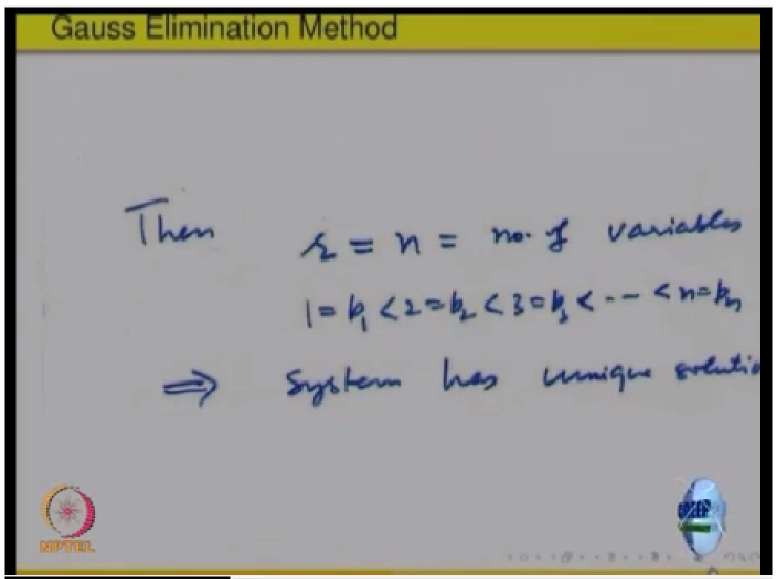

So r=n, the number of variables, right. Here p2=3, p3n, n=3x. This system has a unique solution, right. That is what I was saying. So p1 is 1, p2 is 2, p3 is 3. There are only 3 columns. Each one is having a pivot, right. That means everything will be determined uniquely. Clear? So r=n means unique solution. If r<n, then n-r variables get independent values, arbitrary values. Then infinite solutions come, right. If r<, then some columns will not have pivots. That means those variables will be determined in terms of the other one, okay.

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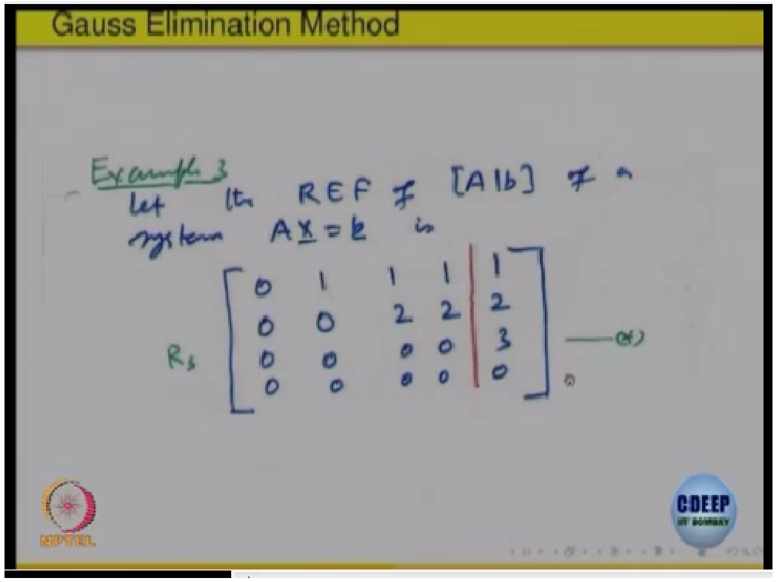

Now let us look at this another one. That means when I reduce the matrix A, aim is keep A and b side by side. Aim is to make A in the row echelon form. Go on changing b accordingly. Do not bother about it. So it gives you, last row is 0. But here comes a row in A which is 0 but on the right hand side, becomes 3. So what will this equation look like? 0=3, that means there is an absurd equation.

One of the equations is never going to have a solution. That means the system cannot have a solution. So if Ap augmented matrix, when A is reduced to row echelon form, you find there is a row, right, where A, all the elements in that row are 0 in A but the corresponding b is not 0, right. Then the system cannot have a, that was an inconsistent I wrote in the beginning itself. All cij's are 0 but, right, br is not equal to 0, right.

So that gives you an inconsistent. So this is the way you will spot an inconsistency. So 3 possibilities arise. One the number of pivots, right, the r is strictly less than n, the number of variables. Then n-r variables will get arbitrary values and remaining r will be determined in terms of these n-r.

Second, each column gets a pivot. n=r. Then unique solution, right. And third, there is a row 0=non-0, right. So in that case, no solution, inconsistent. So 3 possibilities. Again look at 2 variables and 3 variables. 2 lines, either they are parallel, so no solution, they do not meet, right.

Then 2 lines intersect.

They will intersect only at a unique point and then they are coincidental. So when the 2 lines are coincidental, then in the row echelon form, 1 row will be 0; otherwise, relation between 2 variables. So 1 variable will give you the other value. So infinite solutions, right. So perfect similarity comes, okay.

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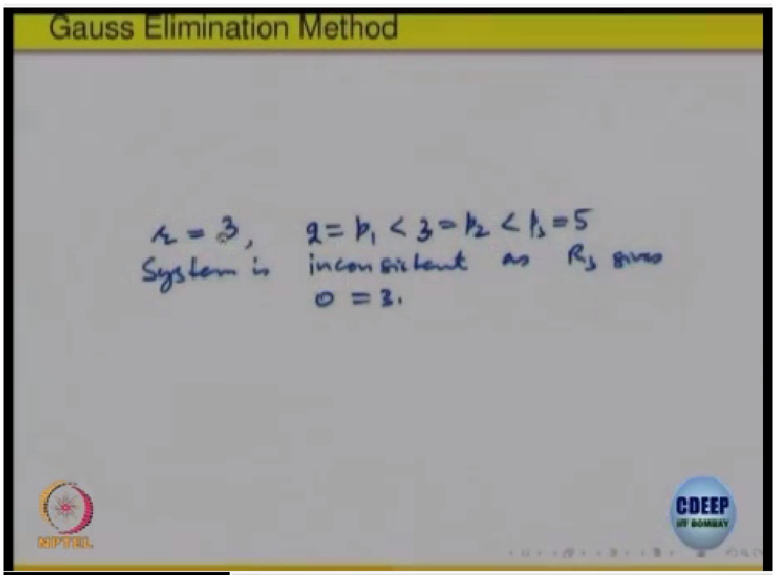

So this says r3=3, p1 is, so let us look at back. p1 is, this is p1, first pivot. 2, second pivot coming at the column 3, right. So p2 is coming at 3, right. Last one is 5. So 0=3 inconsistent, okay.

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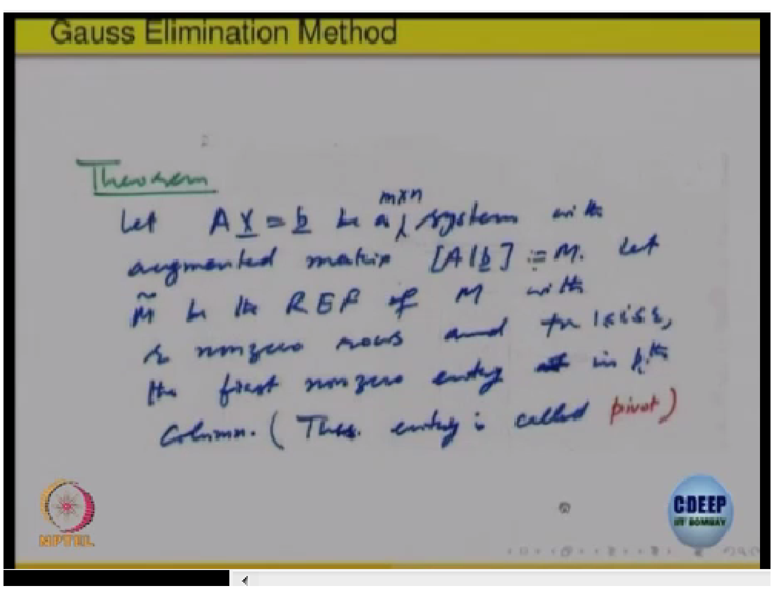

So once again, let us, this row echelon form for solving a system of linear equations. Summarize what we have done. So it says look at the system Ax=b, a m*n system with augmented matrix as A along with b, call it as M, augmented matrix, we are calling it as M. So reduce the matrix to the row echelon form and look at the non-0 rows, r is a number of non-0 rows for this and pivots are coming at pith place, right.

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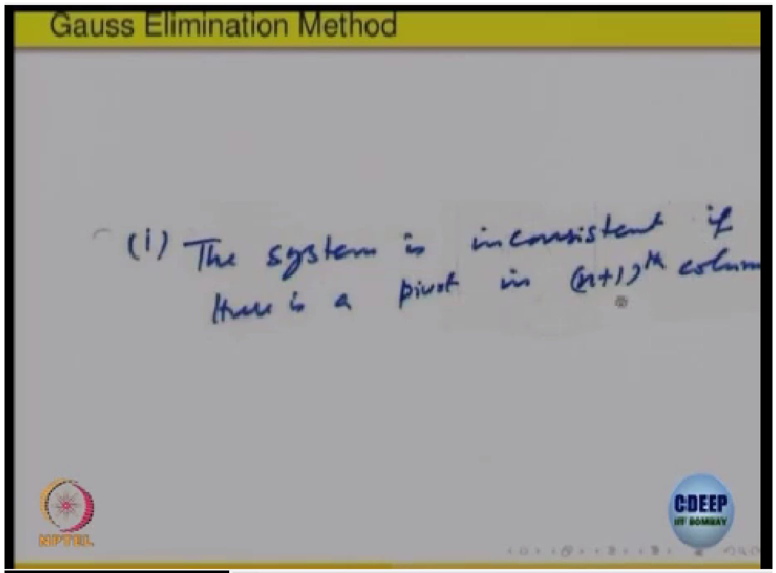

Then what? The system is inconsistent if there is a pivot in that column, right, the b column. There is a non-0. If the pivot comes there, that means all the previous must be 0, right. If in the augmented matrix, there is a pivot lying in the column where b is coming that means there is a non-0 entry in that, right. Everything else on the left side must be, what is the definition of a pivot? Everything on the left is 0. So 0=non-0 that is inconsistent, system is inconsistent. That is what we saw, right.

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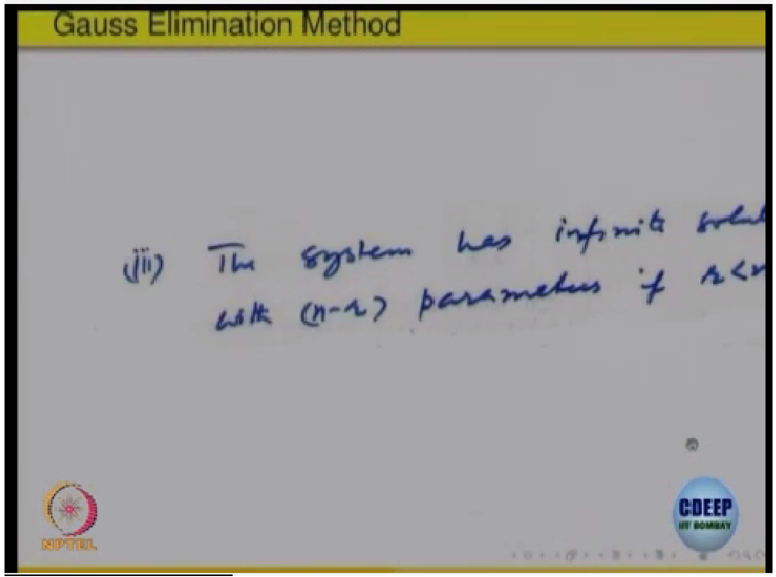

And if r<n, then there are n-r parameters or variables, whatever you want to call them, which will determine a general solution, right.

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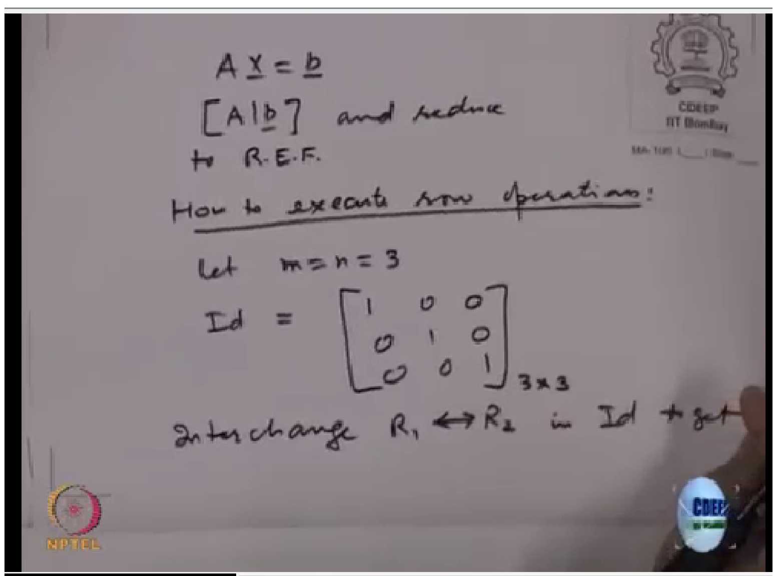

So what we have done is? For a system Ax=b, right, we take A and b and reduce to row echelon form, right. Now reducing a matrix to row echelon form is by scanning and interchanging or multiplying, right. Now how does a computer knows, I cannot give a verbal command, interchange. I have to give it in the form of something computational to the, you understand what I am saying.

If you want to implement this command that interchange 2 rows. How does the computer understand that command? It can understand only multiplication, division, addition, arithmetic, right. So what we are going to do is, interchanging of 2 rows, that is one command. Other is multiplying a matrix by, row by a non-0 scalar, second. And third is adding, right. Adding one row to another.

These ones we want to make it executable commands so that a computer can understand, right. So let us see how do we do it. So let us, how to execute row operations. So that is what we want to understand. So let us do some simple thing. So let us take m=n=3. That is for a square matrix we will take. Example for a 3*3. There is something called identity matrix.

What is that? It is a 3*3 matrix. What does the entries look like? Along diagonal, it is 1. Everything else is 0. So 1 0 0 0 1 0 0 0 1, right. Now let us do those operations on this matrix and see what happens, right. So interchange, so this is R1 and R2. So let us look at that. Let us interchange R1 with R2 in this identity matrix to get. So what we will get? Let us see what do we get?

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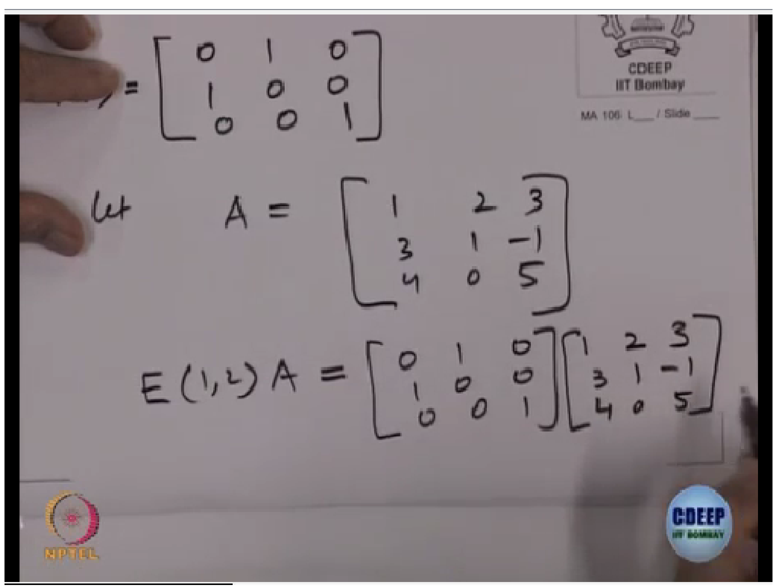

So we will get the matrix 0 1 0, R2 has become R1. 1 0 0 and 0 0 1, right. That is on identity matrix, I have interchanged, done that operation. Now let us take some other matrix A, whatever you like. Let us, okay, let us take 1 2 3 3 1 -1 4 0 5, some arbitrary matrix, right. And let us call this matrix as E12 which we have got after interchanging, right. I want to compute E12*A, I want to compute this. So let us write 0 1 0 1 0 0 0 0 1, 1 2 3 3 1 -1 4 0 5, right. Let us multiply them. What do we get? So =.

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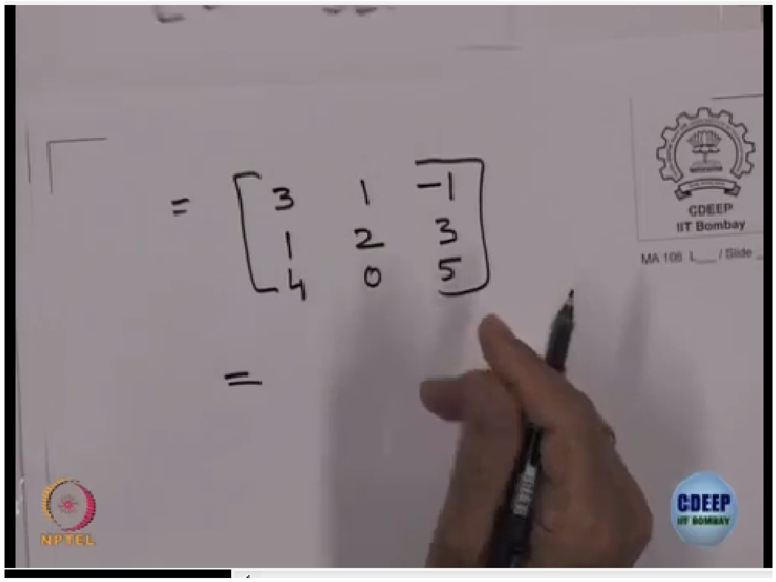

What will be the first entry? This row multiplied by this column, added up, so what I will get. 0*1 is 0+1*3 is 3+0*4, so what is it? 3. Is it okay. What is the next one? This multiplied by the next one, 0*2=0, 1*1=1, so +1. 0*0, so that is 1. What is the next one? 0 1 0 multiplied by, so 0*3=0, 1*-1, that is -1 and 0*5=0, so that is -1. Is it okay? Let us do the next one. 1 0 0, multiplied by the first one. So 1 goes to 1, 0 3 0 0 4 0, so that gives you 1. Next one multiplied by the second one, 1*2=2, 0*1=0, 0*, so that is 2.

And the next one 1*3, both others are 0, so that is equal to 3. The last row, 0 0 1 1 3 4 0 0 and that is, so what does it give me, that gives me 4. 0 0 1 with the second one, 0 2 0 0 1 0 and this gives you 0, that is 0. And the last one, 0*3=0, 0 with -1 is 0, 1 with 5 is 5, right. What is this matrix? This is precisely the matrix A with R1 interchanged with R2, right. So what we are saying is the following. Then if you want to make some operation on a matrix.

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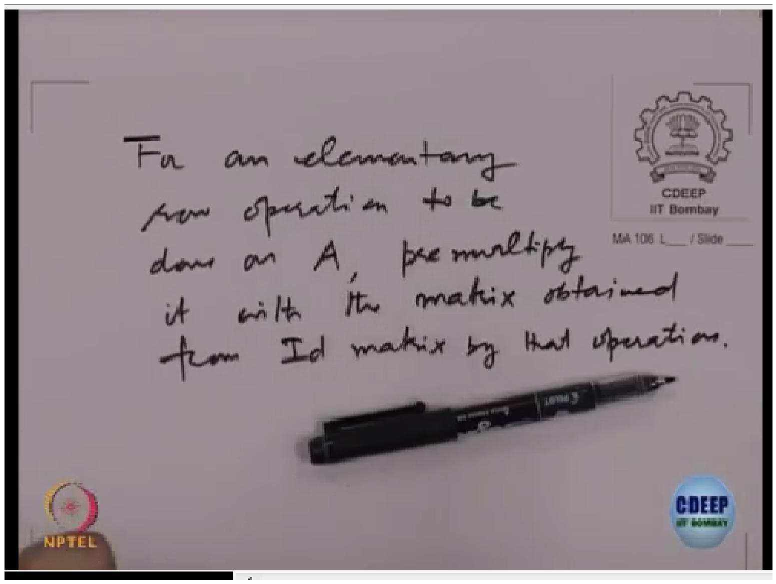

For an elementary row operation to be done on A, premultiply it with the matrix obtained from identity matrix by that operation. So what we are saying is to give a command to a computer, how do I do that operation is take identity matrix, okay. Whatever operation you want elementary row operation you want to be done on A, do it on identity first. We will get a matrix. So store that matrix.

Once you have stored that matrix, multiply the matrix A on the left side by that matrix. The resulting matrix will be doing that operation on A. So if you want to interchange 2 rows, take identity matrix, interchange its rows, you will get a matrix. Premultiply A with it, the resulting matrix will be the one where rows of A are corresponding rows are interchanged.

If you want to add, do the same thing for identity matrix, add those corresponding rows, identity matrix, you get a matrix, premultiply with A, A with this matrix, the corresponding rows will be added. And similarly, right, if you want to multiply and add or interchange, same thing you do here and apply it here. So for all 3 elementary row operations, what are the 3 elementary row operations?

One was interchange of rows. The other was multiplying row by a scalar. And the third one was multiplying and adding. These 3 operations you can do on identity matrix and store them and whenever you want to do these operations on A, premultiply A with that matrix. So computer will store in the computer those things and do that multiplications on the computer to do those operations, right.

So these matrices are called elementary matrices. What are elementary matrices? These are the matrices obtained from the corresponding identity matrix by doing elementary row operations on them of 3 types, right. So there are 3 types of elementary matrices which are used to do elementary row operations on any given matrix, right. So we will continue study of this elementary matrices next time.